J. Korean Math. Soc. **52** (2015), No. 5, pp. 1097–1108
http://dx.doi.org/10.4134/JKMS.2015.52.5.1097

SOME RESULTS OF p-BIHARMONIC MAPS INTO A NON-POSITIVELY CURVED MANIFOLD

Yingbo Han and Wei Zhang

Abstract. In this paper, we investigate p-biharmonic maps $u : (M, g) \to (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. We obtain that if $\int_M |\tau(u)|^{a+p} dv_g < \infty$ and $\int_M |d(u)|^2 dv_g < \infty$, then u is harmonic, where $a \geq 0$ is a nonnegative constant and $p \geq 2$. We also obtain that any weakly convex p-biharmonic hypersurfaces in space form $N(c)$ with $c \leq 0$ is minimal. These results give affirmative partial answer to Conjecture 2 (generalized Chen's conjecture for p-biharmonic submanifolds).

1. Introduction

Harmonic maps play a central roll in geometry. They are critical points of the energy $E(u) = \int_M \frac{|du|^2}{2} dv_g$ for smooth maps between manifolds $u : (M, g) \to (N, h)$ and the Euler-Lagrange equation is that tension field $\tau(u)$ vanishes. Extensions to the notions of p-harmonic maps, F-harmonic maps and f-harmonic maps were introduced and many results have been carried out (for instance, see [1, 2, 3, 8, 23]). In 1983, J. Eells and L. Lemaire [10] proposed the problem to consider the biharmonic maps: they are critical maps of the functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$

We see that harmonic maps are biharmonic maps and even more, minimizers of the bienergy functional. After G. Y. Jiang [15] studied the first and second variation formulas of the bienergy E_2, there have been extensive studies on biharmonic maps (for instance, see [9, 15, 16, 17, 21, 22, 24, 25]). Recently the first author and S. X. Feng in [13] introduced the following functional

$$E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2}) dv_g,$$

Received January 15, 2015.
2010 *Mathematics Subject Classification.* 58E20, 53C21.
Key words and phrases. p-biharmonic maps, p-biharmoinc submanifolds.

where $\tau(u) = -\delta du = \mathrm{trace}\widetilde{\nabla}(du)$. The map u is called an F-biharmonic map if it is a critical point of that F-bienergy $E_{F,2}(u)$, which is a generalization of biharmonic maps, p-biharmonic maps [14] or exponentially biharmonic maps. Notice that harmonic maps are always F-biharmonic by definition. When $F(t) = (2t)^{\frac{p}{2}}$, we have a p-bienergy functional

$$E_{p,2}(u) = \int_M |\tau(u)|^p dv_g,$$

where $p \geq 2$. The Euler-Lagrange equation of $E_{p,2}$ is $\tau_{p,2}(u) = 0$, where $\tau_{p,2}(u)$ is given by (13). A map $u : (M, g) \to (N, h)$ is called a p-biharmonic map if $\tau_{p,2}(u) = 0$. When $u : (M, g) \to (N, h)$ is a p-biharmonic isometric immersion, then M is called a p-biharmonic submanifold in N.

Recently, N. Nakauchi, H. Urakawa and S. Gudmundsson [21] proved that biharmoic maps from a complete Riemannian manifold into a non-positive curved manifold with finite bienergy and energy are harmonic. S. Maeta [20] proved that biharmoic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a + 2)$-bienergy $\int_M |\tau(u)|^{a+2} dv_g < \infty$ $(a \geq 0)$ and energy are harmonic. In this paper, we first obtain the following result:

Theorem 1.1 (cf. Theorem 3.1)**.** *Let $u : (M^m, g) \to (N^n, h)$ be a p-biharmonic map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature and let $a \geq 0$ be a non-negative real constant.*

(i) *If*

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

and the energy is finite, that is,

$$\int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $\mathrm{Vol}(M, g) = \infty$, and*

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

then u is harmonic, where $p \geq 2$.

One of the most interesting problems in the biharmonic theory is Chen's conjecture. In 1988, Chen raised the following problem:

Conjecture 1 ([7])**.** Any biharmonic submanifold in E^n is minimal.

There are many affirmative partial answers to Chen's conjecture.

On the other hand, Chen's conjecture was generalized as follows (cf. [6]): "Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal". There are also many affirmative partial answers to this conjecture.

(a) Any biharmonic submanifold in $H^3(-1)$ is minimal (cf. [5]).

(b) Any biharmonic hypersurfaces in $H^4(-1)$ is minimal (cf. [4])

(c) Any weakly biharmonic hypersurfaces in space form $N^{m+1}(c)$ with $c \leq 0$ is minimal (cf. [18])

(d) Any compact biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [15]).

(e) Any compact F-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [13]).

For p-biharmonic submanifolds, it is natural to consider the following problem.

Conjecture 2. Any p-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

For p-biharmonic submanifolds, we obtain the following result:

Theorem 1.2 (cf. Theorem 4.1)**.** *Let $u : (M^m, g) \to (N^{m+1}, \langle , \rangle)$ be a weakly convex p-biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c \leq 0$. Then u is minimal, where $p \geq 2$.*

These results give affirmative partial answers to the generalized Chen's conjecture for p-biharmonic submanifold.

2. Preliminaries

In this section we give more details for the definitions of harmonic maps, biharmonic maps, p-biharmonic maps and p-biharmonic submanifolds.

Let $u : (M, g) \to (N, h)$ be a map from an m-dimensional Riemannian manifold (M, g) to an n-dimensional Riemannian manifold (N, h). The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} dv_g.$$

The Euler-Lagrange equation of E is

$$\tau(u) = \sum_{i=1}^{m} \{\widetilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)\} = 0,$$

where we denote by ∇ the Levi-Civita connection on (M, g) and $\widetilde{\nabla}$ the induced Levi-civita connection on $u^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame field on (M, g). $\tau(u)$ is called the tension field of u. A map $u : (M, g) \to (N, h)$ is called a harmonic map if $\tau(u) = 0$.

To generalize the notion of harmonic maps, in 1983 J. Eells and L. Lemaire [10] proposed considering the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$

In 1986, G. Y. Jiang [15] studied the first and second variation formulas of the bienergy E_2. The Euler-Lagrange equation of E_2 is

$$\tau_2(u) = -\widetilde{\triangle}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i))du(e_i) = 0,$$

where $\widetilde{\triangle} = \sum_i (\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i} e_i})$ is the rough Laplacian on the section of $u^{-1}TN$ and $R^N(X,Y) = [^N\nabla_X, ^N\nabla_Y] -^N \nabla_{[X,Y]}$ is the curvature operator on N. A map $u : (M,g) \to (N,h)$ is called a biharmonic map if $\tau_2(u) = 0$.

To generalize the notion of biharmoic maps, the first author and S. X. Feng [13] introduced the F-bienergy functional

$$E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2})dv_g,$$

where $F : [0,\infty) \to [0,\infty)$ is a C^3 function such that $F' > 0$ on $(0,\infty)$. The Euler-Lagrange equation of $E_{F,2}$ is

$$\tau_{F,2}(u) = -\widetilde{\triangle}(F'(\frac{|\tau(u)|^2}{2})\tau(u)) - \sum_i R^N(F'(\frac{|\tau(u)|^2}{2})\tau(u), du(e_i))du(e_i) = 0.$$

A map $u : (M,g) \to (N,h)$ is called a F-biharmonic map if $\tau_{F,2}(u) = 0$.

When $F(t) = (2t)^{\frac{p}{2}}$, we have a p-bienergy functional

$$E_{p,2}(u) = \int_M |\tau(u)|^p dv_g,$$

where $p \geq 2$. The Euler-Lagrange equation of $E_{p,2}$ is

$$\tau_{p,2}(u) = -\widetilde{\triangle}(p|\tau(u)|^{p-2}\tau(u)) - \sum_i R^N(p|\tau(u)|^{p-2}\tau(u), du(e_i))du(e_i) = 0.$$

A map $u : (M,g) \to (N,h)$ is called a p-biharmonic map if $\tau_{p,2}(u) = 0$.

Now we recall the definition of p-biharmonic submanifolds (cf. [12]).

Let $u : (M,g) \to (N, h = \langle\cdot,\cdot\rangle)$ be an isometric immersion from an m-dimensional Riemannian manifold into an $m+t$-dimensional Riemannian manifold. We identify $du(X)$ with $X \in \Gamma(TM)$ for each $x \in M$. We also denote by $\langle\cdot,\cdot\rangle$ the induced metric $u^{-1}h$. The Gauss formula is given by

$${}^N\nabla_X Y = \nabla_X Y + B(X,Y), \quad X, Y \in \Gamma(TM),$$

where B is the second fundamental form of M in N. The Weingarten formula is given by

$${}^N\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X \in \Gamma(TM),\ \xi \in \Gamma(T^\perp M),$$

where A_ξ is the shape operator for a unit normal vector field ξ on M, and $\nabla^\perp$ denotes the normal connection on the normal bundle of M in N. For any $x \in M$, the mean curvature vector field H of M at x is given by

$$H = \frac{1}{m}\sum_{i=1}^m B(e_i, e_i).$$

If an isometric $u : (M, g) \to (N, h)$ is p-biharmonic, then M is called a p-biharmonic submanifold in N. In this case, we remark that the tension field $\tau(u)$ of u is written $\tau(u) = mH$, where H is the mean curvature vector field of M. The necessary and sufficient condition for M in N to be p-biharmonic is the following:

$$-\widetilde{\triangle}(|H|^{p-2}H) - \sum_i R^N(|H|^{p-2}H, e_i)e_i = 0. \tag{1}$$

From (1), we obtain the necessary and sufficient condition for M in N to be p-biharmonic as follows:

$$\triangle^\perp(|H|^{p-2}H) - \sum_{i=1}^m B(e_i, A_{|H|^{p-2}H}(e_i)) + [\sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i]^\perp = 0, \tag{2}$$

$$Tr_g(\nabla A_{|H|^{p-2}H}) + Tr_g[A_{\nabla^\perp(|H|^{p-2}H)}(\cdot)] - [\sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i]^\top = 0, \tag{3}$$

where $\triangle^\perp = \sum_{i=1}^m(\nabla^\perp_{e_i}\nabla^\perp_{e_i} - \nabla^\perp_{\nabla_{e_i}e_i})$ is the Laplace operator associated with the normal connection $\nabla^\perp$.

We also need the following lemma.

Lemma 2.1 (Gaffney, [11])**.** *Let (M, g) be a complete Riemannian manifold. If a C^1 a-form α satisfies that $\int_M |\alpha| dv_g < \infty$ and $\int_M(\delta\alpha)dv_g < \infty$, or equivalently, a C^1 vector X defined by $\alpha(Y) = \langle X, Y\rangle$ $(\forall Y \in \Gamma(TM))$ satisfies that $\int_M |X|dv_g < \infty$ and $\int_M div(X)dv_g < \infty$, then*

$$\int_M (-\delta\alpha)dv_g = \int_M div(X)dv_g = 0. \tag{4}$$

3. Main results of p-biharmonic maps

In this section, we obtain the following result.

Theorem 3.1. *Let $u : (M^m, g) \to (N^n, h)$ be a p-biharmonic map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature and let $a \geq 0$ be a non-negative real constant.*

(i) *If*

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

and the energy is finite, that is,

$$\int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If* $\mathrm{Vol}(M, g) = \infty$*, and*

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

then u is harmonic, where $p \geq 2$.

Proof. Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$\begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla \lambda| \leq \frac{C}{r}, & x \in M, \end{cases} \tag{5}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of (M, g). From (13), we have

$$\begin{aligned} & \int_M \langle -\widetilde{\triangle}(|\tau(u)|^{p-2}\tau(u)), \lambda^2 |\tau(u)|^a \tau(u) \rangle dv_g \\ = & \int_M \lambda^2 |\tau(u)|^{a+p-2} \sum_{i=1}^m \langle R^N(\tau(u), du(e_i)) du(e_i), \tau(u) \rangle dv_g \leq 0, \end{aligned} \tag{6}$$

since the sectional curvature of (N, h) is non-positive. From (6), we have

$$\begin{aligned} 0 \geq & \int_M \langle -\widetilde{\triangle}(|\tau(u)|^{p-2}\tau(u)), \lambda^2 |\tau(u)|^a \tau(u) \rangle dv_g \\ = & \int_M \langle \widetilde{\nabla}(|\tau(u)|^{p-2}\tau(u)), \widetilde{\nabla}(\lambda^2 |\tau(u)|^a \tau(u)) \rangle dv_g \\ = & \int_M \sum_{i=1}^m \langle \widetilde{\nabla}_{e_i}(|\tau(u)|^{p-2}\tau(u)), \widetilde{\nabla}_{e_i}(\lambda^2 |\tau(u)|^a \tau(u)) \rangle dv_g \\ = & \int_M \sum_{i=1}^m [\langle |\tau(u)|^{p-2} \widetilde{\nabla}_{e_i}\tau(u) + (p-2)|\tau(u)|^{p-4} \langle \widetilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u), \\ & 2\lambda e_i(\lambda) |\tau(u)|^a \tau(u) + \lambda^2 e_i(|\tau(u)|^a) \tau(u) + \lambda^2 |\tau|^a \widetilde{\nabla}_{e_i}\tau(u) \rangle] dv_g \\ = & \int_M \sum_{i=1}^m [\langle |\tau(u)|^{p-2} \widetilde{\nabla}_{e_i}\tau(u) + (p-2)|\tau(u)|^{p-4} \langle \widetilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u), \\ & 2\lambda e_i(\lambda) |\tau(u)|^a \tau(u) + a\lambda^2 |\tau(u)|^{a-2} \langle \widetilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u) \\ & + \lambda^2 |\tau|^a \widetilde{\nabla}_{e_i}\tau(u) \rangle] dv_g \\ = & \int_M \sum_{i=1}^m 2(p-1)\lambda e_i(\lambda) |\tau(u)|^{a+p-2} \langle \widetilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle dv_g \\ & + \int_M \sum_{i=1}^m [a(p-1) + (p-2)] \lambda^2 |\tau(u)|^{a+p-4} \langle \widetilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle^2 dv_g \\ & + \int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a+p-2} \langle \widetilde{\nabla}_{e_i}\tau(u), \widetilde{\nabla}_{e_i}\tau(u) \rangle dv_g \end{aligned}$$

$$\geq \int_M \sum_{i=1}^m 2(p-1)\lambda e_i(\lambda)|\tau(u)|^{a+p-2}\langle\widetilde{\nabla}_{e_i}\tau(u),\tau(u)\rangle dv_g$$

$$+\int_M \sum_{i=1}^m \lambda^2|\tau(u)|^{a+p-2}\langle\widetilde{\nabla}_{e_i}\tau(u),\widetilde{\nabla}_{e_i}\tau(u)\rangle dv_g, \tag{7}$$

where the inequality follows from $[a(p-1)+(p-2)]\lambda^2|\tau(u)|^{a+p-4}\langle\widetilde{\nabla}_{e_i}\tau(u),\tau(u)\rangle^2 \geq 0$. From (7), we have

$$\int_M \sum_{i=1}^m \lambda^2|\tau(u)|^{a+p-2}\langle\widetilde{\nabla}_{e_i}\tau(u),\widetilde{\nabla}_{e_i}\tau(u)\rangle dv_g$$

$$\leq -2(p-1)\int_M \sum_{i=1}^m \langle\widetilde{\nabla}_{e_i}\tau(u),\lambda e_i(\lambda)|\tau(u)|^{a+p-2}\tau(u)\rangle dv_g. \tag{8}$$

By using Young's inequality, we have

$$-2(p-1)\int_M \sum_{i=1}^m \langle\widetilde{\nabla}_{e_i}\tau(u),\lambda e_i(\lambda)|\tau(u)|^{a+p-2}\tau(u)\rangle dv_g$$

$$\leq \frac{1}{2}\int_M \sum_{i=1}^m \lambda^2|\tau(u)|^{a+p-2}|\widetilde{\nabla}_{e_i}\tau(u)|^2 dv_g + 2(p-1)^2\int_M |\nabla\lambda|^2|\tau(u)|^{a+p}dv_g. \tag{9}$$

From (8) and (9), we have

$$\int_M \sum_{i=1}^m \lambda^2|\tau(u)|^{a+p-2}\langle\widetilde{\nabla}_{e_i}\tau(u),\widetilde{\nabla}_{e_i}\tau(u)\rangle dv_g$$

$$\leq 4(p-1)^2\int_M |\nabla\lambda|^2|\tau(u)|^{a+p}dv_g$$

$$\leq \frac{4(p-1)^2C^2}{r^2}\int_M |\tau(u)|^{a+p}dv_g$$

$$\leq \frac{4(p-1)^2C^2}{r^2}\int_M |\tau(u)|^{a+p}dv_g. \tag{10}$$

By assumption $\int_M |\tau(u)|^{a+p}dv_g < \infty$, letting $r \to \infty$ in (10), we have

$$\int_M \sum_{i=1}^m |\tau(u)|^{a+p-2}\langle\widetilde{\nabla}_{e_i}\tau(u),\widetilde{\nabla}_{e_i}\tau(u)\rangle dv_g = 0.$$

Therefore, we obtain that $|\tau(u)|$ is constant and $\widetilde{\nabla}_X\tau(u) = 0$ for any vector field X on M.

Therefore, if $Vol(M) = \infty$ and $|\tau(u)| \neq 0$, then

$$\int_M |\tau(u)|^{a+p}dv_g = |\tau(u)|^{a+p}\mathrm{Vol}(M) = \infty,$$

which yields a contradiction. Thus, we have $|\tau(u)| = 0$, i.e., u is harmonic. We have (ii).

For (i), assume both $\int_M |\tau(u)|^{a+p} dv_g < \infty$ and $\int_M |du|^2 dv_g < \infty$. Define a 1-from α on M defined by

$$\alpha(X) = |\tau(u)|^{\frac{a+p}{2}-1} \langle du(X), \tau(u)\rangle \tag{11}$$

for any vector $X \in \Gamma(TM)$.

Note here that

$$\begin{aligned}
\int_M |\alpha|^2 dv_g &= \int_M [\sum_{i=1}^m |\alpha(e_i)|^2]^{\frac{1}{2}} dv_g \\
&= \int_M [\sum_{i=1}^m [|\tau(u)|^{\frac{a+p}{2}-1} \langle du(e_i), \tau(u)\rangle]^2]^{\frac{1}{2}} dv_g \\
&\leq \int_M |\tau(u)|^{\frac{a+p}{2}} |du| dv_g \\
&\leq [\int_M |\tau(u)|^{a+p} dv_g]^{\frac{1}{2}} [\int_M |du|^2 dv_g]^{\frac{1}{2}} < \infty. \qquad (12)
\end{aligned}$$

Now we compute

$$\begin{aligned}
-\delta\alpha = & \sum_{i=1}^m (\nabla_{e_i}\alpha)(e_i) = \sum_{i=1}^m [\nabla_{e_i}\alpha(e_i) - \alpha(\nabla_{e_i} e_i)] \\
= & \sum_{i=1}^m \nabla_{e_i} [|\tau(u)|^{\frac{a+p}{2}-1} \langle du(e_i), \tau(u)\rangle] \\
& - \sum_{i=1}^m |\tau(u)|^{\frac{a+p}{2}-1} \langle du(\nabla_{e_i} e_i), \tau(u)\rangle \\
= & \sum_{i=1}^m |\tau(u)|^{\frac{a+p}{2}-1} \langle \widetilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i), \tau(u)\rangle \\
= & |\tau(u)|^{\frac{a+p}{2}+1},
\end{aligned}$$

where the fourth equality follows from that $|\tau(u)|$ is constant and $\widetilde{\nabla}_X \tau(u) = 0$ for $X \in \Gamma(TM)$. Since $\int_M |\tau(u)|^{a+p} dv_g < \infty$ and $|\tau(u)|$ is constant, the function $-\delta\alpha$ is also integrable over M. From this and (12), we can apply Lemma 2.1 for the 1-from α. Therefore we have

$$0 = \int_M (-\delta\alpha) dv_g = \int_M |\tau(u)|^{\frac{a+p}{2}+1} dv_g,$$

so we have $\tau(u) = 0$, that is, u is harmonic. □

4. Main results of *p*-biharmonic hypersurfaces

In this section, we obtain the following result.

Theorem 4.1. *Let* $u : (M^m, g) \to (N^{m+1}, \langle, \rangle)$ *be a weakly convex* p*-biharmonic hypersurface in a space form* $N^{m+1}(c)$ *with* $c \leq 0$*. Then* u *is minimal, where* $p \geq 2$*.*

Proof. Assume that $H = h\nu$, where ν is the unit normal vector field on M. Since M is weakly convex, we have $h \geq 0$. Set $C = \{q \in M : h(q) > 0\}$. We will prove that A is an empty set.

If C is not empty, we see that C is an open subset of M. We assume that C_1 is a nonempty connect component of C. We will prove that $h \equiv 0$ in C_1, thus a contradiction.

Firstly, we prove that h is a constant in C_1.

Let $q \in C_1$ be a point. Choose a local orthonormal frame $\{e_i, i = 1, \dots, m\}$ around q such that $\langle B, \nu \rangle$ is a diagonal matrix and $\nabla_{e_i} e_j|_q = 0$.

From equation (3), we have at q

$$\begin{aligned}
0 &= \langle \sum_{i=1}^{m} (\nabla_{e_i} A_{(h^{p-2}H)})(e_i), e_k \rangle + \langle \sum_{i=1}^{m} A_{\nabla^{\perp}_{e_i}(h^{p-2}H)}(e_i), e_k \rangle \\
&= \sum_{i=1}^{m} e_i \langle A_{(h^{p-2}H)}(e_i), e_k \rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(h^{p-2}H) \rangle \\
&= \sum_{i=1}^{m} e_i \langle h^{p-2}H, B(e_i, e_k) \rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(h^{p-2}H) \rangle \\
&= \sum_{i=1}^{m} \langle h^{p-2}H, \nabla^{\perp}_{e_i} B(e_i, e_k) \rangle + 2\sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(h^{p-2}H) \rangle \\
&= \sum_{i=1}^{m} \langle h^{p-2}H, \nabla^{\perp}_{e_k} B(e_i, e_i) \rangle + 2\sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(h^{p-2}H) \rangle \\
&= m\langle h^{p-2}H, \nabla^{\perp}_{e_k} H \rangle + 2\langle \lambda_k \nu, \nabla^{\perp}_{e_i}(h^{p-2}H) \rangle \\
&= mh^{p-1} e_k(h) + 2(p-1)h^{p-2}\lambda_k e_k(h) \\
&= (mh + 2(p-1)\lambda_k)h^{p-2} e_k(h),
\end{aligned}$$

where λ_k is the kth principle curvature of M at q, which is nonnegative by the assumption that M is weakly convex. Since $(mh + 2(p-1)\lambda_k)h^{p-2} > 0$ at q, we have $e_k(h) = 0$ at q, for $k = 1, \dots, m$, which implies that $\nabla h = 0$ at q. Because q is an arbitrary point in C_1, we have $\nabla h = 0$ in C_1. Therefore we obtain that h is constant in C_1.

Secondly, we prove that h is zero in C_1.

From the equation (2), we have

$$\begin{aligned}
\triangle h^{2p-2} &= \triangle \langle h^{p-2}H, h^{p-2}H \rangle \\
&= 2\langle \triangle^{\perp}(h^{p-2}H), h^{p-2}H \rangle + 2|\nabla^{\perp}(h^{p-2}H)|^2
\end{aligned}$$

$$= 2|\nabla^{\perp}(h^{p-2}H)|^2 + 2\sum_{i=1}^{m}\langle B(A_{h^{p-2}H}e_i, e_i), h^{p-2}H\rangle$$

$$- \sum_{i=1}^{m}\langle R^N(h^{p-2}H, e_i)e_i, h^{p-2}H\rangle$$

$$\geq |\nabla^{\perp}(h^{p-2}H)|^2 + 2\sum_{i=1}^{m}\langle B(A_{h^{p-2}H}e_i, e_i), h^{p-2}H\rangle, \tag{13}$$

where the inequality follows from the sectional curvature of (N, h) is non-positive. Now we state two inequalities:

$$|\nabla^{\perp}(h^{p-2}H)|^2 \geq h^{2p-4}|\nabla^{\perp}H|^2 \tag{14}$$

and

$$\sum_{i=1}^{m}\langle B(A_{h^{p-2}H}e_i, e_i), h^{p-2}H\rangle \geq mh^{2p}. \tag{15}$$

In fact,

$$\begin{aligned}
|\nabla^{\perp}(h^{p-2}H)|^2 &= |(p-2)h^{p-4}\langle\nabla^{\perp}H, H\rangle H + h^{p-2}\nabla^{\perp}H|^2 \\
&= (p-2)^2h^{2p-6}\langle\nabla^{\perp}H, H\rangle^2 + h^{2p-4}|\nabla^{\perp}H|^2 \\
&\quad + 2(p-2)h^{3p-6}\langle\nabla^{\perp}H, H\rangle^2 \\
&\geq h^{2p-4}|\nabla^{\perp}H|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{m}\langle B(A_{h^{p-2}H}e_i, e_i), h^{p-2}H\rangle \\
&= \sum_{i=1}^{m} h^{2p-2}\langle B(A_{\nu}e_i, e_i), \nu\rangle \\
&= \sum_{i=1}^{m} h^{2p-2}\langle A_{\nu}e_i, A_{\nu}e_i\rangle \\
&= \sum_{i,j=1}^{m} h^{2p-2}|\langle B(e_i, e_j), \nu\rangle|^2 \\
&\geq mh^{2p}.
\end{aligned}$$

From (13), (14) and (15), we have

$$\triangle h^{2p-2} \geq 2h^{2p-4}|\nabla^{\perp}H|^2 + 2mh^{2p}.$$

So we have

$$\triangle h^{2p-2} \geq 2mh^{2p}. \tag{16}$$

From equation (16), we have in C_1

$$0 = \triangle h^{2p-2} \geq 2mh^{2p}.$$

We know that $h \equiv 0$ in C_1. This is a contradiction. □

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant Nos.11201400, 11201447); Basic and Frontier Technology Research Project of Henan Province (Grant No.142300410433) and Project for youth teacher of Xinyang Normal University (Grant No.2014-QN-061).

References

[1] M. Ara, *Geometry of F-harmonic maps*, Kodai Math. J. **22** (1999), no. 2, 243–263.
[2] ———, *Instability and nonexistence theorems for F-harmonic maps*, Illinois J. Math. **45** (2001), no. 2, 657–679.
[3] ———, *Stability of F-harmonic maps into pinched manifolds*, Hiroshima Math. J. **31** (2001), no. 1, 171–181.
[4] A. Balmus, S. Montaldo, and C. Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space form*, Math. Nachr. **283** (2010), no. 12, 1696–1705.
[5] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. **130** (2002), 109–123.
[6] R. Caddeo, S. Montaldo, and P. Piu, *On biharmonic maps*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 286–290, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.
[7] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Michigan State University, 1988.
[8] L. F. Cheung and P. F. Leung, *Some results on stable p-harmonic maps*, Glasgow Math. J. **36** (1994), no. 1, 77–80.
[9] Y. J. Chiang, *Developments of Harmonic Maps, Wave Maps and Yang-Mills Fields into Biharmonic Maps, Biwave Maps and Bi-Yang-Mills Fields*, Frontiers in Mathematics, Birkhäuser/Springer, Basel, 2013.
[10] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS, 50, Amer. Math. Soc., 1983.
[11] M. P. Gaffney, *A special Stoke's theorem for complete Riemannian manifold*, Ann. Math. **60** (1954), 140–145.
[12] Y. B. Han, *Some results of p-biharmonic submanifolds in a Riemannian manifold of non-positive curvature*, J. Geometry; doi 10.1007/s00022-015-0259-1.
[13] Y. B. Han and S. X. Feng, *Some results of F-biharmonic maps*, Acta Math. Univ. Comenian. (N.S.) **8**3 (2014), no. 1, 47–66.
[14] P. Hornung and R. Moser, *Intrinsically p-biharmonic maps*, preprint (Opus: University of Bath online publication store).
[15] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402; (the English translation) Note Mate. **28** (2009), 209–232.
[16] E. Loubeau, S. Montaldo, and C. Oniciuc, *The stress-energy tensor for biharmonic maps*, Math. Z. **259** (2008), no. 3, 503–524.
[17] E. Loubeau and C. Oniciuc, *The index of biharmonic maps in spheres*, Compos. Math. **141** (2005), no. 3, 729–745.
[18] Y. Luo, *Weakly convex biharmonic hypersurfaces in nonpositive curvature space forms are minimal*, Results Math. **65** (2014), no. 1-2, 49–56.

[19] ———, *On biharmonic submanifolds in non-positive curved manifolds*, J. Geom. Phys. **88** (2015), 76–87.

[20] S. Maeta, *Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold*, Ann. Global Anal. Geom. **46** (2014), no. 1, 75–85.

[21] N. Nakauchi, H. Urakawa, and S. Gudmundsson, *Biharmonic maps into a Riemannian manifold of non-positive curvature*, Results Math. **63** (2013), 467–474.

[22] C. Oniciuc, *Biharmonic maps between Riemannian manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **48** (2002), no. 2, 237–248.

[23] Y. L. Ou, *On f-harmonic morphisms between Riemannian manifolds*, Chinese Ann. Math. Ser. B **35** (2014), 225–236.

[24] C. Wang, *Remarks on biharmonic maps into spheres*, Calc. Var. Partial Differential Equations **21** (2004), no. 3, 221–242.

[25] ———, *Biharmonic maps from R^4 into Riemannian manifold*, Math. Z. **247** (2004), no. 1, 65–87.

Yingbo Han
College of Mathematics and Information Science
Xinyang Normal University
Xinyang, 464000, Henan, P. R. China
E-mail address: yingbohan@163.com

Wei Zhang
School of Mathematics
South China University of Technology
Guangzhou, 510641, Guangdong, P. R. China
E-mail address: sczhangw@scut.edu.cn