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ON THE CONVERGENCE OF SERIES OF MARTINGALE DIFFERENCES WITH MULTIDIMENSIONAL INDICES

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Abstract. Let $\{X_{\mathbf{n}};\mathbf{n}\succeq\mathbf{1}\}$ be a field of martingale differences taking values in a p-uniformly smooth Banach space. The paper provides conditions under which the series $\sum_{\mathbf{i}\preceq\mathbf{n}}X_{\mathbf{i}}$ converges almost surely and the tail series $\{T_{\mathbf{n}}=\sum_{\mathbf{i}\gg\mathbf{n}}X_{\mathbf{i}};\mathbf{n}\succeq\mathbf{1}\}$ satisfies $\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|=\mathcal{O}_P(b_{\mathbf{n}})$ and $\frac{\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|}{B_{\mathbf{n}}}\xrightarrow{P}0$ for given fields of positive numbers $\{b_{\mathbf{n}}\}$ and $\{B_{\mathbf{n}}\}$. This result generalizes results of A. Rosalsky, J. Rosenblatt [7], [8] and S. H. Sung, A. I. Volodin [11].

1. Introduction

Let $\{X_n;n\geq 1\}$ be a sequence of random variables taking values in a real separable Banach space $\mathbb{E}$ with norm $\|\cdot\|$. If the series $\sum_{i=1}^{\infty}X_i$ converges a.s., then the tail series

$$T_n=\sum_{i=n}^{\infty}X_i,\quad n\geq 1$$

is well-defined and $\sup_{k\geq n}\|T_k\|\xrightarrow{P}0$ as $n\to\infty$.

A. Rosalsky, J. Rosenblatt [7], [8] and S. H. Sung, A. I. Volodin [11] investigated the rate in which $\sup_{k\geq n}\|T_k\|\xrightarrow{P}0$ as $n\to\infty$. Namely, they provided conditions under which $\sup_{k\geq n}\|T_k\|=\mathcal{O}_P(b_n)$ and $\frac{\sup_{k\geq n}\|T_k\|}{B_n}\xrightarrow{P}0$, where (b_n) and (B_n) are given sequences of positive numbers.

The aim of this paper is to extend these results to the case where $\{X_{\mathbf{n}},\mathcal{F}_{\mathbf{n}};\mathbf{n}\in\mathbb{N}^d\}$ is a field of $\mathbb{E}$-valued martingale differences. Under the assumption that $\mathbb{E}$ is a p-uniformly smooth Banach space for some $1\leq p\leq 2$, we will provide sufficient conditions ensuring that $\sum_{\mathbf{n}\succeq\mathbf{1}}X_{\mathbf{n}}$ converges, $\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|=\mathcal{O}_P(b_{\mathbf{n}})$, $\frac{\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|}{B_{\mathbf{n}}}\xrightarrow{P}0$ and $\sum_{\mathbf{n}\succeq\mathbf{1}}\frac{1}{|\mathbf{n}|}P(\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|>\varepsilon a_{\mathbf{n}})<\infty$ for every $\varepsilon>0$, where $T_{\mathbf{n}}=S-\sum_{\mathbf{i}\preceq\mathbf{n}}X_{\mathbf{i}}$, $\{b_{\mathbf{n}}\}$, $\{B_{\mathbf{n}}\},\{a_{\mathbf{n}}\}$ are given fields of positive numbers.

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2. Preliminaries and some useful lemmas

Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Let $\mathbb{E}$ be a real separable Banach space. For a $\mathbb{E}$-valued random variable X and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(X|\mathcal{G})$ is defined analogously to that in the real-random variable case and enjoys similar properties (see [10]).

$(\mathbb{E}, \|\cdot\|)$ is said to be *p-uniformly smooth* ($1 \le p \le 2$) if there exists a finite positive constant C such that for all $\mathbb{E}$-valued martingales $\{S_n; 1 \le n \le m\}$

$$E\|S_m\|^p \le C \sum_{n=1}^{m} E\|S_n - S_{n-1}\|^p. \tag{2.1}$$

Clearly, every separable Banach space is of 1-uniformly smooth. If a real separable Banach space is of p-uniformly smooth for some $1 < p \le 2$, then it is of r-uniformly smooth for all $r \in [1, p)$. A Hilbert space is 2-uniformly smooth and the space L_p is $\min\{p, 2\}$-uniformly smooth (see [5], [6]).

Let d be a positive integer, the set of all nonnegative integer d-dimensional lattice points will be denoted by $\mathbb{N}_0^d$ and the set of all positive integer d-dimensional lattice points will be denoted by $\mathbb{N}^d$. For $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, denote $\mathbf{m} + \mathbf{n} = (m_1 + n_1, \ldots, m_d + n_d)$, $\mathbf{m} - \mathbf{n} = (m_1 - n_1, \ldots, m_d - n_d)$, $|\mathbf{n}| = n_1 \cdot n_2 \cdots n_d$, $\|\mathbf{n}\| = \min\{n_1, \ldots, n_d\}$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}_0^d$, $\bigvee_{i=1}^d (m_i < n_i)$ means that there is at least one of $m_1 < n_1$, $m_2 < n_2$, …, $m_d < n_d$ holds. We write $\mathbf{m} \preceq \mathbf{n}$ (or $\mathbf{n} \succeq \mathbf{m}$) if $m_i \le n_i$, $1 \le i \le d$; $\mathbf{m} \prec \mathbf{n}$ if $\mathbf{m} \preceq \mathbf{n}$ and $\mathbf{m} \ne \mathbf{n}$; $\mathbf{m} \ll \mathbf{n}$ (or $\mathbf{n} \gg \mathbf{m}$) if $\bigvee_{i=1}^d (m_i < n_i)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{E}$ be a real separable Banach space, $\mathcal{B}(\mathbb{E})$ be the σ-algebra of all Borel sets in $\mathbb{E}$. Let $\{X_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ be a (d-dimensional) field of $\mathbb{E}$-valued random variables and $\{\mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ be a (d-dimensional) field of nondecreasing sub-σ-algebras of $\mathcal{F}$ with respect to the partial order $\preceq$ on $\mathbb{N}^d$ such that $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable for all $\mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}$, then $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be an *adapted field.*

Let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ be an adapted field. For $\mathbf{n} \in \mathbb{N}_0^d$ ($\mathbf{m} - \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{M} - \mathbf{1}$), we adopt the convention that $\mathcal{F}_{\mathbf{n}} = \{\emptyset, \Omega\}$ if there exists a positive i ($1 \le i \le d$) such that $n_i = m_i - 1$ and set

$$\mathcal{F}_{\mathbf{n}}^{(i)} = \sigma\{\mathcal{F}_{\mathbf{k}} : \mathbf{k} = (k_1, \ldots, k_d),\ m_j \le k_j \le M_j\ (j \ne i),\ \text{and}\ \ k_i = n_i\}$$

for all $1 \le i \le d$, and $\mathcal{F}_{\mathbf{n}}^* = \sigma\{\mathcal{F}_{\mathbf{n}}^{(i)} : 1 \le i \le d\}$.

The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be a *field of martingale differences* if $E(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{n}-1}^{(i)}) = 0$ for all $\mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}$, $1 \le i \le d$ (see [3]).

The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be *strong adapted* if $E(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{n}-\mathbf{1}}^*)$ is $\mathcal{F}_{\mathbf{n}}^{(i)}$-measurable for all $\mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}$, $1 \le i \le d$.

Remark 2.1. Let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ be a field of martingale differences. Then it is strong adapted. Conversely, let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ be strong adapted, when $d = 1$ then $\{X_n - E(X_n|\mathcal{F}^*_{n-1}),\ \mathcal{F}_n : m \le n \le M\}$ is a sequences of martingale differences, but when $d > 1$ then $\{X_{\mathbf{n}} - E(X_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n-1}}), \mathcal{F}_{\mathbf{n}} : \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is not necessarily a field of martingale differences, because $X_{\mathbf{n}} - E(X_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n-1}})$ may not be $\mathcal{F}_{\mathbf{n}}$-measurable.

The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be *strong* adapted* if $\{X_{\mathbf{n}} I_A, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is strong adapted for all $A \in \sigma(X_{\mathbf{n}})$.

Clearly, when $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is a field of martingale differences, then it is not necessarily a strong* adapted field.

The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be a field of *strong* martingale differences* if it is a strong* adapted and a field of martingale differences.

Remark 2.2. Let $\{X_n, \mathcal{F}_n : m \le n \le M\}$ be a sequence of martingale differences, by $E(X_n I_A|\mathcal{F}^*_{n-1}) = E(X_n I_A|\mathcal{F}_{n-1}) \in \mathcal{F}_{n-1} = \mathcal{F}^{(1)}_{n-1}$. Then $\{X_n, \mathcal{F}_n : m \le n \le M\}$ is a strong* martingale differences.

Example 1. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{d}}\}$ be a field of independent random variables with mean 0. Put $\mathcal{F}_{\mathbf{n}} = \sigma(X_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n})$, then $E(X_{\mathbf{n}}|\mathcal{F}^i_{\mathbf{n-1}}) = 0$, $E(X_{\mathbf{n}} I_A|\mathcal{F}^*_{\mathbf{n-1}}) = 0$ for all $A \in \sigma(X_{\mathbf{n}})$ and $\mathbf{n} \in \mathbb{N}^d$, $1 \le i \le d$. Therefore, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a field of strong* martingale differences.

Example 2. Let $\{X_n, \mathcal{G}_n : n \ge 1\}$ be a sequence of martingale differences and set

$$X_{\mathbf{n}} = X_n \text{ if } \mathbf{n} = (n, n, \ldots, n) \text{ and } X_{\mathbf{n}} = 0 \text{ if } \mathbf{n} \ne (n, n, \ldots, n);$$

$$\mathcal{G}_{\mathbf{n}} = \mathcal{G}_n \text{ if } \mathbf{n} = (n, n, \ldots, n) \text{ and } \mathcal{G}_{\mathbf{n}} = \{\emptyset, \Omega\} \text{ if } \mathbf{n} \ne (n, n, \ldots, n).$$

Let $\mathcal{F}_{\mathbf{n}} = \sigma\{\mathcal{G}_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n}\}$ for all $\mathbf{n} \succeq \mathbf{1}$. Then $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{n} \succeq \mathbf{1}\}$ is a field of martingale differences. Moreover, for all $\mathbf{n} \succeq \mathbf{1}$, then

$$E(X_{\mathbf{n}} I_A|\mathcal{F}^*_{\mathbf{n-1}}) = E(X_n I_A|\mathcal{G}_{n-1}) \in \mathcal{G}_n = \mathcal{F}^i_{\mathbf{n}}$$

if $\mathbf{n} = (n, n, \ldots, n)$ and $E(X_{\mathbf{n}} I_A|\mathcal{F}^*_{\mathbf{n-1}}) = 0 \in \mathcal{F}^i_{\mathbf{n}}$ if otherwise, for all $A \in \sigma(X_{\mathbf{n}})$, so $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{n} \succeq \mathbf{1}\}$ is a field of strong* martingale differences.

Example 3. Let $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a field of independent random variables with mean 0. Put $\mathcal{F}_{\mathbf{n}} = \sigma(Y_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n})$ and $X_{\mathbf{n}} = \prod_{\mathbf{k} \preceq \mathbf{n}} Y_{\mathbf{k}}$, so $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is not a field of independent random variables. If $EX_{\mathbf{n}} < \infty$ for all $\mathbf{n} \succeq \mathbf{1}$, then $E(X_{\mathbf{n}}|\mathcal{F}^i_{\mathbf{n}}) = 0$, $E(X_{\mathbf{n}} I_A|\mathcal{F}^*_{\mathbf{n-1}}) = E(X_{\mathbf{n}} I_A) \in \mathcal{F}^i_{\mathbf{n}}$ for all $A \in \sigma(X_{\mathbf{n}})$, $\mathbf{n} \succeq \mathbf{1}$, $1 \le i \le d$. Therefore, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a field of strong* martingale differences.

In the sequel the following lemmas are useful.

Lemma 2.3. *Let $\mathbb{E}$ be a real separable p-uniformly smooth Banach space for some $1 \le p \le 2$. Then there exits a positive constant C such that for all strong* *adapted fields of* $\mathbb{E}$*-valued random variables* $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{m}\}$*, we have*

$$E \max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{m}} \| \sum_{\mathbf{1}\preceq\mathbf{i}\preceq\mathbf{k}} (X_{\mathbf{i}} - E(X_{\mathbf{i}}|\mathcal{F}^*_{\mathbf{i-1}})\|^p \le C \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{m}} E\|X_{\mathbf{k}}\|^p. \tag{2.2}$$

Proof. We will show that (2.2) holds by induction. Set

$$S_{\mathbf{k}} = \sum_{\mathbf{1}\preceq\mathbf{i}\preceq\mathbf{k}} (X_{\mathbf{i}} - E(X_{\mathbf{i}}|\mathcal{F}^*_{\mathbf{i-1}})).$$

Firstly, for $d = 1$, note that $\{\max_{1\le i\le k} \|S_i\|, \mathcal{F}_k : 1 \le k \le m\}$ is a nonnegative sub-martingale. Applying Doob's inequality and by (2.1), we have (2.2). We assume that (2.2) holds for $d-1$; we will show that it holds for d.

Denote

$$\mathbf{k} = (k_1, \dots, k_{d-1}, k_d) = (\mathbf{k}', k_d);\ \mathbf{m} = (m_1, \dots, m_{d-1}, m_d) = (\mathbf{m}', m_d);$$

with $\mathbf{k}', \mathbf{m}' \in \mathbb{N}^{d-1}$; set $Y_{k_d} = \max_{\mathbf{1}\preceq\mathbf{k}'\preceq\mathbf{m}'} \|S_{(\mathbf{k}';k_d)}\|$ for each $1 \le k_d \le m_d$, we have

$$\begin{aligned}
&E(S_{(\mathbf{k}';k_d)}|\mathcal{F}^d_{(\mathbf{k}';k_d-1)}) \\
&= E(S_{(\mathbf{k}';k_d-1)}|\mathcal{F}^d_{(\mathbf{k}'-\mathbf{1};k_d-1)}) \\
&\quad + \sum_{\mathbf{1}\preceq\mathbf{k}'\preceq\mathbf{m}'} (E(X_{(\mathbf{k}';k_d)} - E(X_{(\mathbf{k}';k_d)}|\mathcal{F}^*_{(\mathbf{k}'-\mathbf{1},k_d-1)})|\mathcal{F}^d_{(\mathbf{k}'-\mathbf{1};k_d-1)})) \\
&= S_{(\mathbf{k}';k_d-1)}
\end{aligned}$$

and by $\{S_{(\mathbf{k}';k_d)}; \mathcal{F}_{(\mathbf{k}';k_d)} : 1 \le k_d \le m_d\}$ being a strong adapted random field, it means that for each $\mathbf{1} \preceq \mathbf{k}' \preceq \mathbf{m}'$ then $\{S_{(\mathbf{k}';k_d)}; \mathcal{F}^d_{(\mathbf{k}';k_d)} : 1 \le k_d \le m_d\}$ is a martingale, and we have that $\{Y_{k_d}; \mathcal{F}_{(\mathbf{k}';k_d)} : 1 \le k_d \le m_d\}$ is a nonnegative sub-martingale sequence. Applying Doob's inequality, we obtain

$$\begin{aligned}
E(\max_{\mathbf{1}\preceq(\mathbf{k}',k_d)\preceq\mathbf{m}} \|S_{(\mathbf{k}';k_d)}\|^p) &= E(\max_{1\le k_d\le m_d} Y^p_{k_d}) \le C.EY^p_{m_d} \\
&= C \cdot E(\max_{\mathbf{1}\preceq\mathbf{k}'\preceq\mathbf{m}'} \|S_{(\mathbf{k}';m_d)}\|^p).
\end{aligned}$$

Set

$$X^{d-1}_{\mathbf{k}'} = \sum_{1\le k_d\le m_d} X_{(\mathbf{k}';k_d)}; \mathcal{F}^{d-1}_{\mathbf{k}'} = \sigma(\mathcal{F}^{d-1}_{(\mathbf{k}';k_d)} : 1 \le k_d \le m_d).$$

Note that $\mathcal{F}^i_{(\mathbf{k}',k_d)} = (\mathcal{F}^{d-1}_{\mathbf{k}'})^i, \mathcal{F}^*_{(\mathbf{k}',k_d)} = (\mathcal{F}^{d-1}_{\mathbf{k}'})^*$ for all $1 \le k_d \le m_d$, $1 \le i \le d-1$, then $\{X^{d-1}_{\mathbf{k}'}; \mathcal{F}^{d-1}_{\mathbf{k}'} : \mathbf{1} \le \mathbf{k}' \le \mathbf{m}'\}$ is a strong adapted field. Therefore, by the induction assumption,

$$\begin{aligned}
&E \max_{\mathbf{1}\preceq(\mathbf{k}',k_d)\preceq\mathbf{m}} \|S_{(\mathbf{k}';k_d)}\|^p \\
&\le C \cdot \sum_{\mathbf{1}\le\mathbf{k}'\le\mathbf{m}'} E\| \sum_{1\le k_d\le m_d} (X_{(\mathbf{k}';k_d)} - E(X_{(\mathbf{k}';k_d)}|\mathcal{F}^*_{(\mathbf{k}'-\mathbf{1},k_d-1)}))\|^p \\
&\le C \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{m}} E\|X_{\mathbf{k}} - E(X_k|\mathcal{F}^*_{\mathbf{k-1}})\|^p \le C \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{m}} E\|X_{\mathbf{k}}\|^p, \qquad \text{by (1.1).}
\end{aligned}$$

□

Remark 2.4. If $\{X_{\mathbf{n}}; \mathbf{n} \succeq \mathbf{1}\}$ is a $\mathbb{E}$-valued martingale difference field, from Lemma 2.3, we obtain Lemma 1.1 in [3] (for $p = q$). Moreover, by Remark 2.1, Lemma 2.3 is stronger than Lemma 1.1 in [3] (for $p = q$).

Lemma 2.5. *Let $\{X_{\mathbf{n}}; \mathbf{n} \succeq \mathbf{1}\}$ be a field of $\mathbb{E}$-valued random variables. Then*

$$P\left(\sup_{\mathbf{k}\succeq\mathbf{m}} \|X_{\mathbf{k}}\| > \epsilon\right) = \lim_{\|\mathbf{n}\|\to\infty} P\left(\max_{\mathbf{m}\preceq\mathbf{k}\preceq\mathbf{n}} \|X_{\mathbf{k}}\| > \epsilon\right), \tag{2.3}$$

$$P\left(\liminf_{\|\mathbf{n}\|\to\infty} \|X_{\mathbf{n}}\| > \epsilon\right) \leq \liminf_{\|\mathbf{n}\|\to\infty} P\left(\|X_{\mathbf{n}}\| > \epsilon\right). \tag{2.4}$$

Proof. 1. Remark that for $d = 1$, by the continuity from below theorem, we have (2.3). Assume that (2.3) holds for $d = D-1 \geq 1$, we with to show that for $d = D$. Let $\mathbf{m} = (m_1, m_2, \ldots, m_d) = (m_1, \mathbf{m_1})$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) = (k_1, \mathbf{k_1})$, $\mathbf{n} = (n_1, n_2, \ldots, n_d) = (n_1, \mathbf{n_1})$, by the continuity from below theorem, we have

$$\begin{aligned} P\left(\sup_{\mathbf{k}\succeq\mathbf{m}} \|X_{\mathbf{k}}\| > \epsilon\right) &= P\left(\lim_{n_1\to\infty} \sup_{\mathbf{k_1}\succeq\mathbf{m_1}} \max_{m_1\leq k_1\leq n_1} \|X_{\mathbf{k}}\| > \epsilon\right) \\ &= \lim_{n_1\to\infty} P\left(\sup_{\mathbf{k_1}\succeq\mathbf{m_1}} \max_{m_1\leq k_1\leq n_1} \|X_{\mathbf{k}}\| > \epsilon\right). \end{aligned}$$

By the induction assumption,

$$\begin{aligned} P\left(\sup_{\mathbf{k}\succeq\mathbf{m}} \|X_{\mathbf{k}}\| > \epsilon\right) &= \lim_{n_1\to\infty} \lim_{\|\mathbf{n_1}\|\to\infty} P\left(\max_{\mathbf{m}\preceq\mathbf{k}\preceq\mathbf{n}} \|X_{\mathbf{k}}\| > \epsilon\right) \\ &= \lim_{\|\mathbf{n}\|\to\infty} P\left(\max_{\mathbf{m}\preceq\mathbf{k}\preceq\mathbf{n}} \|X_{\mathbf{k}}\| > \epsilon\right). \end{aligned}$$

2. By Theorem 8.1.3 of Chow and Teicher [1] and the same argument as in the proof of (2.3), we have (2.4). □

Lemma 2.6. *Let $\{X_{\mathbf{n}}; \mathbf{n} \succeq \mathbf{1}\}$ be a field of $\mathbb{E}$-valued random variables. Then, $X_{\mathbf{n}}$ converges a.s. as $\|\mathbf{n}\| \to \infty$ if only if for all $\varepsilon > 0$,*

$$\lim_{\|\mathbf{n}\|\to\infty} P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|X_{\mathbf{n}+\mathbf{k}} - X_{\mathbf{n}}\| > \varepsilon\right) = 0. \tag{2.5}$$

Proof. Necessity. Suppose that $X_{\mathbf{n}} \to X$ a.s. as $\|\mathbf{n}\| \to \infty$. Then (2.5) holds, by the following inequality

$$\sup_{\mathbf{k}\succeq\mathbf{0}} \|X_{\mathbf{n}+\mathbf{k}} - X_{\mathbf{n}}\| \leq \sup_{\mathbf{m}\succeq\mathbf{n}} \|X_{\mathbf{m}} - X\| + \|X_{\mathbf{n}} - X\|.$$

Sufficiency. Suppose (2.5) holds, let $\mathbf{n'} = (n, n, \ldots, n)$, $\mathbf{k'} = (k, k, \ldots, k)$, we have $n \to \infty$ if and only if $\|\mathbf{n'}\| \to \infty$. Set $Y_n = X_{\mathbf{n'}}$ for all $n \geq 1$. Then for an arbitrary $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{k\geq 0} \|Y_{n+k} - Y_n\| > \varepsilon\right) = \lim_{\|\mathbf{n'}\|\to\infty} P\left(\sup_{\mathbf{k'}\succeq\mathbf{0}} \|X_{\mathbf{n'}+\mathbf{k'}} - X_{\mathbf{n'}}\| > \varepsilon\right) = 0$$

which implies that Y_n converges a.s. to a certain random variable X as $n \to \infty$, i.e., $X_{\mathbf{n}'}$ converges a.s. to X as $\|\mathbf{n}'\| \to \infty$. Now we prove $X_{\mathbf{n}} \to X$ a.s. as $\|\mathbf{n}\| \to \infty$.

For an arbitrary $\varepsilon > 0$,

$$P\left(\sup_{\mathbf{n}\succeq\mathbf{n}'} \|X_{\mathbf{n}} - X\| > \varepsilon\right) \leq P\left(\sup_{\mathbf{n}\succeq\mathbf{n}'} \|X_{\mathbf{n}} - X_{\mathbf{n}'}\| > \varepsilon/2\right) + P\left(\|X_{\mathbf{n}'} - X\| > \varepsilon/2\right) \to 0 \text{ as } n \to \infty,$$

so $X_{\mathbf{n}} \to X$ a.s. as $\|\mathbf{n}\| \to \infty$. □

3. Main results

Let $\{X_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ be a field of random variables in Banach space $\mathbb{E}$. Put $S_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}}$ for all $\mathbf{n} \succeq \mathbf{1}$. The series $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ is said to *converge a.s.* if the field of $\mathbb{E}$-valued random variables $\{S_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ converges a.s.. In this case, put

$$S = \lim_{\|\mathbf{n}\|\to\infty} S_{\mathbf{n}}$$

and

$$T_{\mathbf{n}} = S - S_{\mathbf{n}} = \sum_{\mathbf{k}\gg\mathbf{n}} X_{\mathbf{k}}$$

(set $S_{\mathbf{0}} = 0$). We have

$$\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\| \xrightarrow{P} 0 \quad \text{as} \quad \|\mathbf{n}\| \to \infty.$$

The following theorems provide sufficient conditions a.s. for the convergence of $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ as well as the rate of convergence to 0 of $\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\|$.

Theorem 3.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leq p \leq 2$, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a field of $\mathbb{E}$-valued martingale differences. Let $\{b_{\mathbf{n}}\}$, $\{B_{\mathbf{n}}\}$ be fields of positive constants, such that $b_{\mathbf{n}} = o(1)$, $B_{\mathbf{n}} = o(1)$ as $\|\mathbf{n}\| \to \infty$.*

(1) *If*

$$\sum_{\mathbf{k}\gg\mathbf{n}} E\|X_{\mathbf{k}}\|^p = \mathcal{O}(b_{\mathbf{n}}^p), \tag{3.1}$$

then $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ converges a.s. and

$$\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\| = \mathcal{O}_P(b_{\mathbf{n}}). \tag{3.2}$$

(2) *If*

$$\sum_{\mathbf{k}\gg\mathbf{n}} E\|X_{\mathbf{k}}\|^p = o(B_{\mathbf{n}}^p) \tag{3.3}$$

as $\|\mathbf{n}\| \to \infty$, *then* $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ *converges a.s. and*

$$\frac{\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\|}{B_{\mathbf{n}}} \xrightarrow{P} 0 \tag{3.4}$$

as $\|\mathbf{n}\| \to \infty$.

Proof. (1) Set $S_{\mathbf{n}} = \sum_{\mathbf{k}\succeq\mathbf{n}} X_{\mathbf{k}}$. For an arbitrary $\varepsilon > 0$, set $\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_d) = (\mathbf{n}_i, n_i, \mathbf{n}'_i)$, $\mathbf{k} = (k_1, \ldots, k_i, \ldots, k_d) = (\mathbf{k}_i, k_i, \mathbf{k}'_i)$, $\mathbf{j} = (j_1, \ldots, j_i, \ldots, j_d) = (\mathbf{j}_i, j_i, \mathbf{j}'_i)$ for all $1 \le i \le d$. We have that

$$\begin{aligned}
&P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|S_{\mathbf{n}+\mathbf{k}} - S_{\mathbf{n}}\| > \varepsilon\right)\\
&= P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|\sum_{\mathbf{n}\ll\mathbf{j}\preceq\mathbf{n}+\mathbf{k}} X_{\mathbf{j}}\| > \varepsilon\right)\\
&\le \sum_{i=1}^{d} P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|\sum_{\mathbf{1}\preceq\mathbf{j_i}\preceq\mathbf{n_i}} \sum_{j_i=n_i}^{n_i+k_i} \sum_{\mathbf{1}\preceq\mathbf{j}'\preceq\mathbf{n}'_\mathbf{i}+\mathbf{k}'_\mathbf{i}} X_{(\mathbf{j}_i, j_i, \mathbf{j}'_i)}\| > \varepsilon/d\right).
\end{aligned}$$

Applying the Markov inequality and Lemma 2.3, we obtain

$$\begin{aligned}
&P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|\sum_{\mathbf{1}\preceq\mathbf{j_i}\preceq\mathbf{n_i}} \sum_{j_i=n_i}^{n_i+k_i} \sum_{\mathbf{1}\preceq\mathbf{j}'\preceq\mathbf{n}'_\mathbf{i}+\mathbf{k}'_\mathbf{i}} X_{(\mathbf{j}_i, j_i, \mathbf{j}'_i)}\| > \varepsilon/d\right)\\
&\le \frac{d^p}{\varepsilon^p} E\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|\sum_{\mathbf{1}\preceq\mathbf{j_i}\preceq\mathbf{n_i}} \sum_{j_i=n_i}^{n_i+k_i} \sum_{\mathbf{1}\preceq\mathbf{j}'\preceq\mathbf{n}'_\mathbf{i}+\mathbf{k}'_\mathbf{i}} X_{(\mathbf{j}_i, j_i, \mathbf{j}'_i)}\|^p\right)\\
&\le C \sum_{\mathbf{1}\preceq\mathbf{j_i}\preceq\mathbf{n_i}} \sum_{j_i=n_i}^{\infty} \sum_{\mathbf{j}'\succeq\mathbf{1}} E\|X_{(\mathbf{j}_i, j_i, \mathbf{j}'_i)}\|^p.
\end{aligned}$$

Then, using (3.1) or (3.3), we have that

$$\begin{aligned}
P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|S_{\mathbf{n}+\mathbf{k}} - S_{\mathbf{n}}\| > \varepsilon\right) &\le C \sum_{i=1}^{d} \sum_{\mathbf{1}\preceq\mathbf{j_i}\preceq\mathbf{n_i}} \sum_{j_i=n_i}^{\infty} \sum_{\mathbf{j}'\succeq\mathbf{1}} E\|X_{(\mathbf{j}_i, j_i, \mathbf{j}'_i)}\|^p\\
&= C \sum_{\mathbf{j}\gg\mathbf{n}} E\|X_{\mathbf{j}}\|^p = o(1) \quad \text{as} \quad \|\mathbf{n}\| \to \infty,
\end{aligned}$$

which implies that $S_{\mathbf{n}}$ converges a.s as $\|\mathbf{n}\| \to \infty$ (by Lemma 2.6). Then $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ converges a.s. Thus, the tail series $\{T_{\mathbf{n}} = \sum_{\mathbf{k}\gg\mathbf{n}} X_{\mathbf{k}}\}$ is a well-defined field of random variables.

Next, to prove that (3.1) implies (3.2), observe that for $K > 0$

$$\sup_{\mathbf{n}\succeq\mathbf{1}} P\left(\frac{\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\|}{b_{\mathbf{n}}} > K\right)$$

$$= \sup_{\mathbf{n}\succeq\mathbf{1}} P\left(\sup_{\mathbf{k}\succeq\mathbf{n}} \|\sum_{\mathbf{i}\gg\mathbf{k}} X_{\mathbf{i}}\| > K.b_{\mathbf{n}}\right)$$

$$= \sup_{\mathbf{n}\succeq\mathbf{1}} \lim_{\|\mathbf{N}\|\to\infty} P\left(\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{N}} \lim_{\|\mathbf{M}\|\to\infty} \|\sum_{\mathbf{k}\ll\mathbf{i}\preceq\mathbf{M}} X_{\mathbf{i}}\| > K.b_{\mathbf{n}}\right) \text{ (by Lemma 2.5)}$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \lim_{\|\mathbf{N}\|\to\infty} P\left(\lim_{\|\mathbf{M}\|\to\infty} \max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{N}} \|\sum_{\mathbf{k}\ll\mathbf{i}\preceq\mathbf{M}} X_{\mathbf{i}}\| > K.b_{\mathbf{n}}\right)$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \lim_{\|\mathbf{N}\|\to\infty} \liminf_{\|\mathbf{M}\|\to\infty} P\left(\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{N}} \|\sum_{\mathbf{k}\ll\mathbf{i}\preceq\mathbf{M}} X_{\mathbf{i}}\| > K.b_{\mathbf{n}}\right) \text{ (by Lemma 2.5)}$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \liminf_{\|\mathbf{M}\|\to\infty} P\left(\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}} \|\sum_{\mathbf{k}\ll\mathbf{i}\preceq\mathbf{M}} X_{\mathbf{i}}\| > K.b_{\mathbf{n}}\right)$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \liminf_{\|\mathbf{M}\|\to\infty} \left(P\Big(\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}} \|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} X_{\mathbf{i}}\| > \frac{K.b_{\mathbf{n}}}{2}\Big) + P\Big(\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{M}} X_{\mathbf{i}}\| > \frac{K.b_{\mathbf{n}}}{2}\Big)\right)$$

$$\leq 2 \sup_{\mathbf{n}\succeq\mathbf{1}} \lim_{\|\mathbf{M}\|\to\infty} P\left(\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}} \|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} X_{\mathbf{i}}\| > \frac{K.b_{\mathbf{n}}}{2}\right)$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \frac{2^{p+1}}{K^p b_{\mathbf{n}}^p} \lim_{\|\mathbf{M}\|\to\infty} E \max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}} \|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} X_{\mathbf{i}}\|^p \text{ (by the Markov inequality)}$$

$$\leq \sup_{\mathbf{n}\succeq\mathbf{1}} \frac{2^{p+1}}{K^p b_{\mathbf{n}}^p} \lim_{\|\mathbf{M}\|\to\infty} \sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{M}} E\|X_{\mathbf{i}}\|^p \text{ (by Lemma 2.3)}$$

$$= \sup_{\mathbf{n}\succeq\mathbf{1}} \frac{C}{K^p b_{\mathbf{n}}^p} \sum_{\mathbf{n}\ll\mathbf{i}} E\|X_{\mathbf{i}}\|^p \leq \frac{C}{K^p} \text{ (by (3.1))} \to 0 \text{ as } K\to\infty.$$

(2) The proof that (3.3) implies (3.4) is the same as that in (1). □

Remark 3.2. It should be noted that:

- In the case $d = 1$, Theorem 3.1 reduces to Corollary 2 in [9].
- The proof of Theorem 3.1 closely follows the pattern of Theorem 1 in [8].
- The primary mode of the convergence given by (3.4) of Theorem 3.1 was introduced in [4] in the case $d = 1$ for the tail series of a convergent series of random variables.

Example 4. Let $\{V_{\mathbf{n}}, \mathbf{n}\succeq\mathbf{1}\}$ be a field independent, identically distributed, mean 0 random variables in a p-uniformly smooth Banach space $\mathbb{E}$ $(1\leq p\leq 2)$ such that $E\|V_{\mathbf{1}}\|^p<\infty$, let $\{a_{\mathbf{n}}, \mathbf{n}\succeq\mathbf{1}\}$ be a field of nonzero constants such that $\sum_{\mathbf{n}\succeq\mathbf{1}} |a_{\mathbf{n}}|^p < \infty$, set $X_{\mathbf{n}} = a_{\mathbf{n}} V_{\mathbf{n}}$ for $\mathbf{n}\succeq\mathbf{1}$. Then $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}; \mathbf{n}\succeq\mathbf{1}\}$ is a field of martingale difference. By taking $b_{\mathbf{n}} = (\sum_{\mathbf{i}\gg\mathbf{n}} |a_{\mathbf{i}}|^p)^{1/p}\alpha_{\mathbf{n}}^{-1}$, where $\{\alpha_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ is any field of positive numbers, we have that

$$\frac{\sum_{\mathbf{i}\gg\mathbf{n}} E\|X_{\mathbf{i}}\|^p}{b_{\mathbf{n}}} = \alpha_{\mathbf{n}} E\|X_{\mathbf{1}}\|^p \to 0.$$

If $\sup_{\mathbf{n}\succeq\mathbf{1}} \alpha_{\mathbf{n}} < \infty$, then (3.1) holds, by Theorem 3.1, we have that

$$\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\| = \mathcal{O}_P(b_{\mathbf{n}}).$$

If $\alpha_{\mathbf{n}} \to 0$ as $\|\mathbf{n}\| \to \infty$, then (3.3) holds, by Theorem 3.1, we have that

$$\frac{\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\|}{b_{\mathbf{n}}} \xrightarrow{P} 0 \quad \text{as} \quad \|\mathbf{n}\| \to \infty.$$

Next, we establish the rate of convergence of series of strong* martingale difference fields, with the field of positive Borel function $\{\phi_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ which have a property similar to that of the sequence of functions in [2] of Hong and Tsay, i.e.,

$$C_{\mathbf{n}} \frac{u^{\lambda_{\mathbf{n}}}}{v^{\lambda_{\mathbf{n}}}} \le \frac{\Phi_{\mathbf{n}}(u)}{\Phi_{\mathbf{n}}(v)} \le D_{\mathbf{n}} \frac{u^{\mu_{\mathbf{n}}}}{v^{\mu_{\mathbf{n}}}} \quad \text{for all} \quad u \ge v > 0, \tag{3.5}$$

where $C_{\mathbf{n}} \ge 1$, $D_{\mathbf{n}} \ge 1$, $\lambda_{\mathbf{n}} \ge 1$, $0 < \mu_{\mathbf{n}} \le p$.

Note that the array of functions $\{\Phi_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ with $\Phi_{\mathbf{n}}(x) = x^p$, $p \ge 1$ satisfies the condition (3.5).

Theorem 3.3. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \le p \le 2$, let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a field of $\mathbb{E}$-valued strong* martingale differences. Let $\{\Phi_{\mathbf{n}}; \mathbf{n} \succeq \mathbf{1}\}$ be a field of positive Borel functions which satisfies the conditions* (3.5) *and $\Phi_{\mathbf{n}}(u) \le \Phi_{\mathbf{m}}(u)$ for all $\mathbf{n} \ll \mathbf{m}$. Let $\{b_{\mathbf{n}}\}$, $\{B_{\mathbf{n}}\}$ be fields of positive constants, such that $\Phi_{\mathbf{n}}(b_{\mathbf{n}}) = o(1)$, $\Phi_{\mathbf{n}}(B_{\mathbf{n}}) = o(1)$ as $\|\mathbf{n}\| \to \infty$.*

(1) *If*

$$\sum_{\mathbf{k}\gg\mathbf{n}} A_{\mathbf{k}} E\Phi_{\mathbf{k}}(\|X_{\mathbf{k}}\|) = \mathcal{O}(\Phi_{\mathbf{n}}(b_{\mathbf{n}})), \tag{3.6}$$

where $A_{\mathbf{n}} = \max\{\frac{1}{C_{\mathbf{n}}}, D_{\mathbf{n}}\}$, then $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ converges a.s. and the series $\{T_{\mathbf{n}} = \sum_{\mathbf{k}\gg\mathbf{n}} X_{\mathbf{k}}\}$ satisfies the relation

$$\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\| = \mathcal{O}_P(b_{\mathbf{n}}). \tag{3.7}$$

(2) *If*

$$\sum_{\mathbf{k}\gg\mathbf{n}} A_{\mathbf{k}} E\Phi_{\mathbf{k}}(\|X_{\mathbf{k}}\|) = o((\Phi_{\mathbf{n}}(B_{\mathbf{n}})) \tag{3.8}$$

as $\|\mathbf{n}\| \to \infty$ (where $A_{\mathbf{n}} = \max\{\frac{1}{C_{\mathbf{n}}}, D_{\mathbf{n}}\}$), then $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ converges a.s. and the series $\{T_{\mathbf{n}} = \sum_{\mathbf{k}\gg\mathbf{n}} X_{\mathbf{k}}\}$ obeys the limit law

$$\frac{\sup_{\mathbf{k}\succeq\mathbf{n}} \|T_{\mathbf{k}}\|}{B_{\mathbf{n}}} \xrightarrow{P} 0 \quad \text{as } \|\mathbf{n}\| \to \infty. \tag{3.9}$$

Proof. (1) For each $\mathbf{n} \succeq \mathbf{1}$, set $Y_{\mathbf{n}} = X_{\mathbf{n}} I(\|X_{\mathbf{n}}\| \le 1)$, $Z_{\mathbf{n}} = X_{\mathbf{n}} I(\|X_{\mathbf{n}}\| > 1)$, $U_{\mathbf{n}} = Y_{\mathbf{n}} - E(Y_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}})$, $V_{\mathbf{n}} = Z_{\mathbf{n}} - E(Z_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}})$. $S^1_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n}} U_{\mathbf{k}}$, and $S^2_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n}} V_{\mathbf{k}}$. Then $X_{\mathbf{n}} = U_{\mathbf{n}} + V_{\mathbf{n}}$ and $S_{\mathbf{n}} = S^1_{\mathbf{n}} + S^2_{\mathbf{n}}$. Moreover, since $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ is a field of strong* martingale differences, it is clear that $\{U_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ and $\{V_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ are strong adapted fields.

By the proof of Theorem 3.1, we have

$$\begin{aligned}
P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|S^1_{\mathbf{n}+\mathbf{k}} - S^1_{\mathbf{n}}\| > \varepsilon\right) &= C\sum_{\mathbf{i}\gg\mathbf{n}} E\|Y_{\mathbf{i}}\|^p \le C\sum_{\mathbf{i}\gg\mathbf{n}} E\|Y_{\mathbf{i}}\|^{\mu_{\mathbf{n}}} \\
&\le C\sum_{\mathbf{i}\gg\mathbf{n}} D_{\mathbf{i}}\cdot E\frac{\Phi_{\mathbf{i}}(\|Y_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(1)} \le C\sum_{\mathbf{i}\gg\mathbf{n}} A_{\mathbf{i}}\frac{E\Phi_{\mathbf{i}}(\|X_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(1)} \\
&\le C\frac{1}{\Phi_{\mathbf{1}}(1)}\sum_{\mathbf{i}\gg\mathbf{n}} A_{\mathbf{i}} E\Phi_{\mathbf{n}}(\|X_{\mathbf{i}}\|) < o(1) \quad \text{as} \quad \|\mathbf{n}\|\to\infty.
\end{aligned}$$

Then $S^1_{\mathbf{n}}$ converges a.s. as $\|\mathbf{n}\| \to \infty$ (by Lemma 2.6). Next, by the proof of Theorem 3.1, we have

$$\begin{aligned}
P\left(\sup_{\mathbf{k}\succeq\mathbf{0}} \|S^2_{\mathbf{n}+\mathbf{k}} - S^2_{\mathbf{n}}\| > \varepsilon\right) &= C\sum_{\mathbf{i}\gg\mathbf{n}} E\|Z_{\mathbf{i}}\| \le C\sum_{\mathbf{i}\gg\mathbf{n}} E\|Z_{\mathbf{i}}\|^{\lambda_{\mathbf{n}}} \\
&\le C\sum_{\mathbf{i}\gg\mathbf{n}} \frac{1}{C_{\mathbf{i}}}\cdot E\frac{\Phi_{\mathbf{i}}(\|Z_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(1)} \le C\sum_{\mathbf{i}\gg\mathbf{n}} A_{\mathbf{i}}\frac{E\Phi_{\mathbf{i}}(\|X_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(1)} \\
&\le C\frac{1}{\Phi_{\mathbf{1}}(1)}\sum_{\mathbf{i}\gg\mathbf{n}} A_{\mathbf{i}} E\Phi_{\mathbf{n}}(\|X_{\mathbf{i}}\|) < o(1) \quad \text{as} \quad \|\mathbf{n}\|\to\infty.
\end{aligned}$$

Then $S^2_{\mathbf{n}}$ converges a.s. as $\|\mathbf{n}\| \to \infty$ (by Lemma 2.6). By $S_{\mathbf{n}} = S^1_{\mathbf{n}} + S^2_{\mathbf{n}}$, which implies $S_{\mathbf{n}}$ converges a.s as $\|\mathbf{n}\| \to \infty$, then $\sum_{\mathbf{n}\succeq\mathbf{1}} X_{\mathbf{n}}$ converges a.s. Thus, the tail series $\{T_{\mathbf{n}} = \sum_{\mathbf{k}\gg\mathbf{n}} X_{\mathbf{k}}; \mathbf{n} \succeq \mathbf{1}\}$ is a well-defined field of random variables.

Next, to prove that (3.6) implies (3.7), for each $\mathbf{n} \succeq \mathbf{1}$, and $\mathbf{n} \succeq \mathbf{i}$, set $Y'_{\mathbf{i}} = X_{\mathbf{i}} I(\|X_{\mathbf{i}}\| \le b_{\mathbf{n}})$, $Z'_{\mathbf{i}} = X_{\mathbf{i}} I(\|X_{\mathbf{i}}\| > b_{\mathbf{n}})$, $U'_{\mathbf{n}} = Y'_{\mathbf{n}} - E(Y'_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}})$, $V'_{\mathbf{n}} = Z'_{\mathbf{n}} - E(Z'_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}})$. Then $X_{\mathbf{n}} = U'_{\mathbf{n}} + V'_{\mathbf{n}}$. Moreover, by $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ being a strong* martingale difference field, then it is clear that $\{U'_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ and $\{V'_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \succeq \mathbf{1}\}$ are strong adaped fields.

By the proof of Theorem 3.1, we observe that for $K > 0$,

$$\begin{aligned}
&\sup_{\mathbf{n}\succeq\mathbf{1}} P\left(\frac{\sup_{\mathbf{k}\succeq\mathbf{n}}\|T_{\mathbf{k}}\|}{b_{\mathbf{n}}} > K\right) \\
&\le 2\sup_{\mathbf{n}\succeq\mathbf{1}}\liminf_{\|\mathbf{M}\|\to\infty} P\left(\frac{\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}}\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} X_{\mathbf{i}}\|}{b_{\mathbf{n}}} > \frac{K}{2}\right) \\
&\le 2\sup_{\mathbf{n}\succeq\mathbf{1}}\lim_{\|\mathbf{M}\|\to\infty}\left(P\left(\frac{\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}}\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} U'_{\mathbf{i}}\|}{b_{\mathbf{n}}} > \frac{K}{4}\right)\right. \\
&\qquad\left. + P\left(\frac{\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}}\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}} V'_{\mathbf{i}}\|}{b_{\mathbf{n}}} > \frac{K}{4}\right)\right)
\end{aligned}$$

$$\le 2\sup_{\mathbf{n}\succeq\mathbf{1}}\lim_{\mathbf{M}\to\infty}\left(\frac{4^p}{K^p b_{\mathbf{n}}^p}E\Big\{\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}}\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}}U'_{\mathbf{i}}\|^p\Big\}\right.$$
$$\left.+\frac{4}{Kb_{\mathbf{n}}}E\Big\{\max_{\mathbf{n}\preceq\mathbf{k}\preceq\mathbf{M}}\|\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{k}}V'_{\mathbf{i}}\|\Big\}\right)$$
$$\le 2\sup_{\mathbf{n}\succeq\mathbf{1}}\lim_{\|\mathbf{M}\|\to\infty}\left(\frac{4^p}{K^p b_{\mathbf{n}}^p}\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{M}}E\|U'_{\mathbf{i}}\|^p\right.$$
$$\left.+\frac{4}{Kb_{\mathbf{n}}}\sum_{\mathbf{n}\ll\mathbf{i}\preceq\mathbf{M}}E\|V'_{\mathbf{i}}\|\right)\quad\text{(by Lemma 2.3)}$$
$$\le C\sup_{\mathbf{n}\succeq\mathbf{1}}\left(\frac{1}{K^p}\sum_{\mathbf{n}\ll\mathbf{i}}\frac{E\|Y'_{\mathbf{i}}\|^p}{b_{\mathbf{n}}^p}+\frac{1}{K}\sum_{\mathbf{n}\ll\mathbf{i}}\frac{E\|Z'_{\mathbf{i}}\|}{b_{\mathbf{n}}}\right)$$
$$\le C\sup_{\mathbf{n}\succeq\mathbf{1}}\left(\frac{1}{K^p}\sum_{\mathbf{n}\ll\mathbf{i}}\frac{E\|Y'_{\mathbf{i}}\|^{\mu_{\mathbf{n}}}}{b_{\mathbf{n}}^{\mu_{\mathbf{n}}}}+\frac{1}{K}\sum_{\mathbf{n}\ll\mathbf{i}}\frac{E\|Z'_{\mathbf{i}}\|^{\lambda_{\mathbf{n}}}}{b_{\mathbf{n}}^{\lambda_{\mathbf{n}}}}\right)$$
$$\le C\sup_{\mathbf{n}\succeq\mathbf{1}}\left(\frac{1}{K^p}\sum_{\mathbf{i}\gg\mathbf{n}}D_{\mathbf{i}}\cdot E\frac{\Phi_{\mathbf{i}}(\|Y'_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(b_{\mathbf{n}})}+\frac{1}{K}\sum_{\mathbf{i}\gg\mathbf{n}}\frac{1}{C_{\mathbf{i}}}\cdot E\frac{\Phi_{\mathbf{i}}(\|Z'_{\mathbf{i}}\|)}{\Phi_{\mathbf{i}}(b_{\mathbf{n}})}\right)$$
$$\le C\sup_{\mathbf{n}\succeq\mathbf{1}}\frac{1}{\Phi_{\mathbf{n}}(b_{\mathbf{n}})}\left(\frac{1}{K^p}+\frac{1}{K}\right)\sum_{\mathbf{i}\gg\mathbf{n}}A_{\mathbf{i}}E\Phi_{\mathbf{n}}(\|X_{\mathbf{i}}\|)<o(1)\quad\text{as}\quad K\to\infty.$$

(2) The proof that (3.8) implies (3.9) is the same as that in (1). □

When $d=1$, by Remark 2.1, we have the following corollary.

Corollary 3.3.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1\le p\le 2$, $\{X_n,\mathcal{F}_n;n\in\mathbb{N}\}$ be a sequence of $\mathbb{E}$-valued martingale differences. Let $\{\Phi_n;n\ge 1\}$ be a sequence of positive Borel functions which satisfies the following two conditions*

$$C_n\frac{u^{\lambda_n}}{v^{\lambda_n}}\le\frac{\Phi_n(u)}{\Phi_n(v)}\le D_n\frac{u^{\mu_n}}{v^{\mu_n}}\quad\textit{for all}\quad u\ge v>0,$$

where $C_n\ge 1$, $D_n\ge 1$, $\lambda_n\ge 1$, $0<\mu_n\le p$,

$$\Phi_n(u)\le\Phi_m(u)\quad\textit{for all}\quad n>m.$$

Let $\{b_n\}$, $\{B_n\}$ be sequences of positive constants, such that $\Phi_n(b_n)=o(1)$, $\Phi_n(B_n)=o(1)$.

(1) *If*

$$\sum_{k\ge n+1}A_kE\Phi_k(\|X_k\|)=\mathcal{O}(\Phi_n(b_n)),$$

then $\sum_{n\geq 1} X_n$ *converges a.s. and*

$$\sup_{k\geq n+1} \|T_k\| = \mathcal{O}_P(b_n),$$

where $T_n = \sum_{k\geq n+1} X_k$.

(2) *If*

$$\sum_{k\geq n+1} A_k E\Phi_k(\|X_k\|) = o((\Phi_n(B_n)),$$

then $\sum_{n\geq 1} X_n$ *converges a.s. and*

$$\frac{\sup_{k\geq n+1} \|T_k\|}{B_n} \xrightarrow{P} 0,$$

where $T_n = \sum_{k\geq n+1} X_k$.

Remark 3.4. Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of real-valued independent random variables with $EX_n = 0$, $n \geq 1$. Let $\{g_n(x), n \geq 1\}$ be a sequence of functions defined on $[0, \infty)$ such that

$$0 \leq g_n(0) \leq g_n(x),\ 0 < g_n(x) \uparrow \infty \quad \text{as } n \uparrow \infty \quad \text{for each } x > 0$$

and

$$\frac{g_n(x)}{x} \uparrow, \quad \frac{g_n(x)}{x^p} \downarrow \quad \text{on} \quad (0, \infty), \quad n \geq 1, \quad \text{for some } 1 < p \leq 2.$$

In Corollary 3.3.1, taking $\Phi_n = g_n$ for all $n \geq 1$, with $\lambda_n = 1$, $\mu_n = p$, $C_n = 1$, $D_n = 1$, $n \geq 1$, we obtain Theorem 2 in [11].

Finally, we establish the rate of complete convergence of the tail series of martingale difference fields.

Theorem 3.5. *Let* $\mathbb{E}$ *be a* p*-uniformly smooth Banach space for some* $1 \leq p \leq 2$, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ *be a field of* $\mathbb{E}$*-valued martingale differences. Let* $\{a_{\mathbf{n}}\}$ *be field of positive constants, such that either* $a_{\mathbf{n}} \leq a_{\mathbf{m}}$ *for all* $\mathbf{n} \preceq \mathbf{m}$ *or* $a_{\mathbf{n}} \geq a_{\mathbf{m}}$ *for all* $\mathbf{n} < \mathbf{m}$ *and* $\sup_n a_{\mathbf{2^n}}/a_{\mathbf{2^{n+1}}} \leq M < \infty$. *If*

$$\sum_{\mathbf{n}\succeq \mathbf{1}} \varphi(\mathbf{n}) E\|X_{\mathbf{n}}\|^p < \infty, \tag{3.10}$$

where $\varphi(\mathbf{n}) = \sum_{\mathbf{2^k}\ll \mathbf{n}} \frac{1}{a_{\mathbf{2^k}}}$, *then for all* $\varepsilon > 0$,

$$\sum_{\mathbf{n}\succeq \mathbf{1}} \frac{1}{|\mathbf{n}|} P(\sup_{\mathbf{k}\succeq \mathbf{n}} \|T_{\mathbf{k}}\| > \varepsilon a_{\mathbf{n}}) < \infty. \tag{3.11}$$

Proof. For all $\mathbf{n} \succeq \mathbf{2}$ then $\varphi(\mathbf{n}) \geq \frac{1}{b_{\mathbf{2}}} > 0$, so $\sum_{\mathbf{k}\gg \mathbf{n}} E\|X_{\mathbf{k}}\|^p = o(1)$ as $\|\mathbf{n}\| \to \infty$. By proof of Theorem 3.1, we have $\sum_{\mathbf{n}\succeq \mathbf{1}} X_{\mathbf{n}}$ converges a.s. Thus, the tail series $\{T_{\mathbf{n}} = \sum_{\mathbf{k}\gg \mathbf{n}} X_{\mathbf{k}}; \mathbf{n} \succeq \mathbf{1}\}$ is a well-defined field of random variables.

Next, to prove that (3.10) implies (3.11), we note that

$$\sum_{\mathbf{n}\succeq \mathbf{1}} \frac{1}{|\mathbf{n}|} P(\sup_{\mathbf{k}\succeq \mathbf{n}} \|T_{\mathbf{k}}\| > \varepsilon a_{\mathbf{n}}) = \sum_{\mathbf{n}\succeq \mathbf{0}} \sum_{\mathbf{2^i}\preceq \mathbf{n}\preceq \mathbf{2^{i+1}}} \frac{1}{|\mathbf{i}|} P(\sup_{\mathbf{k}\succeq \mathbf{i}} \|T_{\mathbf{k}}\| > \varepsilon a_{\mathbf{i}})$$

$$\leq \sum_{\mathbf{n}\succeq \mathbf{0}} P(\sup_{\mathbf{k}\succeq \mathbf{2^n}} \|T_{\mathbf{k}}\| > \frac{1}{M}\cdot \varepsilon a_{\mathbf{2^n}}).$$

Applying Lemma 2.3, Lemma 2.5, the Markov inequality and the same argument as the proof of Theorem 3.1, we see

$$\sum_{\mathbf{n}\succeq \mathbf{1}} \frac{1}{|\mathbf{n}|} P(\sup_{\mathbf{k}\succeq \mathbf{n}} \|T_{\mathbf{k}}\| > \varepsilon b_{\mathbf{n}}) = \sum_{\mathbf{n}\succeq \mathbf{0}} \frac{M^p}{\varepsilon^p \cdot a^p_{\mathbf{2^n}}} \sum_{\mathbf{i}\gg \mathbf{2^n}} E\|X_{\mathbf{i}}\|^p$$
$$\leq C \sum_{\mathbf{n}\succeq \mathbf{1}} \varphi(\mathbf{n}) \|X_{\mathbf{n}}\|^p < \infty.$$

□

Corollary 3.5.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leq p \leq 2$, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a field of $\mathbb{E}$-valued martingale differences. If*

$$\sum_{\mathbf{n}\succeq \mathbf{1}} E\|X_{\mathbf{n}}\|^p < \infty, \tag{3.12}$$

then for all $\alpha > 0$, $\varepsilon > 0$,

$$\sum_{\mathbf{n}\succeq \mathbf{1}} \frac{1}{|\mathbf{n}|} P(\sup_{\mathbf{k}\succeq \mathbf{n}} \|T_{\mathbf{k}}\| > \varepsilon |\mathbf{n}|^\alpha) < \infty. \tag{3.13}$$

Proof. Put $a_{\mathbf{n}} = |\mathbf{n}|^\alpha$ then $\varphi(\mathbf{n}) \leq \sum_{\mathbf{n}\succeq \mathbf{1}} \frac{1}{|2^{\alpha \mathbf{n}}|} = (\sum_{n=1}^\infty \frac{1}{2^{\alpha n}})^d < \infty$ so (3.12) implies (3.10). By Theorem 3.5 we get (3.13). □

Corollary 3.5.2. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leq p \leq 2$, $\{X_n, \mathcal{F}_n; n \in \mathbb{N}\}$ be a sequence of $\mathbb{E}$-valued martingale differences. If*

$$\sum_{n=1}^\infty E\|X_n\|^p \log_2 n < \infty, \tag{3.14}$$

then for all $\alpha > 0, \varepsilon > 0$,

$$\sum_{n=1}^\infty \frac{1}{n} P(\sup_{k\geq n} \|T_k\| > \varepsilon) < \infty. \tag{3.15}$$

Proof. Put $a_n = 1$ then for $d = 1$ we have $\varphi(n) \leq \log_2 n$. Hence (3.14) implies (3.10). By Theorem 3.5 we obtain (3.15). □

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