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THE SYMMETRY OF spin$^{\mathbb{C}}$ DIRAC SPECTRUMS ON RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. It is well-known that the spectrum of a spin$^{\mathbb{C}}$ Dirac operator on a closed Riemannian spin$^{\mathbb{C}}$ manifold M^{2k} of dimension $2k$ for $k \in \mathbb{N}$ is symmetric. In this article, we prove that over an odd-dimensional Riemannian product $M_1^{2p} \times M_2^{2q+1}$ with a product spin$^{\mathbb{C}}$ structure for $p \geq 1$, $q \geq 0$, the spectrum of a spin$^{\mathbb{C}}$ Dirac operator given by a product connection is symmetric if and only if either the spin$^{\mathbb{C}}$ Dirac spectrum of M_2^{2q+1} is symmetric or $(e^{\frac{1}{2}c_1(L_1)}\hat{A}(M_1))[M_1] = 0$, where L_1 is the associated line bundle for the given spin$^{\mathbb{C}}$ structure of M_1.

1. Introduction

This article is a generalization of the paper [7] to a spin$^{\mathbb{C}}$ Dirac operator on a spin$^{\mathbb{C}}$ manifold. Let (M^n, g) be an n-dimensional closed Riemannian manifold with a spin$^{\mathbb{C}}$ structure given by an associated complex line bundle L with $c_1(L) \equiv w_2(M^n) \bmod 2$. Here c_1 is the first Chern class and w_2 is the 2-nd Stiefel-Whitney class. Let A be a $U(1)$-connection on the line bundle L. This combined with the Levi-Civita connection of g induces a covariant derivative

$$\nabla^A : \Gamma(\Sigma(M, L)) \to \Gamma(T^*M \otimes \Sigma(M, L))$$

in the associated spinor bundle $\Sigma(M, L)$. The associated spin$^{\mathbb{C}}$ Dirac operator D^A is the composition of the covariant derivative ∇^A and Clifford multiplication γ

$$D^A = \gamma \circ \nabla^A : \Gamma(\Sigma(M, L)) \to \Gamma(T^*M \otimes \Sigma(M, L)) \to \Gamma(\Sigma(M, L)).$$

Then D^A is a self-adjoint elliptic operator of first-order. Therefore the spectrum $\mathrm{Spec}(D^A)$ of D^A is discrete and real. The behavior of $\mathrm{Spec}(D^A)$ generally depends on the $U(1)$-connection A, the metric, and the spin$^{\mathbb{C}}$ structure. For general properties of Dirac operators we refer to [4, 8].

Definition 1.1. The $\mathrm{Spec}(D^A)$ is called symmetric, if the following conditions hold:

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(1) There exits $-\lambda \in \operatorname{Spec}(D^A)$ whenever $\lambda \in \operatorname{Spec}(D^A)$.
(2) The multiplicity of $-\lambda$ is equal to that of λ.

If n is even, then the volume form μ of (M^n, g) anti-commutes with the spin$^{\mathbb{C}}$ Dirac operator D^A

$$D^A \circ \mu = -\mu \circ D^A.$$

Thus, in this case $\operatorname{Spec}(D^A)$ is symmetric.

Since D^A is elliptic, each eigenspace is finite-dimensional. The asymmetry of $\operatorname{Spec}(D^A)$ on an odd-dimensional manifold was investigated by Atiyah, Patodi and Singer [1] via the *eta funtion* defined as

$$\eta_{D^A}(s) := \sum_{\lambda \neq 0} \frac{\operatorname{sign}(\lambda)}{|\lambda|^s}, \ s \in \mathbb{C},$$

where λ runs through the eigenvalues according to their multiplicities. The series $\eta_{D^A}(s)$ converges for sufficiently large $\operatorname{Re}(s)$ and has the meromorphic continuation to the whole $\mathbb{C}$ with $\eta_{D^A}(0)$ finite. They showed that the value $\eta_{D^A}(0)$, called the *eta invariant* of D^A, appears as a global correction term for the index theorem for compact manifolds with boundary. Note that $\eta_{D^A}(s) \equiv 0$ is a necessary condition for the symmetry of $\operatorname{Spec}(D^A)$.

In this paper, we prove a necessary and sufficient condition for the symmetry of $\operatorname{Spec}(D^A)$ on odd-dimensional Riemannian spin$^{\mathbb{C}}$ product manifolds using the ideas mainly adapted from that of E. C. Kim [7]. We will take a convention that a superscript on a manifold denotes its dimension.

Theorem 1.2. *Let* $(Q^n := M_1^{2p} \times M_2^{2q+1}, h := g_1 + g_2)$ *be a Riemannian product of two closed Riemannian* spin$^{\mathbb{C}}$ *manifolds* (M_1^{2p}, g_1), $p \geq 1$, *and* (M_2^{2q+1}, g_2), $q \geq 0$. *Let* $\pi_1 : Q^n \to M_1^{2p}$ *and* $\pi_2 : Q^n \to M_2^{2q+1}$ *be the natural projections.*

Suppose that M_1 *(resp.* M_2*) is equipped with a spin*$^{\mathbb{C}}$ *structure given by a complex line bundle* L_1 *(resp.* L_2*) with* $c_1(L_1) \equiv w_2(M_1^{2p}) \bmod 2$ *(resp.* $c_1(L_2) \equiv w_2(M_2^{2q+1}) \bmod 2$*), and* $M_1 \times M_2$ *is equipped with the product* spin$^{\mathbb{C}}$ *structure given by the complex line bundle* $L = \pi_1^*(L_1) \otimes \pi_2^*(L_2)$.

Let A_1 *(resp.* A_2*) be a* $U(1)$*-connection on* L_1 *(resp.* L_2*), and* $A = \pi_1^*(A_1) + \pi_2^*(A_2)$ *be a connection of* L. *Let* $D_{M_1}^{A_1}, D_{M_2}^{A_2}$, *and* D^A *be the associated* spin$^{\mathbb{C}}$ *Dirac operators of* (M_1^{2p}, g_1), (M_2^{2q+1}, g_2), *and* (Q^n, h), *respectively. Then we have the following*:

$\operatorname{Spec}(D^A)$ *is symmetric if and only if either* $\operatorname{Spec}(D_{M_2}^{A_2})$ *is symmetric or* $(e^{\frac{1}{2}c_1(L_1)}\hat{A}(M_1))[M_1] = 0$, *where* $\hat{A}(M_1)$ *denotes the* $\hat{A}$*-class of* TM_1.

As a result auxiliary, we generalize real and quaternionic structures of spinor bundles on spin manifolds to complex anti-linear mappings between $\Sigma(M, L)$ and $\Sigma(M, -L)$ on spin$^{\mathbb{C}}$ manifolds, and study several variations on product spin$^{\mathbb{C}}$ manifolds. These may be used as tools for studying the spectrum of a spin$^{\mathbb{C}}$ Dirac operator on a product spin$^{\mathbb{C}}$ manifold.

2. Preliminaries

This section is divided into two parts. In the first part, we explain that the spinor bundle of a Riemannian spin$^{\mathbb{C}}$ product manifold has a natural tensor-product splitting. In the second part, we prove the decomposition property of L^2-sections of a vector bundle given by the tensor product of two vector bundles.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}^{\mathbb{C}}(k_1+k_2) & \hookleftarrow & \mathrm{Spin}^{\mathbb{C}}(k_1)\otimes_{S^1}\mathrm{Spin}^{\mathbb{C}}(k_2) \ni (\pm a_1 \otimes_{\mathbb{Z}_2} \pm e^{\frac{i\theta_1}{2}}) \otimes_{S^1} (\pm a_2 \otimes_{\mathbb{Z}_2} \pm e^{\frac{i\theta_2}{2}}) \\ \downarrow \pi & & \downarrow \\ SO(k_1+k_2)\oplus S^1 & \hookleftarrow & SO(k_1)\oplus SO(k_2)\oplus S^1 \ni (a_1, a_2, e^{i(\theta_1+\theta_2)}), \end{array}$$

where π is a 2-fold covering map.

Let P_{M_1}, P_{M_2}, and P_Q be the $SO(k_1)$-, $SO(k_2)$-, and $SO(k_1+k_2)$-principal bundles of positively oriented orthonormal frames of $(M_1^{k_1}, g_1)$, $(M_2^{k_2}, g_2)$, and $(Q^n := M_1^{k_1} \times M_2^{k_2}, h := g_1 + g_2)$, respectively. Let L_1 (resp. L_2) be a complex line bundle with $c_1(L_1) \equiv w_2(M_1^{k_1})$ mod 2 (resp. $c_1(L_2) \equiv w_2(M_2^{k_2})$ mod 2), and let $L = \pi_1^* L_1 \otimes \pi_2^* L_2$, where $\pi_1 : Q \to M_1$ and $\pi_2 : Q \to M_2$ are the natural projections. To denote the double cover of a principal bundle, we will put $\widetilde{\ }$. Then the above pointwise diagram globalizes over the whole manifold to give the following commutative diagram:

$$\begin{array}{ccc} \widetilde{P_Q \oplus L} & \hookleftarrow & \pi_1^*(\widetilde{P_{M_1} \oplus L_1}) \otimes_{S^1} \pi_2^*(\widetilde{P_{M_2} \oplus L_2}) \\ \downarrow \pi & & \downarrow \\ P_Q \oplus (\pi_1^* L_1 \otimes \pi_2^* L_2) & \hookleftarrow & (\pi_1^* P_{M_1} \oplus \pi_2^* P_{M_2}) \oplus (\pi_1^* L_1 \otimes \pi_2^* L_2). \end{array}$$

Thus, the $\mathrm{Spin}^{\mathbb{C}}(k_1+k_2)$-principle bundle $\widetilde{P_Q \oplus L}$ over (Q^n, h) reduces to the $\mathrm{Spin}^{\mathbb{C}}(k_1) \otimes_{S^1} \mathrm{Spin}^{\mathbb{C}}(k_2)$-principal bundle $\pi_1^*(\widetilde{P_{M_1} \oplus L_1}) \otimes \pi_2^*(\widetilde{P_{M_2} \oplus L_2})$ in a π-equivariant way.

Let $(E_1, \dots, E_{k_1})$ and $(F_1, \dots, F_{k_2})$ be local orthonormal frames on $(M_1^{k_1}, g_1)$ and $(M_2^{k_2}, g_2)$, respectively. We identify $(E_1, \dots, E_{k_1})$ and $(F_1, \dots, F_{k_2})$ with their lifts to (Q^n, h). We may then regard $(E_1, \dots, E_{k_1}, F_1, \dots, F_{k_2})$ as a local orthonormal frame on (Q^n, h). Let $\mu_1 = E^1 \wedge \cdots \wedge E^{k_1}$, $E^t := g_1(E_t, \cdot)$, and $\mu_2 = F^1 \wedge \cdots \wedge F^{k_2}$, $F^l := g_2(F_l, \cdot)$, be the volume forms of $(M_1^{k_1}, g_1)$ and $(M_2^{k_2}, g_2)$, respectively, as well as their lifts to (Q^n, h).

Now let's assume that at least one of k_1 and k_2 is even. Say, $k_1 = 2p$ for $p \in \mathbb{N}$. Using the following Clifford action on the tensor product vector space [7, Section 2], one can extend the $\mathrm{Spin}^{\mathbb{C}}(k_1) \otimes_{S^1} \mathrm{Spin}^{\mathbb{C}}(k_2)$-action on $\triangle_{k_1} \otimes \triangle_{k_2}$ to the $\mathrm{Spin}^{\mathbb{C}}(k_1+k_2)$-action:

$$E_t \cdot (\varphi_1 \otimes \varphi_2) = (E_t \cdot \varphi_1) \otimes \varphi_2, \ t = 1, \dots, 2p, \tag{2.1}$$

$$F_l \cdot (\varphi_1 \otimes \varphi_2) = (\sqrt{-1})^p (\mu_1 \cdot \varphi_1) \otimes (F_l \cdot \varphi_2), \ l = 1, \dots, k_2, \tag{2.2}$$

where $\varphi_1 \in \triangle_{k_1}$ and $\varphi_2 \in \triangle_{k_2}$. If $k_2 = 2p$, the Clifford actions are defined as:

$$E_t \cdot (\varphi_1 \otimes \varphi_2) = (E_t \cdot \varphi_1) \otimes (\sqrt{-1})^p (\mu_2 \cdot \varphi_2),\ t = 1, \ldots, k_1,$$
$$F_l \cdot (\varphi_1 \otimes \varphi_2) = \varphi_1 \otimes (F_l \cdot \varphi_2),\ l = 1, \ldots, 2p.$$

Hence we can conclude that the associated vector bundle $\pi_1^*(\Sigma(M_1, L_1)) \otimes \pi_2^*(\Sigma(M_2, L_2))$ for the principal bundle $\pi_1^*(\widetilde{P_{M_1} \oplus L_1}) \otimes_{S^1} \pi_2^*(\widetilde{P_{M_2} \oplus L_2})$ is also an associated vector bundle for $\widetilde{P_Q \oplus L}$.

Recall that there is a unique spinor bundle $\Sigma(M_1 \times M_2, L)$ on $M_1 \times M_2$, if $k_1 + k_2$ is even, and there exist two of them, if $k_1 + k_2$ is odd. Suppose that $\dim M_1 = 2p$ and $\dim M_2 = 2q + 1$ for $p \geq 1$ and $q \geq 0$. Then we have two non-isomorphic spinor bundles

$$\Sigma(M_1, L_1) \otimes \Sigma'(M_2, L_2) \quad \text{and} \quad \Sigma(M_1, L_1) \otimes \Sigma''(M_2, L_2)$$

on $M_1 \times M_2$, where $\Sigma(M_1, L_1)$ is a unique spinor bundle on M_1, and $\Sigma'(M_2, L_2)$, $\Sigma''(M_2, L_2)$ denote two non-isomorphic spinor bundles corresponding to $\Sigma(M_2, L_2)$ on M_2 (more precisely, they are the eigenbundles of $+(\sqrt{-1})^{q+1}$ and $-(\sqrt{-1})^{q+1}$ under the action of μ_2, respectively). One can easily check that $\Sigma(M_1, L_1) \otimes \Sigma'(M_2, L_2)$ and $\Sigma(M_1, L_1) \otimes \Sigma''(M_2, L_2)$ are the eigenbundles of $+(\sqrt{-1})^{p+q+1}$ and $-(\sqrt{-1})^{p+q+1}$ under the action of the volume form $\mu_1 \wedge \mu_2$ on $M_1 \times M_2$, respectively.

Since the tensor product bundle $\pi_1^*(\Sigma(M_1, L_1)) \otimes \pi_2^*(\Sigma(M_2, L_2))$ and the spinor bundle $\Sigma(M_1 \times M_2, L = \pi_1^* L_1 \otimes \pi_2^* L_2)$ have the same dimension, we have proved the first part of the following:

Lemma 2.3. *If at least one of k_1 and k_2 is even, then*

$$\Sigma(M_1 \times M_2, L = \pi_1^* L_1 \otimes \pi_2^* L_2) = \pi_1^*(\Sigma(M_1, L_1)) \otimes \pi_2^*(\Sigma(M_2, L_2)).$$

If both k_1 and k_2 are odd, then

$$\begin{aligned} &\Sigma(M_1 \times M_2, L = \pi_1^* L_1 \otimes \pi_2^* L_2) \\ &= (\pi_1^*(\Sigma'(M_1, L_1)) \oplus \pi_1^*(\Sigma''(M_1, L_1))) \otimes \pi_2^*(\Sigma(M_2, L_2)). \end{aligned}$$

The proof of the second part can be done in the same as the spin case, whose details can be found in [7, Section 4].

Moreover the connections on the LHS of the previous diagram are induced ones from the RHS, which are subbundles. A spin$^{\mathbb{C}}$ connection on

$$\pi_1^*(\widetilde{P_{M_1} \oplus L_1}) \otimes_{S^1} \pi_2^*(\widetilde{P_{M_2} \oplus L_2}),$$

which is lifted from downstairs is given by the tensor-product. Therefore, letting A_i for $i = 1, 2$ and $A = \pi_1^*(A_1) + \pi_2^*(A_2)$ be connections on L_i and L respectively, and $\nabla^{A_1}, \nabla^{A_2}$, and ∇^A be the spinor derivatives of $\Sigma(M_1, L_1)$, $\Sigma(M_2, L_2)$, and $\Sigma(M_1 \times M_2, L)$ respectively, we have

$$\nabla_X^A (\varphi_1 \otimes \varphi_2) = (\nabla_{\pi_{1*}(X)}^{A_1} \varphi_1) \otimes \varphi_2 + \varphi_1 \otimes (\nabla_{\pi_{2*}(X)}^{A_2} \varphi_2) \tag{2.4}$$

for $\varphi_i \in \Gamma(\pi_i^*(\Sigma(M_i, L_i)))$ and $X \in T(M_1 \times M_2)$.

Now we prove some analysis lemmas on general vector bundles.

Lemma 2.5. *Let E and F be hermitian vector bundles over M_1, M_2, respectively, and $\pi_1^*E \otimes \pi_2^*F$ be the induced bundle over $M_1 \times M_2$, where each $\pi_l : M_1 \times M_2 \to M_l$ for $l = 1, 2$ is the natural projection. Let $L^2(\pi_1^*E \otimes \pi_2^*F)$, $L^2(E)$, $L^2(F)$ be the completion, with respect to the L^2-norm, of $C^0(\pi_1^*E \otimes \pi_2^*F)$, $C^0(E)$, $C^0(F)$, respectively. Then we have*

$$L^2(\pi_1^*E \otimes \pi_2^*F) = \overline{\pi_1^*(L^2(E)) \otimes \pi_2^*(L^2(F))},$$

where the over-line denotes the L^2-completion.

Proof. Let $\{\varphi_\alpha(x)\}$ and $\{\psi_\beta(y)\}$ be orthonormal bases for $\pi_1^*(L^2(E))$ and $\pi_2^*(L^2(F))$ respectively. Then $\{\varphi_\alpha(x) \otimes \psi_\beta(y)\}$ forms an orthonormal set in $L^2(\pi_1^*E \otimes \pi_2^*F)$, and hence

$$L^2(\pi_1^*E \otimes \pi_2^*F) \supseteq \overline{\pi_1^*(L^2(E)) \otimes \pi_2^*(L^2(F))}.$$

To prove the reverse direction, we will prove that $\{\varphi_\alpha(x) \otimes \psi_\beta(y)\}$ is actually a maximal orthonormal set, i.e., basis. Let $\operatorname{rank}(E) = m$ and $f \in C^0(\pi_1^*E \otimes \pi_2^*F)$. Take $U \times M_2 \subset M_1 \times M_2$, where U is a small open neighborhood of a point in M_1. Since $f|_{x \times \{M_2\}}$ for each $x \in U$ is continuous and hence in $L^2(F)$, the section f on $U \times M_2$ is expressed as

$$\begin{aligned} f(x, y) = (f_1(x, y), \ldots, f_m(x, y)) &= \sum_\beta a_\beta(x)\psi_\beta(y) \\ &= (\sum_\beta a_{\beta,1}(x)\psi_\beta(y), \ldots, \sum_\beta a_{\beta,m}(x)\psi_\beta(y)). \end{aligned}$$

Since f is continuous, we have the continuity of

$$a_{\beta,k}(x) = \langle f_k(x, y), \psi_\beta(y)\rangle_{L^2(F)} = \int_{M_2} \langle f_k(x, y), \psi_\beta(y)\rangle_F \ dy,$$

where $\langle \cdot, \cdot \rangle_F$ is the hermitian inner product on F. Applying the Hölder's inequality, we obtain

$$\begin{aligned} \int_U |a_{\beta,k}(x)|^2 dx &\le \int_U (\int_{M_2} |f_k(x, y)|_F |\psi_\beta(y)|_F dy)^2 dx \\ &\le \int_U (\int_{M_2} |f_k(x, y)|_F^2 dy \int_{M_2} |\psi_\beta(y)|_F^2 dy) \ dx \\ &= \int_U (\int_{M_2} |f_k(x, y)|_F^2 dy) \ dx < \infty, \end{aligned}$$

where the finiteness is due to the fact that $f \in L^2(\pi_1^*E \otimes \pi_2^*F)$ and hence $f_k \in L^2(\pi_1^*E|_{U(p)} \otimes \pi_2^*F)$. This implies that $a_{\beta,k}$ and hence a_β are locally in L^2. Moreover, since $f \in \Gamma(\pi_1^*E \otimes \pi_2^*F)$ and $\psi_\beta \in \Gamma(\pi_2^*F)$, we have

$$a_\beta(x) = \int_{M_2} \langle f(x, y), \psi_\beta(y)\rangle_F \ dy \in \Gamma(\pi_1^*E).$$

Therefore, we can write

$$a_\beta(x) = \sum_\alpha c_{\alpha\beta}\varphi_\alpha(x)$$

for $c_{\alpha\beta} \in \mathbb{C}$, and hence f can be expressed as

$$f(x, y) = \sum_\beta \sum_\alpha c_{\alpha\beta}\varphi_\alpha(x) \otimes \psi_\beta(y).$$

Because the subset of continuous sections is dense in the space of L^2-sections, the proof is completed. □

Remark 2.6. Lemma 2.5 remains valid when E and F are real vector bundles with Riemannian metrics over M_1 and M_2, respectively.

Corollary 2.7. *Let $D_{M_1} : E \to E$ (resp. $D_{M_2} : F \to F$) denote a linear self-adjoint elliptic differential operator on a complex vector bundle over a closed manifold M_1 (resp. M_2) and let $\Gamma_\rho(D_{M_j})$, for $j \in \{1, 2\}$, denote the space of all eigenvectors of D_{M_j} with eigenvalue $\rho \in \mathbb{R}$. If $D_{M_1} \otimes \mathtt{Id} + \mathtt{Id} \otimes D_{M_2} : \pi_1^*E \otimes \pi_2^*F \to \pi_1^*E \otimes \pi_2^*F$ as an operator on $M_1 \times M_2$ is elliptic, then we have*

$$\Gamma_\gamma(D_{M_1} \otimes \mathtt{Id} + \mathtt{Id} \otimes D_{M_2}) = \bigoplus_{\gamma=\chi+\nu} (\pi_1^*(\Gamma_\chi(D_{M_1})) \otimes \pi_2^*(\Gamma_\nu(D_{M_2}))).$$

Proof. " $\supseteq$ " part is obvious, and we will show the other direction. Note that $D_{M_1} \otimes \mathtt{Id} + \mathtt{Id} \otimes D_{M_2}$ is also self-adjoint. Since the unit norm eigenvectors of D_{M_1}, D_{M_2}, and $D_{M_1} \otimes \mathtt{Id} + \mathtt{Id} \otimes D_{M_2}$ form orthonormal bases of $L^2(E)$, $L^2(F)$, and $L^2(\pi_1^*E \otimes \pi_2^*F)$, respectively, the proof follows from Lemma 2.5. □

Lemma 2.8. *Let $D : E \to E$ be a linear self-adjoint elliptic operator, where E is a complex vector bundle over a closed manifold M. Then all the eigenvectors of D^2 come from the eigenvectors of D, and the squares of the eigenvalues of D are exactly the eigenvalues of D^2.*

Proof. Obviously, the eigenvector of D is the eigenvector of D^2. Since the unit norm eigenvectors of D form an orthonormal basis of $L^2(E)$, and the same is true for the unit norm eigenvectors of D^2, the conclusion follows. □

3. Real and quaternionic structures

Let's first review some basic properties of real or quaternionic structures j_0 and j_1 of the spinor representation. For more details, the readers are referred to [3, 4, 6, 7, 9].

The real Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$ is multiplicatively generated by the standard basis $\{e_1, \dots, e_n\}$ of the Euclidean space $\mathbb{R}^n$ subject to the relations $e_i^2 = -1$ for all $i \leq n$ and $e_ie_j = -e_je_i$ for all $i \neq j$. Note that the dimension of $\mathrm{Cl}(\mathbb{R}^n)$ is 2^n. The complexification $\mathrm{Cl}(\mathbb{R}^n; \mathbb{C}) := \mathrm{Cl}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the matrix algebra $M(2^m; \mathbb{C})$ for $n = 2m$ and to the matrix algebra $M(2^m; \mathbb{C}) \oplus M(2^m; \mathbb{C})$ for $n = 2m + 1$. For an explicit isomorphism map for $n \geq 2$, we refer to [5, Section 1] (or [7, Section 2]). Following them, let us denote by $u(\epsilon) \in \mathbb{C}^2$ the vector

$$u(\epsilon) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\epsilon\sqrt{-1} \end{pmatrix}, \quad \epsilon = \pm 1. \tag{3.1}$$

Then

$$u(\epsilon_1, \ldots, \epsilon_m) := u(\epsilon_1) \otimes \ldots, \otimes u(\epsilon_m), \quad m = [\frac{n}{2}] \geq 1, \tag{3.2}$$

form an orthonormal basis for the spinor space $\triangle_n := \mathbb{C}^{2^m}$, $m = [\frac{n}{2}] \geq 1$, with respect to the standard hermitian inner product.

Definition 3.3. The complex-antilinear mappings $j_0, j_1 : \triangle_n \to \triangle_n$ defined, in the notations of (3.1) and (3.2), by

$$j_0 u(\epsilon_1, \ldots, \epsilon_m) = (\sqrt{-1})^{\sum_{\alpha=1}^m \alpha\epsilon_\alpha} u(-\epsilon_1, \ldots, -\epsilon_m),$$
$$j_1 u(\epsilon_1, \ldots, \epsilon_m) = (\sqrt{-1})^{\sum_{\alpha=1}^m (m-\alpha+1)\epsilon_\alpha} u(-\epsilon_1, \ldots, -\epsilon_m), \; m = [\frac{n}{2}],$$

are called the j_0-structure and j_1-structure, respectively.

The following facts are well-known [3, 7]. Fix $m = [\frac{n}{2}]$.

(A) $j_0 \circ e_k = e_k \circ j_0$ for all $k = 1, \ldots, 2m$ and $j_0 \circ e_{2m+1} = (-1)^{m+1} e_{2m+1} \circ j_0$. Thus, the mapping $j_0 : \triangle_n \to \triangle_n$ is $\mathrm{Spin}(n)$-equivariant for $n, n \not\equiv 1 \bmod 4$.

(B) $j_1 \circ e_l = (-1)^{m+1} e_l \circ j_1$ for all $l = 1, \ldots, n$. Thus, the mapping $j_1 : \triangle_n \to \triangle_n$ is $\mathrm{Spin}(n)$-equivariant for all $n \geq 2$.

(C) $j_0 \circ j_0 = j_1 \circ j_1 = (-1)^{m(m+1)/2}$ and $j_0 \circ j_1 = j_1 \circ j_0$.

(D) $\langle j_0(\psi), j_0(\varphi) \rangle = \langle j_1(\psi), j_1(\varphi) \rangle = \langle \varphi, \psi \rangle, \varphi, \psi \in \triangle_n$, where $\langle \cdot, \cdot \rangle$ is the standard hermitian inner product on $\triangle_n$.

Thus, j_0 (for $n \not\equiv 1 \bmod 4$) and j_1 give real (resp. quaternionic) structures on $\triangle_n$ as $\mathrm{Spin}(n)$-representations, if $m \equiv 0, 3 \bmod 4$ (resp. $m \equiv 1, 2 \bmod 4$).

Let us now fix a local trivialization of $\Sigma(M, L)$ on a $\mathrm{spin}^{\mathbb{C}}$ manifold M^n. Namely, let $\cup_\alpha U_\alpha$ be an open covering of M for which there exits a system of transition functions $\{g_{\alpha_1\alpha_2} : U_{\alpha_1} \cap U_{\alpha_2} \to \mathrm{Spin}^{\mathbb{C}}(n) = \mathrm{Spin}(n) \otimes_{\mathbb{Z}_2} S^1\}$. Define $\overline{g_{\alpha_1\alpha_2}} : U_{\alpha_1} \cap U_{\alpha_2} \to \mathrm{Spin}^{\mathbb{C}}(n)$ by

$$x \mapsto f(x) \otimes_{\mathbb{Z}_2} \overline{h(x)},$$

where $f(x) \otimes_{\mathbb{Z}_2} h(x) = g_{\alpha_1\alpha_2}(x)$ for $x \in U_{\alpha_1} \cap U_{\alpha_2}, f(x) \in \mathrm{Spin}(n)$ and $h(x) \in S^1$. Then $\overline{g_{\alpha_1\alpha_2}}$ is the transition function of a local trivialization for $\Sigma(M, -L)$. By the property (A), $j_0 \circ (f(x) \otimes_{\mathbb{Z}_2} h(x)) = (f(x) \otimes_{\mathbb{Z}_2} \overline{h(x)}) \circ j_0$ for $n \not\equiv 1 \bmod 4$. Also, by the property (B), $j_1 \circ (f(x) \otimes_{\mathbb{Z}_2} h(x)) = (f(x) \otimes_{\mathbb{Z}_2} \overline{h(x)}) \circ j_1$ for $n \geq 2$.

We then have the following commutative diagram:

$$\begin{array}{ccc} U_{\alpha_1}\times \triangle_n & \xrightarrow{g_{\alpha_1\alpha_2}} & U_{\alpha_2}\times \triangle_n \\ \Big\downarrow j_r & & \Big\downarrow j_r \\ U_{\alpha_1}\times \triangle_n & \xrightarrow{\overline{g_{\alpha_1\alpha_2}}} & U_{\alpha_2}\times \triangle_n, \end{array}$$

where $r = 0$ with $n \not\equiv 1 \bmod 4$, or $r = 1$ with $n \geq 2$. Thus, the mapping j_r is compatible with the transition functions of $\Sigma(M, L)$ and $\Sigma(M, -L)$ so that the j_0- and j_1-structure can be globalized to mappings $j_0, j_1 : \Sigma(M, L) \to \Sigma(M, -L)$, and we can carry all the properties (A)–(D) over to $\Sigma(M, L)$. Be aware that the mapping j_0 is well-defined for $n \not\equiv 1 \bmod 4$, and j_1 is well-defined for all $n \geq 2$.

Lemma 3.4.

$$D^{-A} \circ j_0 = j_0 \circ D^A \quad \textit{for} \quad n \not\equiv 1 \bmod 4,$$

and

$$D^{-A} \circ j_1 = (-1)^{m+1} j_1 \circ D^A.$$

Proof. Let $(\omega_{i,j})$ be the $\mathfrak{so}(n)$-valued 1-form on U_α coming from the Levi-Civita connection of (M^n, g). The spinor derivative ∇^A with respect to a $U(1)$-connection A in the bundle L is locally expressed as

$$\begin{aligned} j_k(\nabla^A_X s) &= j_k(X(s) + \frac{1}{2}(\sqrt{-1}A(X) + \sum_{i<j} \omega_{j,i}(X) e_i \cdot e_j) \cdot s) \\ &= X(j_k(s)) + \frac{1}{2}(-\sqrt{-1}A(X) + \sum_{i<j} \omega_{j,i}(X) e_i \cdot e_j) \cdot j_k(s) \\ &= \nabla^{-A}_X j_k(s), \end{aligned}$$

where $s \in \Gamma(\Sigma(M, L))$, $X \in \Gamma(TM)$, and $k = 1, 2$. Thus, we have

$$\nabla^{-A} \circ j_0 = j_0 \circ \nabla^A \quad \text{and} \quad \nabla^{-A} \circ j_1 = j_1 \circ \nabla^A. \tag{3.5}$$

Now the conclusion immediately follows from the properties (A) and (B). □

As a corollary, we have proved that if m is odd (resp. even), then $\mathrm{Spec}(D^A) = \mathrm{Spec}(D^{-A})$ (resp. $-\mathrm{Spec}(D^{-A})$) with the same multiplicities.

Now we take up the case of Theorem 1.2. Using (2.1), (2.2), and (2.4), one can easily verify the following formulas:

$$D^A(\varphi_1 \otimes \varphi_2) = (D^{A_1}_{M_1}\varphi_1) \otimes \varphi_2 + (\sqrt{-1})^p(\mu_1 \cdot \varphi_1) \otimes (D^{A_2}_{M_2}\varphi_2), \tag{3.6}$$

$$(D^A)^2(\varphi_1 \otimes \varphi_2) = ((D^{A_1}_{M_1})^2\varphi_1) \otimes \varphi_2 + \varphi_1 \otimes ((D^{A_2}_{M_2})^2\varphi_2) \tag{3.7}$$

for $\varphi_i \in \Gamma(\pi_i^*(\Sigma(M_i, L_i)))$.

Consider the *partial* spin$^{\mathbb{C}}$ Dirac operators $D_+^{A_1}, D_-^{A_2}$ acting on sections $\psi \in \Gamma(\Sigma(Q,L))$ of the spinor bundle over (Q^n,h):

$$D_+^{A_1}\psi = \sum_{k=1}^{2p} E_k \cdot \nabla_{E_k}^{A_1}\psi, \quad D_-^{A_2}\psi = \sum_{l=1}^{2q+1} F_l \cdot \nabla_{F_l}^{A_2}\psi.$$

Define the *twist* $\widetilde{D}^A$ of the spin$^{\mathbb{C}}$ Dirac operator $D^A = D_+^{A_1} + D_-^{A_2}$ by

$$\widetilde{D}^A = D_+^{A_1} - D_-^{A_2}.$$

By (2.1) and (2.2), for $i = 1,2$

$$D_+^{A_1} \circ \mu_i = -\mu_i \circ D_+^{A_1}, \quad D_-^{A_2} \circ \mu_i = \mu_i \circ D_-^{A_2},$$

and

$$D^A \circ \mu_i = -\mu_i \circ \widetilde{D}^A, \quad \widetilde{D}^A \circ \mu_i = -\mu_i \circ D^A. \tag{3.8}$$

Since the Spin$^{\mathbb{C}}(2p+2q+1)$-principle bundle $\widetilde{P_Q \oplus L}$ over

$$(Q^n := M_1^{2p} \times M_2^{2q+1}, h)$$

reduces to the Spin$^{\mathbb{C}}(2p) \otimes_{S^1}$ Spin$^{\mathbb{C}}(2q+1)$-principal bundle $\pi_1^*(\widetilde{P_{M_1} \oplus L_1}) \otimes \pi_2^*(\widetilde{P_{M_2} \oplus L_2})$, the complex-antilinear mapping $j^* : \triangle_{2p+2q+1} \to \triangle_{2p+2q+1}$ defined by

$$j^*u(\epsilon_1,\ldots,\epsilon_p,\epsilon_{p+1},\ldots,\epsilon_{p+q}) = \{j_0u(\epsilon_1,\ldots,\epsilon_p)\} \otimes \{j_1u(\epsilon_{p+1},\ldots,\epsilon_{p+q})\},$$

combining the j_0- and j_1-structure in Definition 3.3, globalizes to mapping $j^* : \Sigma(Q,L) \to \Sigma(Q,-L)$. Then the mapping j^* is well-defined for all $p \geq 1,\ q \geq 1$. Using the properties (A) and (B) below Definition 3.3, the formulas (2.1) and (2.2), we have the following:

$$j^* \circ E_k = E_k \circ j^*,\ k = 1,\ldots,2p, \quad j^* \circ F_l = (-1)^{p+q+1} F_l \circ j^*,\ l = 1,\ldots,2q+1.$$

From the formulas (2.4) and (3.5), it follows that

$$\nabla^{-A} \circ j^* = j^* \circ \nabla^A,$$

and hence

$$D_+^{-A_1} \circ j^* = j^* \circ D_+^{A_1}, \quad D_-^{-A_2} \circ j^* = (-1)^{p+q+1} j^* \circ D_-^{A_2}. \tag{3.9}$$

Similarly we also define complex-antilinear mappings $\widehat{j}^*,\ j_0^*,\ j_1^* : \Sigma(Q,L) \to \Sigma(Q,-L)$ as

$$\widehat{j}^*u(\epsilon_1,\ldots,\epsilon_p,\epsilon_{p+1},\cdots,\epsilon_{p+q}) = \{j_1u(\epsilon_1,\ldots,\epsilon_p)\} \otimes \{j_0u(\epsilon_{p+1},\ldots,\epsilon_{p+q})\},$$
$$j_0^*u(\epsilon_1,\ldots,\epsilon_p,\epsilon_{p+1},\ldots,\epsilon_{p+q}) = \{j_0u(\epsilon_1,\ldots,\epsilon_p)\} \otimes \{j_0u(\epsilon_{p+1},\ldots,\epsilon_{p+q})\},$$
$$j_1^*u(\epsilon_1,\ldots,\epsilon_p,\epsilon_{p+1},\ldots,\epsilon_{p+q}) = \{j_1u(\epsilon_1,\ldots,\epsilon_p)\} \otimes \{j_1u(\epsilon_{p+1},\ldots,\epsilon_{p+q})\}.$$

The mappings $\widehat{j}^*$ and j_0^* are well-defined when q is odd, and j_1^* is well-defined for all $p \geq 1,\ q \geq 1$. One can easily check that

$$D_+^{-A_1} \circ \widehat{j}^* = (-1)^{p+1}\, \widehat{j}^* \circ D_+^{A_1}, \quad D_-^{-A_2} \circ \widehat{j}^* = (-1)^p\, \widehat{j}^* \circ D_-^{A_2}, \tag{3.10}$$

$$D_+^{-A_1} \circ j_0^* = j_0^* \circ D_+^{A_1}, \qquad D_-^{-A_2} \circ j_0^* = (-1)^p j_0^* \circ D_-^{A_2}, \tag{3.11}$$

$$D_+^{-A_1} \circ j_1^* = (-1)^{p+1} j_1^* \circ D_+^{A_1}, \quad D_-^{-A_2} \circ j_1^* = (-1)^{p+q+1} j_1^* \circ D_-^{A_2}. \tag{3.12}$$

Putting these together, we can produce various operators which anti-commute with spin$^{\mathbb{C}}$ Dirac operators:

Proposition 3.13. *Under the assumptions of Theorem 1.2 and with the above notations, for* $i = 1, 2$:

1. *For* $p \geq 1$ *and* $q \geq 0$, $D^A \circ (\mu_i \circ D_+^{A_1}) = -(\mu_i \circ D_+^{A_1}) \circ D^A$.
2. *Let* $p \geq 1$ *and* $q \geq 1$. *If* p *and* q *are either both even or both odd, then* $D^{-A} \circ (\mu_i \circ j^*) = -(\mu_i \circ j^*) \circ D^A$.
3. *Let* $p \geq 1$ *and* $q \geq 1$. *If* p *and* q *are odd, then* $D^{-A} \circ (\mu_i \circ \widehat{j}^*) = -(\mu_i \circ \widehat{j}^*) \circ D^A$ *and* $D^{-A} \circ (\mu_i \circ j_0^*) = -(\mu_i \circ j_0^*) \circ D^A$.
4. *Let* $p \geq 1$ *and* $q \geq 1$. *If* p *and* q *are even, then* $D^{-A} \circ j_1^* = -j_1^* \circ D^A$.

Proof. Since the Riemann curvatures $R(E_k, F_l, \cdot, \cdot) = 0$ vanish and the Clifford multiplication anti-commutes, we have

$$D_-^{A_2} D_+^{A_1} + D_+^{A_1} D_-^{A_2} = 0. \tag{3.14}$$

Using (3.8) and (3.14), we obtain

$$\begin{aligned} D^A \circ (\mu_i \circ D_+^{A_1}) &= -\mu_i \circ \widetilde{D}^A \circ D_+^{A_1} \\ &= -\mu_i \circ ((D_+^{A_1})^2 - D_-^{A_2} \circ D_+^{A_1}) \\ &= -\mu_i \circ ((D_+^{A_1})^2 + D_+^{A_1} \circ D_-^{A_2}) \\ &= -\mu_i \circ D_+^{A_1} \circ D^A. \end{aligned}$$

Suppose that p and q are either both even or both odd. By (3.8) and (3.9), we have

$$\begin{aligned} D^{-A} \circ (\mu_i \circ j^*) &= -(\mu_i \circ \widetilde{D}^{-A}) \circ j^* = -\mu_i \circ (D_+^{-A_1} - D_-^{-A_2}) \circ j^* \\ &= -\mu_i \circ j^* \circ (D_+^{A_1} + D_-^{A_2}) = -(\mu_i \circ j^*) \circ D^A. \end{aligned}$$

For the statement (3), we assume that p and q are odd. By (3.8), (3.10) and (3.11), we see that

$$\begin{aligned} D^{-A} \circ (\mu_i \circ \widehat{j}^*) &= -(\mu_i \circ \widetilde{D}^{-A}) \circ \widehat{j}^* = -(\mu_i \circ \widehat{j}^*) \circ D^A \quad \text{and} \\ D^{-A} \circ (\mu_i \circ j_0^*) &= -(\mu_i \circ \widetilde{D}^{-A}) \circ j_0^* = -(\mu_i \circ j_0^*) \circ D^A. \end{aligned}$$

The statement (4) follows from (3.12). □

4. Proof of Theorem 1.2

By Corollary 2.7 and (3.7), we see that the eigenvalues of $(D^A)^2$ are all possible sums of one eigenvalue of $(D_{M_1}^{A_1})^2$ and one of $(D_{M_2}^{A_2})^2$.

From now on, we omit the projections and rewrite the formula of Corollary 2.7 as

$$\Gamma_\gamma((D^A)^2) = \bigoplus_{\gamma=\chi+\nu} (\Gamma_\chi((D^{A_1}_{M_1})^2) \otimes \Gamma_\nu((D^{A_2}_{M_2})^2)). \tag{4.1}$$

Using the decomposition

$$\Sigma(M, L) = \Sigma^+(M, L) \oplus \Sigma^-(M, L),$$

where

$$\Sigma^\pm(M_1, L_1) := \{\varphi \in \Sigma(M_1, L_1) \mid \mu_1 \cdot \varphi = \pm(\sqrt{-1})^p \varphi\},$$

we have a decomposition $\Sigma(Q, L) = \Sigma^+(Q, L) \oplus \Sigma^-(Q, L)$ due to the action of the volume form $\mu_1 = E^1 \wedge \cdots \wedge E^{2p}$,

$$\Sigma^\pm(Q, L) := \Sigma^\pm(M_1, L_1) \otimes \Sigma(M_2, L_2).$$

The positive part ψ^+ (resp. negative part ψ^-) of $\psi \in \Gamma(\Sigma(Q, L))$ is in fact equal to

$$\psi^\pm = \frac{1}{2}\psi \pm \frac{1}{2}(-\sqrt{-1})^p \mu_1 \cdot \psi.$$

Lemma 4.2. *Let $\Gamma^\pm_0(D^{A_1}_{M_1})$ be the space of all positive* (*resp. negative*) *harmonic spinors of $D^{A_1}_{M_1}$. For any $\lambda \neq 0 \in \mathrm{Spec}(D^A)$, define a complex vector space*

$$H_\lambda := \{\psi \in \Gamma_\lambda(D^A) \mid D^{A_1}_+ \psi = 0,\ D^{A_2}_- \psi = \lambda\psi\}.$$

Then, in the notation (4.1)*, we have*

$$H_\lambda = \{\Gamma^+_0(D^{A_1}_{M_1}) \otimes \Gamma_{(-1)^p\lambda}(D^{A_2}_{M_2})\} \oplus \{\Gamma^-_0(D^{A_1}_{M_1}) \otimes \Gamma_{-(-1)^p\lambda}(D^{A_2}_{M_2})\}.$$

Proof. By (3.6), it is enough to show that

$$H_\lambda \subset \{\Gamma^+_0(D^{A_1}_{M_1}) \otimes \Gamma_{(-1)^p\lambda}(D^{A_2}_{M_2})\} \oplus \{\Gamma^-_0(D^{A_1}_{M_1}) \otimes \Gamma_{-(-1)^p\lambda}(D^{A_2}_{M_2})\}.$$

Suppose that $\psi \in H_\lambda$. Then $\psi \in \Gamma_{\lambda^2}((D^A)^2)$ and (4.1) implies that

$$\psi = \sum_{k,l} c_{k,l}\varphi_{0,k} \otimes \varphi_{\lambda^2,l},\ c_{k,l} \neq 0 \in \mathbb{C}, \tag{4.3}$$

is a finite linear combination of tensor products of some $\varphi_{0,k} \in \Gamma_0((D^{A_1}_{M_1})^2)$ and some $\varphi_{\lambda^2,l} \in \Gamma_{\lambda^2}((D^{A_2}_{M_2})^2)$. By the decomposition of $\Sigma(Q, L)$, we can rewrite (4.3) as

$$\psi = \sum_{k,l} c_{k,l}\varphi^+_{0,k} \otimes \varphi_{\lambda^2,l} + \sum_{k,l} c_{k,l}\varphi^-_{0,k} \otimes \varphi_{\lambda^2,l},$$

where $\varphi^\pm_{0,k} \in \Gamma^\pm_0((D^{A_1}_{M_1})^2)$. By Lemma 2.8, the eigenvalue λ^2 of $(D^{A_2}_{M_2})^2$ comes from the eigenvalue λ or $-\lambda$ of $D^{A_2}_{M_2}$. Since $D^{A_2}_- \psi = \lambda\psi$, one can easily see from (3.6)

$$\psi \in \{\Gamma^+_0(D^{A_1}_{M_1}) \otimes \Gamma_{(-1)^p\lambda}(D^{A_2}_{M_2})\} \oplus \{\Gamma^-_0(D^{A_1}_{M_1}) \otimes \Gamma_{-(-1)^p\lambda}(D^{A_2}_{M_2})\}. \qquad \square$$

Proof of Theorem 1.2. At first, using Proposition 3.13(1), one can define a map:

$$f : \Gamma_\lambda(D^A) \cap H_\lambda^\perp \longrightarrow \Gamma_{-\lambda}(D^A) \cap H_{-\lambda}^\perp \text{ defined by } \psi \mapsto \mu_1 \cdot D_+^{A_1}\psi,$$

where $H_\lambda^\perp$ is the orthogonal complement of H_λ. Note that f is bijective via the inverse map

$$f^{-1} := (-1)^p (D_+^{A_1})^{-1} \cdot \mu_1.$$

By Lemma 4.2,

$$H_{-\lambda} = \{\Gamma_0^+(D_{M_1}^{A_1}) \otimes \Gamma_{-(-1)^p\lambda}(D_{M_2}^{A_2})\} \oplus \{\Gamma_0^-(D_{M_1}^{A_1}) \otimes \Gamma_{(-1)^p\lambda}(D_{M_2}^{A_2})\}. \tag{4.4}$$

The symmetry of $\mathrm{Spec}(D^A)$ holds if and only if $\dim_{\mathbb{C}} H_\lambda = \dim_{\mathbb{C}} H_{-\lambda}$ for any $\lambda \neq 0 \in \mathrm{Spec}(D^A)$. Letting $\dim_{\mathbb{C}}(\Gamma_0^+(D_{M_1}^{A_1})) = a_1$, $\dim_{\mathbb{C}}(\Gamma_0^-(D_{M_1}^{A_1})) = a_2$, $\dim_{\mathbb{C}}(\Gamma_{(-1)^p\lambda}(D_{M_2}^{A_2})) = b_{1,\lambda}$, and $\dim_{\mathbb{C}}(\Gamma_{-(-1)^p\lambda}(D_{M_2}^{A_2})) = b_{2,\lambda}$, Lemma 4.2 and (4.4) give

$$\begin{aligned}\dim_{\mathbb{C}} H_\lambda - \dim_{\mathbb{C}} H_{-\lambda} &= (a_1 - a_2)(b_{1,\lambda} - b_{2,\lambda})\\ &= (\mathrm{ind}\ (D_{M_1}^{A_1}|_{\Sigma^+(M_1,L_1)}))(b_{1,\lambda} - b_{2,\lambda})\\ &= (e^{\frac{1}{2}c_1(L_1)}\hat{A}(M_1))[M_1](b_{1,\lambda} - b_{2,\lambda}),\end{aligned}$$

where the last equality is due to the Atiyah-Singer index theorem [8]. Now the desired conclusion follows immediately. □

5. Examples

In this section, applying Theorem 1.2, we present an important example. Let $D^0 := i\partial_\theta$ be the Dirac operator of $(S^1,\ g_2 = d\theta^2)$. Then the eigenvectors of unit L^2-norm are

$$\frac{e^{in\theta}}{\sqrt{2\pi}} \quad \text{for } n \in \mathbb{Z}$$

with multiplicity 1, and hence $\mathrm{Spec}(D^0) = \mathbb{Z}$. Thus $\mathrm{Spec}(D^0)$ is symmetric, and since

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\mathrm{sign}(n)}{|n|^s} = 0 \quad \text{for } \mathrm{Re}(s) \gg 1,$$

$\eta_{D^0}(s) = 0$ for all $s \in \mathbb{C}$. In [7], it is shown that the Dirac spectrum of $(M_1^{2p} \times S^1,\ g_1 + g_2)$ for a spin manifold M_1^{2p} is symmetric. We can generalize this to the spin$^{\mathbb{C}}$ case:

Example 5.1. Note that $D^{-iad\theta} = D^0 + a$ for $a \in \mathbb{R}$ is a spin$^{\mathbb{C}}$ Dirac operator of $(S^1,\ g_2 = d\theta^2)$. Thus

$$\mathrm{Spec}(D^{-iad\theta}) = \{n + a \mid n \in \mathbb{Z}\},$$

and hence $\mathrm{Spec}(D^{-iad\theta})$ is symmetric if and only if $a \in \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$. In this case, by Theorem 1.2, the spin$^{\mathbb{C}}$ Dirac spectrum of $(M_1^{2p} \times S^1,\ g_1 + g_2)$ for any product spin$^{\mathbb{C}}$ structure is symmetric, and hence the corresponding eta invariant vanishes.

The eta invariant of a spin$^{\mathbb{C}}$ Dirac operator on a product spin$^{\mathbb{C}}$ manifold $M_1^2 \times S^1$ appears when computing the dimension of the moduli space of Seiberg-Witten equations on a cylindrical-ended 4-manifold with asymptotic boundary equal to $M_1^2 \times S^1$. For details, the readers are referred to [10].

For general a, it is known that the eta invariant of $D^{-iad\theta}$ is $1 - 2a$. (See [10, Example 4.1.7] or [2].)

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