J. Korean Math. Soc. **52** (2015), No. 5, pp. 1003–1021
http://dx.doi.org/10.4134/JKMS.2015.52.5.1003

COMPLETION OF HANKEL PARTIAL CONTRACTIONS OF NON-EXTREMAL TYPE

In Hyoun Kim, Seonguk Yoo, and Jasang Yoon

Abstract. A matrix completion problem has been exploited amply because of its abundant applications and the analysis of contractions enables us to have insight into structure and space of operators. In this article, we focus on a specific completion problem related to Hankel partial contractions. We provide concrete necessary and sufficient conditions for the existence of completion of Hankel partial contractions for both extremal and non-extremal types with lower dimensional matrices. Moreover, we give a negative answer for the conjecture presented in [8]. For our results, we use several tools such as the Nested Determinants Test (or Choleski's Algorithm), the Moore-Penrose inverse, the Schur product techniques, and a congruence of two positive semi-definite matrices; all these suggest an algorithmic approach to solve the contractive completion problem for general Hankel matrices of size $n \times n$ in both types.

1. Introduction

A *partial matrix* is a square array in which some entries are specified and others are not. A *completion* of a partial matrix is a choice of values for the unspecified entries. A *matrix completion problem* asks whether a given partial matrix has a completion of a desired type. For example, the positive definite completion problem asks which partial Hermitian matrices have a positive definite completion. For a 2×2 partial operator matrix $A \equiv \left(\begin{smallmatrix} B & C \\ D & X \end{smallmatrix}\right)$, we say that X is a solution for A, if A is a completion of a desired type. These completion problems have been studied by G. Arsene and A. Gheondea [1], by C. Davis, W. Kahan and H. Weinberger [10] (see also [4] and [9]), by C. Foiaş

Received November 27, 2014; Revised March 20, 2015.
2010 *Mathematics Subject Classification.* Primary 47B35, 15A83, 15A60, 15A48; Secondary 47A30, 15-04.
Key words and phrases. Hankel partial contraction, contractive completion, extremal type, non-extremal type.
The first named author was partially supported by the National Research Foundation of Korea Grant funded by the Korean Government (MOE) (Grant Number: 2103R1A1A2009365). The second named author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (MOE) (Grant Number: 2103R1A1A2009365) and by the Brain Korea 21 Program of the National Research Foundation of Korea (Grant Number: 22A20130012598).

©2015 Korean Mathematical Society

and A. Frazho [11] (using Redheffer products), by S. Parrott [20], and by Y. L. Shmul'yan and R. N. Yanovskaya [24]; a solution is also implicit in the work of W. Arveson [2] (see also [17] and [22]). A *Hankel matrix* is a square matrix with constant skew-diagonals. A *Toeplitz matrix* is a square matrix in which each descending diagonal from left to right is constant. Hankel and Toeplitz matrices have a long history (see, for instance, [16]) and have given rise to important recent applications in a variety of areas. A matrix completion problem is due, in particular, to its many applications, e.g., in probability and statistics (e.g. entropy methods for missing data, see, for instance, [12] and [13]), chemistry (e.g. the molecular conformation problem [5]), numerical analysis (e.g. optimization, see, for instance, [19]), electrical engineering (e.g. data transmission, coding and image enhancement, see, for instance, [3]) and geophysics (seismic reconstruction problems, see, for instance, [14]). A *Hankel Partial Contraction* (HPC) is a square Hankel matrix such that not all of its entries are determined, but in which every well-defined submatrix (completely determined submatrix) is a contraction (in the sense that their operator norms are at most 1). In this article, we study whether a HPC can be completed to a contraction or not when the upper left triangle is known. That is, given real numbers $a_1, \ldots, a_n$, let

$$
\begin{aligned}
H_n &\equiv H_n(a_1, a_2, \ldots, a_n; x_1, \ldots, x_{n-1}) \\
&:= \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix}
\end{aligned} \tag{1}
$$

be a Hankel matrix, where $x_1, \ldots, x_{n-1}$ are real numbers to be determined. We then consider:

Problem 1.1. Find the necessary and sufficient conditions on the given real numbers $a_1, a_2, \ldots, a_n$ as in (1) to guarantee the existence of a contractive Hankel completion.

We say that Problem 1.1 is *well-posed* if H_n is partially contractive, and that it is *soluble* if H_n is contractive for some $x_1, \ldots, x_{n-1}$. We also say that H_n is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$.

In [8, Section 4], R. Curto, S. Lee and the third named author of this paper found necessary and sufficient conditions for the existence of contractive completion of HPC's of the extremal type for 4×4 matrices. In this paper, we improve and extend the main results in [8, Section 4] to the non-extremal type for 4×4 matrices and extremal type for 5×5 matrices, respectively. We also give a negative answer to the conjecture presented in [8, Remark 4.5]. We find concrete necessary and sufficient conditions for the existence of completion of 4×4 and 5×5 Hankel partial contractions using the Nested Determinants Test (or Choleski's Algorithm), the Moore-Penrose inverse of a matrix, the Schur product techniques of matrices, and the congruence of the positivity for two matrices. All these techniques may allow us to solve, algorithmically, the contractive completion problem for the non-extremal type of 5×5 Hankel matrices and more.

Acknowledgement. The authors are deeply indebted to the referee for several helpful suggestions. The portions of the proof of some results were obtained using calculations with the software tool *Mathematica* [26].

2. Preliminaries

For the reader's convenience, in this section, we gathered several auxiliary results which are needed for the proofs of the main results in this article. We first recall that an $n \times n$ matrix $M_{n\times n}$ is a contraction if and only if the matrix

$$P_n := P_n(M_{n\times n}) := I - M_{n\times n}M_{n\times n}^*$$

is positive semi-definite (in symbols, $P_n \geq 0$), where I is the identity matrix and $M_{n\times n}^*$ is the adjoint of $M_{n\times n}$. In order to check the positivity of P_n, we use the following version of the Nested Determinants Test.

Lemma 2.1 ([6]). *Assume*

$$P := (p_{ij})_{i,j=1}^n := \begin{pmatrix} u & \mathbf{t} \\ \mathbf{t}^* & P_0 \end{pmatrix},$$

where P_0 is an $(n-1) \times (n-1)$ matrix, $\mathbf{t}$ is a row vector, and u is a real number.

(i) *If P_0 is invertible, then* $\det P = \det P_0(u - \mathbf{t}P_0^{-1}\mathbf{t}^*)$.

(ii) *If P_0 is invertible and positive, then $P \geq 0 \Longleftrightarrow (u - \mathbf{t}P_0^{-1}\mathbf{t}^*) \geq 0 \Longleftrightarrow$* $\det P \geq 0$.

(ii) *If $u > 0$, then $P \geq 0 \Longleftrightarrow P_0 - \mathbf{t}^*u^{-1}\mathbf{t} \geq 0$.*

(iv) *If $P \geq 0$ and $p_{ii} = 0$ for some i, $1 \leq i \leq n$, then $p_{ij} = p_{ji} = 0$ for all $j = 1, \ldots, n$.*

We next consider:

Lemma 2.2 ([23]). *Let $M \equiv \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right)$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then,*

$$M \geq 0 \text{ if and only if there exists } W \text{ such that } \begin{cases} A \geq 0, \\ B = AW, \text{ and} \\ C \geq W^*AW. \end{cases}$$

For a $m \times n$ matrix A, the *Moore-Penrose inverse* of A is defined as the unique $n \times m$ matrix $A^\dagger$ satisfying all of the following four conditions:

(i) $AA^\dagger A = A$;

(ii) $A^\dagger AA^\dagger = A^\dagger$;

(iii) $\left(AA^\dagger\right)^* = AA^\dagger$;

(iv) $\left(A^\dagger A\right)^* = A^\dagger A$.

The following result is a variant of Lemma 2.2.

Lemma 2.3 ([7, Lemma 1.2])**.** *Let* $M \equiv \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right)$ *be a finite matrix. Then,* $M \geq 0$ *if and only if the following three conditions hold*:
(i) $A \geq 0$;
(ii) $\operatorname{ran} B \subseteq \operatorname{ran} A$; *and*
(iii) $C \geq B^* A^\dagger B$, *where* $A^\dagger$ *is the Moore-Penrose inverse of* A.

The following result suggests that we could complete $\left(\begin{smallmatrix} A & B \\ C & * \end{smallmatrix}\right)$ as a contraction provided that $\begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} A & C \end{pmatrix}^T$ are contractive, so that it makes some contribution to establish main results.

Lemma 2.4 (cf. [10], [20])**.** *If* $\begin{pmatrix} A & B \end{pmatrix}$ *and* $\begin{pmatrix} A & C \end{pmatrix}^T$ *are contractions, then there exists a matrix* D *such that the matrix* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *is a contraction as well.*

Here, we pose to introduce matrices whose positive semi-definiteness and determinant play an important role in getting our main results. For $-1 \leq a, b, c, d \leq 1$, we let

$$H_{22}(x) := \begin{pmatrix} a & b \\ b & x \end{pmatrix}, H_{23}(x) := \begin{pmatrix} a & b & c \\ b & c & x \end{pmatrix},$$

$$H_{24}(x) := \begin{pmatrix} a & b & c & d \\ b & c & d & x \end{pmatrix}, H_{33}(x) := \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & x \end{pmatrix}, \; H_{32} := \begin{pmatrix} b & c \\ c & d \\ d & e \end{pmatrix}$$

and define a matrix-valued function $P(A) := I - AA^*$, where I is the identity matrix of the same size as AA^*. We also let

$$P_{22}(x) := P(H_{22}(x)), \quad P_{23}(x) := P(H_{23}(x)),$$
$$P_{24}(x) := P(H_{24}(x)), \quad P_{33}(x) := P(H_{33}(x)), \text{ and } R_{23} := P(H_{32}).$$

Then, we investigate some connections among the matrices given above:

Lemma 2.5. *If* $1 - a^2 - b^2 > 0$ *and* $\det P_{22}(c) = 0$, *then* $\det P_{23}(x) \geq 0$ *if and only if* $x = -\frac{bc(a+c)}{1-a^2-b^2}$.

Proof. It is a straightforward calculation from

$$\det P_{22}(c) = 0 \iff \left(1 - a^2 - b^2\right)\left(1 - b^2 - c^2\right) = b^2 (a+c)^2 \tag{2}$$

and

$$\det P_{23}(x) = -(1 - a^2 - b^2)\left(x + \frac{bc(a+c)}{1 - a^2 - b^2}\right)^2. \tag{3}$$

□

Corollary 2.6. *If* $1 - a^2 - b^2 - c^2 > 0$, *then the following holds*:
(i) *for some* $-1 \leq x \leq 1$, *if* $\det P_{23}(x) = 0$, *then* $\det P_{22}(c) \geq 0$;
(ii) *for some* $-1 \leq x \leq 1$, $\det P_{23}(x) > 0$ *if and only if* $\det P_{22}(c) > 0$;
(iii) *if* $\det P_{22}(c) = 0$, *then there is* x *such that* $\det P_{23}(x) = 0$. *Indeed,* $x = -\frac{bc(a+c)}{1-a^2-b^2}$.

For the next auxiliary results, we observe by Lemma 2.3(iii) that

$$\begin{aligned} H_{33}(x) \text{ is a HPC} &\iff f_1(x) := \alpha_1 x^2 + \beta_1 x + \gamma_1 \geq 0; \\ H_{24}(x) \text{ is a HPC} &\iff f_2(x) := \alpha_2 x^2 + \beta_2 x + \gamma_2 \geq 0, \end{aligned}$$

where the coefficients of $f_1(x)$ and $f_2(x)$ are

$\alpha_1 := -\frac{\det P_{22}(c)}{1-a^2-b^2-c^2}$, $\beta_1 := \frac{2\left(-ac^2-b^2c^3+ac^4-2bcd+2b^3cd-2abc^2d-ab^2d^2-cd^2+a^2cd^2\right)}{1-a^2-b^2-c^2}$,
$\gamma_1 := \frac{\left(1-b^2+ac-c^2-d^2\right)^2-\left(a+c-c^3+2bcd-ad^2\right)^2}{1-a^2-b^2-c^2}$,
$\alpha_2 := -\left(1-a^2-b^2-c^2\right)$, $\beta_2 := -2d\,(ab+bc+cd)$, and
$\gamma_2 := \det P_{23}\,(d) - d^2\left(1-b^2-c^2-d^2\right)$, respectively.

Let $\mathcal{S}_+(i)$ and $\mathcal{S}_-(i)$ be the solution sets in $[-1,1]$ of $f_i(x) \geq 0$ and $f_i(x) < 0$, respectively, where $f_i(x) := \alpha_i x^2 + \beta_i x + \gamma_i$; $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$.

Lemma 2.7. *If* $1-a^2-b^2-c^2-d^2 > 0$, *then the following statements are true*:

(i) *the discriminant* $\beta_1^2 - 4\alpha_1\gamma_1$ *of the quadratic equation* $f_1(x)$ *is*

$$\beta_1^2 - 4\alpha_1\gamma_1 = \frac{4\left(\det P_{23}\,(d)\right)^2}{\left(1-a^2-b^2-c^2\right)^2};$$

(ii) *the discriminant* $\beta_2^2 - 4\alpha_2\gamma_2$ *of the quadratic equation* $f_2(x)$ *is*

$$\beta_2^2 - 4\alpha_2\gamma_2 = 4\left(1-a^2-b^2-c^2-d^2\right)\det P_{23}\,(d)\,.$$

Proof. It is a straightforward calculation. □

Furthermore, if $1-a^2-b^2-c^2-d^2 > 0$, then for $i = 1, 2$, we have

$$\mathcal{S}_+\,(i) \neq \emptyset \Longrightarrow \det P_{23}\,(d) \geq 0. \tag{4}$$

We conclude this section with introducing a helpful tool used in the proof of main results:

Lemma 2.8. *Problem 1.1 is soluble for* $H_4 \equiv H_4(a,b,c,d;x,y,z)$ *if and only if there exists* x *satisfying both inequalities*

$$\|H_{24}\,(x)\| \leq 1 \quad \text{and} \quad \|H_{33}\,(x)\| \leq 1. \tag{5}$$

Proof. It is clear from Lemma 2.4. □

3. Partially contractive Hankel matrices of extremal type: The 4 × 4 case

Since $\|S\| \leq \|T\|$, if S is a submatrix of the matrix T with $\|T\| \leq 1$, then S is again a contraction. Thus, a necessary condition for a partial matrix T to be a contraction is that each submatrix must be a contraction. We call a partial matrix meeting this necessary condition a *partial contraction* (or *well-posed condition*).

In this section, we improve the main results in [8]; Theorem 3.2 given below covers the results in [8, Theorems 4.2, 4.3, and 4.4] at a time. We need to introduce another matrices to establish Theorem 3.2; let

$$H_{34}(x,y) := \begin{pmatrix} a & b & c & d \\ b & c & d & x \\ c & d & x & y \end{pmatrix} \quad \text{and} \quad H_{43}(x,y) := \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & x \\ d & x & y \end{pmatrix}.$$

Also, let $C_1(a,b,c,d) := (a+c)(b+d)(ad - ab - bc - cd)$ and $C_2(a,b,c,d) := |(a+c)(b+d)| - |ac+bd|$. We next present more concrete conditions for the solubility of H_4 according to the values of d:

Theorem 3.1 ([8]). *Assume $d = 0$. Then, Problem 1.1 is soluble for H_4 if and only if*

$$b(a+c) = 0.$$

We also have:

Theorem 3.2. *Assume $d \neq 0$. Then, Problem 1.1 is soluble for H_4 if and only if the following two conditions hold*:
(i) $C_1(a,b,c,d) \geq 0$ *and*
(ii) $C_2(a,b,c,d) \geq 0$.

Proof. We assume that Problem 1.1 is soluble for H_4. Then, by Lemma 2.2 and (5), we have

$$\|H_{33}(x)\| \leq 1 \text{ if and only if } C_1(a,b,c,d) \geq 0 \text{ and } C_2(a,b,c,d) \geq 0. \tag{6}$$

On the other hand, we assume $C_1(a,b,c,d) \geq 0$ and $C_2(a,b,c,d) \geq 0$. By Lemma 2.3, we note

$$\begin{aligned} \|H_{24}(x)\| \leq 1 &\iff P_{24}(x) \geq 0 \\ &\iff x = \frac{ab+bc+cd}{d} \text{ and } C_1(a,b,c,d) \geq 0. \end{aligned} \tag{7}$$

By (5), (7), and (6), we observe that if we choose x such that

$$|x| = \left|\frac{ab+bc+cd}{d}\right| \leq |a|,$$

then $\|H_{24}(x)\| \leq 1$ and $\|H_{33}(x)\| \leq 1$, simultaneously. Thus, by Lemma 2.8, Problem 1.1 is soluble for H_4, as desired. ☐

4. Partially cntractive Hankel matrices of non-extremal type: The 4 × 4 case

We now pay attention to the non-extremal type for 4×4 Hankel matrices of the form $H_4 \equiv H_4(a,b,c,d;x,y,z)$ (that is, when $a^2 + b^2 + c^2 + d^2 < 1$). Consider the solubility of Problem 1.1 for a Hankel matrix H_4 which is well-posed. For $i = 1, 2$, suppose $\alpha_i < 0$ and $\beta_i, \gamma_i \in \mathbb{R}$. We recall that $\mathcal{S}_+(i)$

(resp. $\mathcal{S}_-(i)) \subseteq [-1,1]$ is the solution set of the quadratic inequality equation $f_i(x) = \alpha_i x^2 + \beta_i x + \gamma_i \geq 0$ (resp. $f_i(x) < 0$). We next let

$$k_1 := \tfrac{-c(ac+bd)-\left(1-a^2-b^2-c^2\right)}{1-a^2-b^2} \quad \text{and} \quad k_2 := \tfrac{-c(ac+bd)+\left(1-a^2-b^2-c^2\right)}{1-a^2-b^2}.$$

Lemma 4.1. *Assume that $H_{33}(x)$ is well-posed, $a^2+b^2+c^2 < 1$, and $\det P_{22}(c) = 0$. Then, $H_{33}(x)$ is contractive for $k_1 \leq x \leq k_2$.*

Proof. After applying Lemma 2.5, Corollary 2.6, and Theorem 3.2 in [8] to $H_{33}(x)$, we get the desired results. □

Here, we present one of the main results using the classification according to the values of $\det P_{22}(c)$:

Theorem 4.2. *Assume $\det P_{22}(c) = 0$. Then, Problem 1.1 is soluble for H_4 if and only if*

$$k_1 \leq \frac{2b^2 c\,(a+c)^2}{(1-a^2-b^2)^2} \leq k_2.$$

Proof. ($\Longrightarrow$) Suppose that Problem 1.1 is soluble for H_4. Then, by (5) and Lemma 4.1, we have that $H_{33}(x)$ admits a contractive completion and, in particular, x is given by

$$k_1 \leq x \leq k_2. \tag{8}$$

Since $a^2+b^2+c^2+d^2 < 1$, it follows from (5) and Lemma 2.7 that

$$\|H_{24}(x)\| \leq 1 \Longleftrightarrow \det P_{24}(x) \geq 0 \Longleftrightarrow \mathcal{S}_+(2) \neq \emptyset. \tag{9}$$

Since $\det P_{22}(c) = 0$, by Lemma 2.5 and Lemma 4.1, we obtain $d = -\frac{bc(a+c)}{1-a^2-b^2}$. Note

$$\beta_2^2 - 4\alpha_2\gamma_2 = 4\left(1-a^2-b^2-c^2-d^2\right)\det P_{23}(d) = 0,$$

which leads to

$$\mathcal{S}_+(2) \neq \emptyset \Longleftrightarrow x = \frac{2b^2 c\,(a+c)^2}{(1-a^2-b^2)^2}. \tag{10}$$

Therefore, by (8), (9), and (10), H_4 admits a contractive completion only if

$$k_1 \leq \frac{2b^2 c\,(a+c)^2}{(1-a^2-b^2)^2} \leq k_2,$$

as desired.

($\Longleftarrow$) Suppose that $k_1 \leq \frac{2b^2c(a+c)^2}{(1-a^2-b^2)^2} \leq k_2$. Put $x := \frac{2b^2c(a+c)^2}{(1-a^2-b^2)^2}$. Then, the proof of Lemma 4.1 allows us to see

$$\|H_{33}(x)\| \leq 1. \tag{11}$$

By (10), we also have $\mathcal{S}_+(2) \neq \emptyset$. Next, by (9), we also obtain

$$\|H_{24}(x)\| \leq 1. \tag{12}$$

Therefore, by (5), (11), and (12), we know that H_4 has a contractive completion. □

We next have:

Theorem 4.3. *Assume* $\det P_{22}(c) > 0$. *Then, Problem 1.1 is soluble for* H_4 *if and only if*

$$\mathcal{S}_+(1) \cap \mathcal{S}_+(2) \neq \emptyset.$$

Proof. ($\Longrightarrow$) Suppose that Problem 1.1 is soluble for H_4. Then, by (5), we have

$$\begin{aligned} \|H_{33}(x)\| \leq 1 &\iff P_3(H_{33}(x)) \geq 0 \\ &\iff \det P_{23}(d) \geq 0 \quad \text{and} \quad \det P_3(H_{33}(x)) \geq 0, \end{aligned} \tag{13}$$

which implies $\mathcal{S}_+(1) \neq \emptyset$. Since $a^2 + b^2 + c^2 + d^2 < 1$, by Lemma 2.7, we observe that the discriminant $\beta_2^2 - 4\alpha_2\gamma_2$ of the quadratic equation $f_2(x)$ is not negative. Thus, we get

$$\|H_{24}(x)\| \leq 1 \Longleftrightarrow \mathcal{S}_+(2) \neq \emptyset. \tag{14}$$

Therefore, by (13) and (14), we have $\mathcal{S}_+(1) \cap \mathcal{S}_+(2) \neq \emptyset$, as desired.

($\Longleftarrow$) Suppose that $\mathcal{S}_+(1) \cap \mathcal{S}_+(2) \neq \emptyset$. Then, both sets $\mathcal{S}_+(1)$ and $\mathcal{S}_+(2)$ are non-empty. Since $a^2+b^2+c^2+d^2 < 1$, it follows from (4) that $\det P_{23}(d) \geq 0$.

By Lemma 2.7 again, we can see that the discriminant of the quadratic equation $f_1(x)$ is nonnegative. That is,

$$\beta_1^2 - 4\alpha_1\gamma_1 = \frac{4\left(\det P_{23}(d)\right)^2}{\left(1 - a^2 - b^2 - c^2\right)^2} \geq 0.$$

Due to $\det P_3(H_{33}(x)) = f_1(x)$, we know

$$\det P_3(H_{33}(x)) \geq 0 \Longleftrightarrow \mathcal{S}_+(1) \neq \emptyset. \tag{15}$$

Since $\mathcal{S}_+(2) \neq \emptyset$, (14) leads us to see

$$\|H_{24}(x)\| \leq 1. \tag{16}$$

Therefore, by (5), (14), (15), and (16), we conclude that H_4 admits a contractive completion. $\square$

5. Partially contractive Hankel matrices of extremal type: The 5 × 5 case

In this section, we focus on the extremal case for $H_5 = H(a, b, c, d, e; x, y, z, w)$. Our approach requires that we split the analysis into two cases ($e = 0$ and $e > 0$), because we get a similar result using the repeated calculations in the proofs of Theorems 5.2, 5.3, 5.4, and 5.5 given below for the case $e < 0$. Consider the solubility of Problem 1.1 for a Hankel matrix H_5, which is well-posed. By Lemma 2.4 and Lemma 2.8, we first observe that Problem 1.1 is soluble for H_5 if and only if there exist x and y such that we simultaneously have

$$\|H_{25}(x)\| \leq 1, \ \|H_{34}(x)\| \leq 1, \ \|H_{35}(x, y)\| \leq 1, \text{ and } \ \|H_{44}(x, y)\| \leq 1, \tag{17}$$

where

$$H_{25}(x) := \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & x \end{pmatrix}, \; H_{34}(x) := \begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & x \end{pmatrix},$$

$$H_{35}(x,y) := \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & x \\ c & d & e & x & y \end{pmatrix}, \text{ and } H_{44}(x,y) := \begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & x \\ d & e & x & y \end{pmatrix}.$$

To obtain our results, we let

$$P_{25}(x) := P(H_{25}(x)), \quad P_{34}(x) := P(H_{34}(x)),$$
$$P_{35}(x) := P(H_{35}(x)), \quad \text{and } P_{44}(x) := P(H_{44}(x)).$$

We have relied heavily on the Nested Determinant Test so far for checking positivity of matrices; however, we need to use different approaches in this section: For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, where $(A \circ B)_{i,j} := (A)_{i,j} (B)_{i,j}$ for $1 \le i, j \le n$. The following result is well known: If $A \ge 0$ and $B \ge 0$, then $A \circ B \ge 0$ [21]. Recall that two matrices $A, B \in M_n(\mathbb{R})$ are called *congruent* if there exists an invertible matrix $Q \in M_n(\mathbb{R})$ such that $B = Q^T A Q$. The following result is also well known: $A \ge 0 \Longleftrightarrow Q^T A Q \ge 0$ [15]; the facts will be used to prove Theorem 5.5.

We are ready to consider the first case:

The case $e = 0$. Using the Nested Determinants Test in Lemma 2.1 and eliminating the common factors in matices $P_{25}(x)$, $P_{34}(x)$, $P_{35}(x,y)$ and $P_{44}(x,y)$, respectively, we can show

$$\|H_{25}(x)\| \le 1 \iff \{|x| \le |a| \text{ and } ab + bc + cd = 0 \;, \tag{18}$$

$$\|H_{34}(x)\| \le 1 \iff \begin{cases} ab + bc + cd = ac + bd + dx = 0 \text{ and} \\ A(x) := \begin{pmatrix} a^2 & ab \\ ab & a^2 + b^2 - x^2 \end{pmatrix} \ge 0, \end{cases} \tag{19}$$

(20)

$$\|H_{35}(x,y)\| \le 1 \iff \begin{cases} ab + bc + cd = ac + bd + dx = 0 \text{ and} \\ B(x,y) := \begin{pmatrix} a^2 - x^2 & ab - xy \\ ab - xy & a^2 + b^2 - x^2 - y^2 \end{pmatrix} \ge 0, \end{cases}$$

and

$$\|H_{44}(x,y)\| \le 1 \tag{21}$$

$$\iff \begin{cases} ac + bd + dx = ad + cx + dy = 0 \text{ and} \\ C(x,y) := \begin{pmatrix} a^2 & -c(b+d) & ac \\ -c(b+d) & a^2 + b^2 - x^2 & -cd - xy \\ ac & -cd - xy & 1 - d^2 - x^2 - y^2 \end{pmatrix} \ge 0. \end{cases}$$

Then, we have:

Theorem 5.1. *Assume $e = 0$. Then, Problem 1.1 is soluble for H_5 if and only if one of the following two conditions holds*:

(i) $d = 0$ *and* $ac = b(a + c) = 0$;

(ii) $d \neq 0$, $ab + bc + cd = 0$, *and* $|ac + bd| \leq |ad|$.

Proof. ($\Longrightarrow$) We assume that Problem 1.1 is soluble for H_5. Then, by (18), we see at once $ab + bc + cd = 0$ and $|x| \leq |a|$.

Subcase $d = 0$. By (19), we have $ac = 0$.

If $a = c = 0$, then we have $b(a + c) = 0$.

If $a = 0$ and $c \neq 0$, then by (20), $b = 0$, so that $b(a + c) = 0$.

If $a \neq 0$ and $c = 0$, then by (20) again, $b = 0$. Thus, we have $b(a + c) = 0$. Therefore, we have the desired conditions.

Subcase $d \neq 0$. By (19), we must choose $x = \frac{-ac-bd}{d}$. Thus, we have $ab + bc + cd = 0$ and $A\left(\frac{-ac-bd}{d}\right) \geq 0 \Longleftrightarrow |ac + bd| \leq |ad|$.

Since $x = \frac{-ac-bd}{d}$, by (21), we have $y = \frac{ac^2+bcd-ad^2}{d^2}$. Hence, we have the desired conditions.

($\Longleftarrow$) Suppose that the condition (i) holds. For any x with $|x| \leq |a|$, the conditions (18) and (19) are satisfied. If $a = c = 0$, then by (20), we get $x = 0$ and $|y| = 1$. Hence, direct calculations show $\|H_{35}(0, \pm 1)\| \leq 1$ and $\|H_{44}(0, \pm 1)\| \leq 1$.

If $a = 0$ and $c \neq 0$, then by (19) and (20), we have $x = y = 0$ which show $\|H_{35}(0, 0)\| \leq 1$ and$\|H_{44}(0, 0)\| \leq 1$.

If $a \neq 0$ and $c = 0$, then by (20) and (21), we have $|x| \leq 1$ and $|y| \leq \left(1 - x^2\right)$ which hold $B(x, y) \geq 0$ and $C(x, y) \geq 0$, simultaneously. Therefore, we have $\|H_{35}(x, y)\| \leq 1$ and $\|H_{44}(x, y)\| \leq 1$. In each case given above, Problem 1.1 is soluble for H_5.

Suppose that the condition (ii) holds. Put $x = \frac{-ac-bd}{d}$. Then, we obtain $\left\|H_{25}\left(\frac{-ac-bd}{d}\right)\right\| \leq 1$ and $\left\|H_{34}\left(\frac{-ac-bd}{d}\right)\right\| \leq 1$. We also put $y = \frac{ac^2+bcd-ad^2}{d^2}$. Then, the Nested Determinants Test in Lemma 2.1 implies

$$\left\|H_{35}\left(\frac{-ac-bd}{d}, \frac{ac^2+bcd-ad^2}{d^2}\right)\right\| \leq 1 \text{ and } \left\|H_{44}\left(\frac{-ac-bd}{d}, \frac{ac^2+bcd-ad^2}{d^2}\right)\right\| \leq 1,$$

because $\det B\left(\frac{-ac-bd}{d}, \frac{ac^2+bcd-ad^2}{d^2}\right) = \det C\left(\frac{-ac-bd}{d}, \frac{ac^2+bcd-ad^2}{d^2}\right) = 0$. Thus, the conditions (18), (19), (20), and (21) hold. Therefore, by (17), Problem 1.1 is also soluble for H_5, so that our proof is complete. $\square$

The case $e > 0$. Direct calculations (i.e., the Nested Determinants Test in Lemma 2.1 and eliminating the common factors in matices $P_{25}(x)$, $P_{34}(x)$, $P_{35}(x, y)$ and $P_{44}(x, y)$) imply

$$\|H_{25}(x)\| \leq 1 \iff x = -\frac{ab+bc+cd+de}{e} \quad \text{and} \quad |x| \leq |a|, \tag{22}$$

$$\|H_{34}(x)\| \leq 1 \tag{23}$$

$$\iff \begin{pmatrix} \frac{a^2e^2-(ab+bc+cd+de)^2}{e^2} & m(x) \\ m(x) & \frac{(a^2+b^2-x^2)e^2-(ac+bd+ce+dx)^2}{e^2} \end{pmatrix} \geq 0,$$

(24) $$\|H_{35}(x,y)\| \leq 1$$
$$\iff \begin{cases} x = -\frac{ab+bc+cd+de}{e}, \ y = \frac{abd+bcd+cd^2+d^2e-ace-bde-ce^2}{e^2}, \\ \begin{pmatrix} a^2-x^2 & ab-xy \\ ab-xy & a^2+b^2-x^2-y^2 \end{pmatrix} \geq 0, \end{cases}$$

and

(25) $$\|H_{44}(x,y)\| \leq 1 \iff M := \begin{pmatrix} f(x) & g(x,y) & h(x,y) \\ g(x,y) & j(x,y) & k(x,y) \\ h(x,y) & k(x,y) & \ell(x,y) \end{pmatrix} \geq 0,$$

where

$$m(x) := \frac{e^2(bc+cd+de+ex)-(ab+bc+cd+de)(ac+bd+ce+dx)}{e^2},$$

$$f(x) := a^2-x^2,\ g(x,y) := ab-xy,\ j(x,y) := a^2+b^2-x^2-y^2$$

$$h(x,y) := \frac{ace+adx+bex+dxy}{e},\ k(x,y) := \frac{abe+bce+ady+bey+cxy-exy+dy^2}{e}, \text{ and}$$

$$\ell(x,y) := \frac{a^2e^2-a^2d^2+c^2e^2-abde-2c(ad+be)x-(c^2+e^2)x^2-2d(ad+be+cx)y-(d^2+e^2)y^2}{e^2}.$$

Then, we have:

Theorem 5.2. *Assume $e > 0$ and $a = 0$. Then, Problem 1.1 is soluble for H_5 if and only if the following two conditions hold*:
(i) $bc + cd + de = 0$ *and*
(ii) $|bd + ce| \leq |be|$.

Proof. ($\Longrightarrow$) We suppose that Problem 1.1 is soluble for H_5. Since $a = 0$, by (22), we have $x = 0$ and $bc + cd + de = 0$. By (23), we note the following:

(26) $$\|H_{34}(x)\| \leq 1 \iff |bd + ce| \leq |be|.$$

Therefore, we have two desired conditions.

($\Longleftarrow$) Suppose that the two conditions $bc + cd + de = 0$ and $|bd + ce| \leq |be|$ hold. Put $x = 0$. Then, by (26), we have $\|H_{34}(0)\| \leq 1$. Put $y = -\frac{bd+ce}{e}$. Then, a direct calculation shows

$$\left\|H_{35}\left(0, -\tfrac{bd+ce}{e}\right)\right\| \leq 1.$$

If $c = 0$, then $d = y = 0$ and by (25), we have $\|H_{44}(0,0)\| \leq 1$.

If $c \neq 0$, then $b = \frac{-d(c+e)}{c}$, so $\left\|H_{44}\left(0, -\frac{bd+ce}{e}\right)\right\| \leq 1$. Thus, by the analysis given above, the conditions (22), (23), (24), and (25) hold. Therefore, by (17), Problem 1.1 is soluble for H_5. $\square$

We define $s := ab + bc + cd + de + ea$ and this notation is to be used in the next three theorems. Note

$$\|H_{25}(x)\| \leq 1 \iff x = -\frac{s - ea}{e} \text{ and } |x| \leq |a|.$$

Then, we have:

Theorem 5.3. *Assume $e > 0$, $a \neq 0$, and $s = 0$. Then, Problem 1.1 is soluble for H_5 if and only if the following three conditions hold*:

(i) $a + d \neq 0$;
(ii) $b = c$;
(iii) $ab + bd + da = 0$.

Proof. ($\Longrightarrow$) We suppose that Problem 1.1 is soluble for H_5. By (22), we must choose $x = a$. By (23), we have $be = ac + bd + ce + da$. By (24), we must choose $y = b$.

If $a + d = 0$, then $ab + bc + cd = 0 \iff ac = b(a + c)$. By (25), we have $ac = bc$. Since $ac = b(a + c)$ and $a \neq 0$, we have $b = c = 0$ which implies $a = 0$. Note that our assumption is $a \neq 0$, so that $a = 0$ contradicts our assumption. Therefore, $a = -d$ case does not occur.

If $a + d \neq 0$. Then, by (25) we have $b = c$ which implies $be = ac + bd + ce + da \iff ab + bd + da = 0$. Thus, we have three desired conditions.

($\Longleftarrow$) Suppose that the conditions (i), (ii), (iii) hold. Choose $x = a$ and $y = b$. Since $e > 0$ and $s = 0$, by the Nested Determinants Test in Lemma 2.1, we have $\|H_{25}(a)\| \leq 1$, $\|H_{34}(a)\| \leq 1$, $\|H_{35}(a, b)\| \leq 1$, and $\|H_{44}(a, b)\| \leq 1$. Thus, the conditions (22), (23), (24), and (25) hold. Therefore, by (17), Problem 1.1 is soluble for H_5. $\square$

Theorem 5.4. *Assume $e > 0$, $a \neq 0$, and $s = 2ea$. Then, Problem 1.1 is soluble for H_5 if and only if one of the following three conditions hold*:

(i) $a - d \neq 0$;
(ii) $b = 0$;
(iii) $ad = c(a + e)$.

Proof. ($\Longrightarrow$) We suppose that Problem 1.1 is soluble for H_5. Then, by (22), we must choose $x = -a$.

If $a = d$, then we have $ab + bc + cd = 0$. Thus, by (23), we obtain $a^2 = ab + ac + be + ce$. By (24), we must choose $y = b$ and obtain $a^2 = ab + ac - be + ce$. Thus, we have $b = 0$ which implies $cd = 0$ and $a^2 = c(a + e)$. Since $a \neq 0$, that is, $d \neq 0$, $cd = 0 \Longrightarrow c = 0$ which means $a = 0$. Note that our assumption is $a \neq 0$, so that $a = 0$ contradicts our assumption. Therefore, $a = d$ case does not occur.

If $a \neq -d$, then, by (23), we have $ad = ac + bd - be + ce$. By (24), we must choose $y = b$ and have $ad = ac + bd + be + ce$. Thus, we have $b = 0$ which implies $ad = c(a + e)$.

($\Longleftarrow$) Suppose that the conditions (i), (ii), (iii) hold. Choose $x = -a$ and $y = b$. Then, direct calculations show that $H_{25}(-a) = \mathbf{0}$ and $H_{34}(-a) = \mathbf{0}$, where $\mathbf{0}$ is a zero matrix. Since $e > 0$ and $s = 2ea$, we have $\|H_{35}(-a, b)\| \le 1$. Since $a \ne 0$, we note $d = \frac{c(a+e)}{a}$ and $s - 2ea = 0 \Longrightarrow \frac{ac^2 - a^2 e + ace + c^2 e + ce^2}{a} = 0$. By the Nested Determinants Test in Lemma 2.1, we obtain $\|H_{44}(-a, b)\| \le 1$. Therefore, the conditions (22), (23), (24), and (25) also hold. Hence, by (17), Problem 1.1 is soluble for H_5. $\square$

For the next result, let $t := ac + ad + bd + be + ce$ and $v := ac - ad + 2ae + bd - be + ce - s$. We also let $w_1(s) := s^2 - (ad + 2ae + be)s + aet$ and $w_2(s) := s^2 + (ad - 2ae + be)s - aet$, where $s = ab + bc + cd + de + ea$. Then, we have:

Theorem 5.5. *Assume $e > 0$, $a \ne 0$, $s \ne 0$, and $s \ne 2ea$. Then, Problem 1.1 is soluble for H_5 if and only if the following three conditions hold*:
(i) $s(2ae - s) > 0$;
(ii) $w_1(s) w_2(s) \ge 0$;
(iii) $v(s + t) \ge 0$.

Proof. ($\Longrightarrow$) Suppose that Problem 1.1 is soluble for H_5. By (22) and the assumption given above, we must choose $x = \frac{ae-s}{e}$. Note

$$\|H_{34}(x)\| \le 1 \iff \begin{pmatrix} \frac{1}{e^2} & \frac{1}{e^3} \\ \frac{1}{e^3} & \frac{1}{e^4} \end{pmatrix} \circ L(x) \ge 0, \tag{27}$$

where $\circ$ means the Schur product,

$$L(x) := \begin{pmatrix} \ell_{11} & \ell_{12}(x) \\ \ell_{21}(x) & \ell_{22}(x) \end{pmatrix}, \ \ell_{11} := s(2ae - s),$$

$$\ell_{12}(x) = \ell_{21}(x) := ae(t - ad) - (ac + bd + ce)s + (ae - s)dx, \text{ and}$$

$$\begin{aligned} \ell_{22}(x) := &-(d^2 + e^2)x^2 - 2(ac + bd + ce)dx \\ &-a^2c^2 - 2abcd - b^2d^2 - 2ac^2e - 2bcde + a^2e^2 + b^2e^2 - c^2e^2. \end{aligned}$$

Since $\begin{pmatrix} \frac{1}{e^2} & \frac{1}{e^3} \\ \frac{1}{e^3} & \frac{1}{e^4} \end{pmatrix} \ge 0$, we get $\|H_{34}(x)\| \le 1 \Longleftrightarrow L(x) \ge 0$. A direct calculation shows that

$$\begin{aligned} &\left\|H_{34}\left(\tfrac{ea-s}{e}\right)\right\| \le 1 \\ \iff &L\left(\frac{ea-s}{e}\right) \ge 0 \\ \iff &(2ae - s)s \ge 0 \ \text{ and } \ \det L\left(\tfrac{ea-s}{e}\right) = w_1(s)\, w_2(s) \ge 0. \end{aligned} \tag{28}$$

By (24), we must choose $y = -\frac{ace+ade+bde+ce^2-ds}{e^2}$. Note

$$\left\|H_{35}\left(\tfrac{ea-s}{e}, -\tfrac{ace+ade+bde+ce^2-ds}{e^2}\right)\right\| \le 1 \iff w_1(s)\, w_2(s) \ge 0. \tag{29}$$

Thus, we have

$$
\begin{aligned}
& \left\| H_{44}\left(x,y\right)\right\| \leq 1 \\
(30) \qquad \Longleftrightarrow & \begin{pmatrix} M_{11} & M_{12}\left(x,y\right) \\ M_{21}\left(x,y\right) & M_{22}\left(x,y\right) \end{pmatrix} \geq 0 \\
\Longleftrightarrow & \; Z\left(x,y\right) := M_{22}\left(x,y\right) - M_{21}\left(x,y\right)\left(M_{11}\right)^{\dagger} M_{12}\left(x,y\right) \geq 0,
\end{aligned}
$$

where $\left(M_{11}\right)^{\dagger}$ is the Moore-Penrose inverse of M_{11},

$$M_{11} := \begin{pmatrix} 1-a^2-b^2-c^2-d^2 & -ab-bc-cd-de \\ -ab-bc-cd-de & 1-b^2-c^2-d^2-e^2 \end{pmatrix},$$

$$M_{12}\left(x,y\right) = M_{21}\left(x,y\right) := \begin{pmatrix} -ac-bd-ce-dx & -ad-be-cx-dy \\ -ad-be-cx-dy & -bd-ce-dx-ey \end{pmatrix},$$

and $M_{22}\left(x,y\right) := \begin{pmatrix} 1-c^2-d^2-e^2-x^2 & -cd-de-ex-xy \\ -cd-de-ex-xy & 1-d^2-e^2-x^2-y^2 \end{pmatrix}.$

Now, we consider a congruence for the positivity of two matrices:

$$
\begin{aligned}
(31) \qquad & Z\left(\tfrac{ea-s}{e}, -\tfrac{ace+ade+bde+ce^2-ds}{e^2}\right) \geq 0 \\
\Longleftrightarrow & \; Q^T Z\left(\tfrac{ea-s}{e}, -\tfrac{ace+ade+bde+ce^2-ds}{e^2}\right) Q \geq 0,
\end{aligned}
$$

where $Q := \left(\begin{smallmatrix} e & 0 \\ 0 & e^2 \end{smallmatrix}\right)$. Hence, by (31), we obtain

$$
\begin{aligned}
& \left\| H_{44}\left(\tfrac{ea-s}{e}, -\tfrac{ace+ade+bde+ce^2-ds}{e^2}\right)\right\| \leq 1 \\
\Longleftrightarrow & \; Q^T Z\left(\tfrac{ea-s}{e}, -\tfrac{ace+ade+bde+ce^2-ds}{e^2}\right) Q \geq 0 \\
(32) \qquad \Longleftrightarrow & \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \geq 0 \\
\Longleftrightarrow & \; \tfrac{w_1(s)w_2(s)}{s(2ae-s)} \geq 0 \;\text{ and }\; s\left(2ae-s\right)v(s+t) \geq 0,
\end{aligned}
$$

where

$n_{11} := \frac{w_1(s)w_2(s)}{s(2ae-s)}$, $n_{12} = n_{21} := -\frac{w_1(s)\left(ds^2+\left(ad^2-ace-2ade-ce^2\right)s+a(e-d)et\right)}{s(2ae-s)}$, and
$n_{22} := \frac{w_1(s)\left(\left(d^2+e^2\right)s^2-\left(de(2ac+2ad+bd-ae+2ce)+(2a-b)e^3-ad^3\right)s-aet(e-d)^2\right)}{(2ae-s)s}.$

Therefore, (28), (29), and (32) satisfy the conditions (i), (ii), (iii).

($\Longleftarrow$) Suppose that all of the following three conditions (i), (ii), (iii) hold. We first choose

$$x = \frac{ea-s}{e} \quad \text{and} \quad y = -\frac{ace+ade+bde+ce^2-ds}{e^2}.$$

Then, by (27), (28), and (29), $\left\|H_{25}\left(x\right)\right\| \leq 1$, $\left\|H_{34}\left(x\right)\right\| \leq 1$, and $\left\|H_{35}\left(x,y\right)\right\| \leq 1$ are satisfied. By (32), $\left\|H_{44}\left(x,y\right)\right\| \leq 1$ also holds. Therefore, by (17), we have that Problem 1.1 is soluble for H_5, as desired. $\square$

6. Applications: examples and an answer to a conjecture

It is well known that Problem 1.1 is always soluble for $H_3 \equiv H_3(a,b,c;x,y)$, and that there exist real numbers a,b,c,d such that $H_4 \equiv H_4(a,b,c,d;x,y,z)$ is partially contractive but not contractive for all choices of x,y,z; in both instances, the results are theoretical in nature [18]. In [8, Example 5.2], the authors give concrete real numbers a,b,c,d such that H_4 is the extremal type and partially contractive but not contractive for any choices of x,y,z. In this section, we provide concrete examples in both extremal and non-extremal types for $H_4(a,b,c,d;x,y,z)$ and $H_5(a,b,c,d,e;x,y,z,w)$ which are not soluble for any x,y,z,w.

Example 6.1. For $a \in \left(0, \frac{1}{2}\right)$, we let $(a,b,c,d) = \left(a, \sqrt{\frac{1}{2}}, \sqrt{\frac{1-2a^2}{2}}, 0\right)$. Then, we have

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Furthermore, we can see that H_4 is partially contractive but not contractive for any choices of x,y,z.

Proof. By Theorem 3.1, $a^2 + b^2 + c^2 + d^2 = 1$ and $b(a+c) = \frac{2a+\sqrt{2-4a^2}}{2\sqrt{2}} \neq 0$ imply that H_4 is not soluble for any x,y,z. $\square$

Example 6.2. For $a \in \left(0, \frac{1}{3}\right)$, we let $(a,b,c,d) = \left(a, \sqrt{\frac{1}{2}}, \sqrt{\frac{1-3a^2}{2}}, 0\right)$. Then, we have

$$a^2 + b^2 + c^2 + d^2 < 1.$$

Furthermore, we can have that H_4 is partially contractive but not contractive for any choices of x,y,z.

Proof. Note $a^2 + b^2 + c^2 + d^2 < 1$, $\det P_{22}(c) > 0$, and $\det P_{23}(0) < 0$ on $\left(0, \sqrt{\frac{1}{3}}\right)$. Thus, Lemma 4 and Theorem 4.3 imply that H_4 is not soluble for any x,y,z. $\square$

It is about to introduce concrete examples for $H_5 \equiv H_5(a,b,c,d,e;x,y,z,w)$ which is not soluble for any x,y,z,w. We first consider the extremal type for H_5.

Example 6.3. For $b \in (0,1)$, we let $(a,b,c,d,e) = \left(0, b, \sqrt{1-b^2}, 0, 0\right)$. Then, we have

$$a^2 + b^2 + c^2 + d^2 + e^2 = 1.$$

Furthermore, we can have that H_5 is partially contractive but not contractive for any choices of x,y,z,w.

Proof. Since $(a,b,c,d,e) = \left(0, b, \sqrt{1-c^2}, 0, 0\right)$, it is clear that $a^2 + b^2 + c^2 + d^2 + e^2 = 1$. A direct calculation shows $ab + bc > 0$, so that by Theorem 4.3, H_5 is not soluble for any x,y,z,w. $\square$

We next consider the non-extremal type for H_5. For this, we let $\rho := a^2 + b^2 + c^2 + d^2 + e^2 < 1$ and $\sigma := ac + bd + ce$. Also, let

$$\begin{aligned}
&\alpha_5 := \rho - 1 - e^2, \qquad \alpha_6 := -\det P_{23}(d), \qquad \beta_5 := 2e(ea - s),\\
&\beta_6 := 2d\sigma\left(\rho - 1 - a^2\right) - 2abe\left(\rho - 1 - e^2\right),\\
&\gamma_5 := e^2(\rho - 1 - a^2) + \det P_{24}(e), \text{ and}\\
&\gamma_6 := \sigma^2\left(\rho - 1 - a^2\right) - \left(\rho - 1 - e^2\right)\left(\det R_{23} + e\left(1 - c^2 - d - e^2\right)\right).
\end{aligned}$$

Recall the algebraic set $\mathcal{S}_+(i)$ and the function $f_i(x)$ from Section 2. Then, we have:

Theorem 6.4. *Let $s - ea = ab + bc + cd + de = 0$. Then, Problem 1.1 is soluble for H_5 only if $\mathcal{S}_+(5) \cap \mathcal{S}_+(6) \neq \emptyset$.*

Proof. Since $s - ea = 0$, Lemma 2.2 implies

$$\begin{aligned}
&\left\|H_{25}(x)\right\| \leq 1\\
\Longleftrightarrow\ & f_5(x) \geq 0 \qquad\qquad (33)\\
\Longleftrightarrow\ & \frac{-\sqrt{(1-\rho)\det P_{24}(e)}}{e^2 - \rho} \leq x \leq \frac{\sqrt{(1-\rho)\det P_{24}(e)}}{e^2 - \rho}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|H_{34}(x)\right\| \leq 1\\
\Longleftrightarrow\ & f_6(x) \geq 0 \qquad\qquad (34)\\
\Longleftrightarrow\ & \frac{-\beta_6 - \sqrt{\beta_6^2 - 4\alpha_6\gamma_6}}{\alpha_6} \leq x \leq \frac{-\beta_6 + \sqrt{\beta_6^2 - 4\alpha_6\gamma_6}}{\alpha_6},
\end{aligned}$$

where $\beta_6^2 - 4\alpha_6\gamma_6 \geq 0$. Thus, by (33) and (34), we have that Problem 1.1 is soluble for H_4 only if $\mathcal{S}_+(5) \cap \mathcal{S}_+(6) \neq \emptyset$. □

Next, we have:

Example 6.5. For $a \in \left(0, \frac{1}{5}\right)$, we let $(a, b, c, d, e) = \left(a, 0, \frac{3}{4}, 0, \frac{1}{10}\right)$. Then, we have

$$a^2 + b^2 + c^2 + d^2 + e^2 < 1.$$

Furthermore, we can see that H_5 is partially contractive but not contractive for any choices of x, y, z, w.

Proof. Since $(a, b, c, d, e) = \left(a, 0, \frac{3}{4}, 0, \frac{1}{10}\right)$ and $a \in \left(0, \frac{1}{5}\right)$, it is clear that $a^2 + b^2 + c^2 + d^2 + e^2 < 1$. A direct calculation shows $\det P_{24}(e) < 0$, so that by (33), we get $\mathcal{S}_+(5) = \emptyset$. Thus, by Theorem 6.4, H_5 is not soluble for any x, y, z, w. □

We now conclude this section with giving a negative answer to the conjecture presented in [8, Remark 4.5]; the authors expected that the solution set of Problem 1.1,

$$\left\{(x, y, z) \in \mathbb{R}^3 : H_4 \equiv H_4(a, b, c, d; x, y, z) \text{ is contractive}\right\}, \qquad (35)$$

is a prism in $\mathbb{R}^3$ when (a, b, c, d) is not extremal. However, we can show that there is a solution set which is other than a prism.

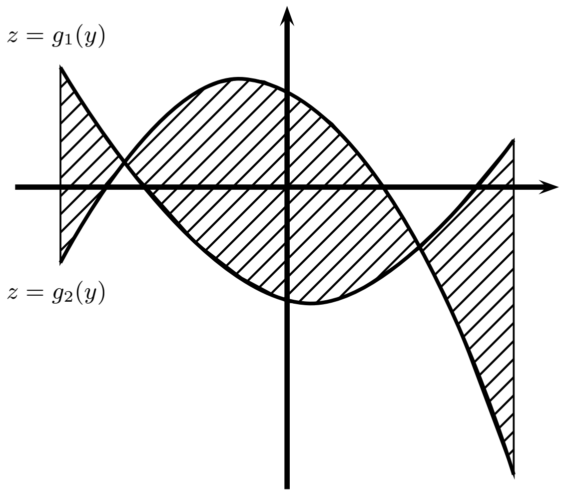

FIGURE 1. The solution set as in (36).

Example 6.6. The solution set $\mathcal{S}$ of Problem 1.1 for H_4,

$$\mathcal{S} := \left\{(x, y, z) \in \mathbb{R}^3 : H_4 \equiv H_4\left(\frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}; 0, y, z\right) \text{ is contractive}\right\}, \tag{36}$$

is not a polygon in yz-plane (see Figure 1), so $\mathcal{S}$ is not a prism.

Proof. We first note $a^2 + b^2 + c^2 + d^2 = \frac{541}{1600} < 1$, that is, (a, b, c, d) is not extremal. Direct calculations show

$$\det P_{22}(c) = \tfrac{18357}{20480},\ f_1(x) = -\tfrac{91785}{93376}x^2 - \tfrac{10231}{46688}x + \tfrac{4731}{11672},$$

$$f_2(x) = -\tfrac{1459}{1600}x^2 - \tfrac{27}{160}x + \tfrac{42621}{102400}.$$

Thus, after solving the following system of inequalities

$$\begin{cases} f_1(x) \geq 0 \\ f_2(x) \geq 0, \end{cases}$$

we have

$$\mathcal{S}_+(1) \cap \mathcal{S}_+(2) = \left\{x : -\tfrac{332}{435} \leq x \leq \tfrac{114}{211}\right\} \neq \emptyset$$

which implies that $H_4 \equiv H_4\left(\frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}; x, y, z\right)$ is soluble by Theorem 4.3. To investigate the solution set as in (35) in detail, we put $x = 0$ and check the positive semi-definiteness of the matrix $I - H_4^* H_4$. By the Nested Determinants Test in Lemma 2.1, we can see

$$I - H_4^* H_4 \geq 0 \Longleftrightarrow h_1(y, z)\, h_2(y, z) \leq 0, \tag{37}$$

where $h_1(y,z) := 332z - \left(435y^2 - 80y - 252\right)$ and $h_2(y,z) := 114z - (-211y^2 - 80y + 74$. We now observe that if the solution set as in (35) is a prism, then the projection of the solution set as in (35) onto the yz-plane must be a polygon. We let

$$g_1(y) := \tfrac{435y^2-80y-252}{332} \quad \text{and} \quad g_2(y) := \tfrac{-211y^2-80y+74}{114}.$$

Then, the solution to the inequality $h_1(y,z)\,h_2(y,z) \le 0$ is

$$g_1(y) \le z \le g_2(y) \quad \text{or} \quad g_2(y) \le z \le g_1(y)$$

which depends on the sign of the difference between $g_1(y)$ and $g_2(y)$. Indeed, if $y_1 \le y \le y_2$, where

$$y_1 := \tfrac{2\left(-2180-3\sqrt{44808878}\right)}{59821} \quad \text{and} \quad y_2 := \tfrac{2\left(-2180+3\sqrt{44808878}\right)}{59821},$$

then we have $g_1(y) \le g_2(y)$, so that the set

$$\{(y,z) : g_1(y) \le z \le g_2(y)\}$$

solves the inequality $h_1(y,z)\,h_2(y,z) \le 0$. On the other hand, if $-1 \le y \le y_1$ or $y_2 \le y \le 1$, then we have $g_2(y) \le g_1(y)$, and hence the set

$$\{(y,z) : g_2(y) \le z \le g_1(y)\}$$

is the solution of the inequality $h_1(y,z)\,h_2(y,z) \le 0$. After summarizing the explanation just given above to the solution set as in (36), we obtain Figure 1. As shown in Figure 1, the projection of the solution set as in (35) is not a polygon. Therefore, the solution set as in (35) is not a prism, as desired. □

References

[1] G. Arsene and A. Gheondea, *Completing matrix contractions*, J. Operator Theory **7** (1982), no. 1, 179–189.

[2] W. Arveson, *Interpolation problems in nest algebra*, J. Funct. Anal. **20** (1975), no. 3, 208–233.

[3] J. Bowers, J. Evers, L. Hogben, S. Shaner, K Sinder, and A. Wangsness, *On completion problems for various classes of P-matrices*, Linear Algebra Appl. **413** (2006), no. 2-3, 342–354.

[4] M. G. Crandall, *Norm preserving extensions of linear transformations on Hilbert space*, Proc. Amer. Math. Soc. **21** (1969), 335–340.

[5] G. M. Crippen and T. F. Havel, *Distance Geometry and Molecular Conformation*, Wiley, New York, 1988.

[6] R. Curto, C. Hernández, and E. de Oteyza, *Contractive completions of Hankel partial contractions*, J. Math. Anal. Appl. **203** (1996), no. 2, 303–332.

[7] R. Curto and W. Y. Lee, *Joint hyponormality of Toeplitz pairs*, Mem. Amer. Math. Soc. **150** (2001) no. 712, 65 pp.

[8] R. Curto, S. H. Lee, and J. Yoon, *Completion of Hankel partial contractions of extremal type*, J. Math. Phys. **53** (2012), 123526.

[9] C. Davis, *An extremal problem for extensions of a sesquilinear form*, Linear Algebra Appl. **13** (1976), no. 1-2, 91–102.

[10] C. Davis, W. M. Kahan, and H. F. Weinberger, *Norm-preserving dilations and their application to optimal error bounds*, SIAM J. Numer. Anal. **19** (1982), no. 3, 445–469.

[11] C. Foiaş and A. E. Frazho, *Redheffer products and the lifting of contractions on Hilbert space*, J. Operator Theory **11** (1984), no. 1, 193–196.
[12] R. M. Gray, *On unbounded Toeplitz matrices and nonstationary time series with an application to information theory*, Inform. Control **24** (1974), 181–196.
[13] R. M. Gray and L. D. Davisson, *An Introduction to Statistical Signal Processing*, Cambridge University Press, London, 2005.
[14] L. Hogben, *Matrix completion problems for pairs of related classes of matrices*, Numer. Linear Algebra Appl. **373** (2003), 13–19.
[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, London, 1985.
[16] I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms: Algebraic Theory*, Birkhäuser-Verlag, Boston, 1982.
[17] C. R. Johnson and L. Rodman, *Completion of partial matrices to contractions*, J. Funct. Anal. **69** (1986), no. 2, 260–267.
[18] ______, *Completion of Toeplitz partial contractions*, SIAM J. Matrix Anal. Appl. **9** (1988), no. 2, 159–167.
[19] M. Laurent, *A connection between positive semidefinite and Euclidean distance matrix completion problems*, Linear Algebra Appl. **273** (1998), 9–22.
[20] S. Parrott, *On a quotient norm and Sz.-Nagy-Foiaş lifting theorem*, J. Funct. Anal. **30** (1978), no. 3, 311–328.
[21] V. Paulsen, *Completely bounded maps and dilations*, Pitmam Research Notes in Mathematics Series, vol. 146, Longman Sci. Tech., New York, 1986.
[22] S. Power, *The distance to upper triangular operators*, Math. Proc. Cambridge Philos. Soc. **88** (1980), no. 2, 327–329.
[23] J. L. Smul'jan, *An operator Hellinger integral*, Mat. Sb. (N.S.) **49** (1959), 381–430 (in Russian).
[24] Y. L. Shmul'yan and R. N. Yanovskaya, *Blocks of a contractive operator matrix*, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **25** (1981), no. 7, 72–75; (English translation) Soviet Math. (Iz. VUZ) **25** (1981), 82–86.
[25] H. J. Woerdeman, *Strictly contractive and positive completions for block matrices*, Linear Algebra Appl. **136** (1990), 63–105.
[26] Wolfram Research, Inc., *Mathematica*, Version 9, Wolfram Research Inc., Champaign, IL, 2013.

In Hyoun Kim
Department of Mathematics
Incheon National University
Incheon 406-772, Korea
E-mail address: `ihkim@incheon.ac.kr`

Seonguk Yoo
Department of Mathematics
Inha University
Incheon 402-751, Korea
E-mail address: `seyoo73@gmail.com`

Jasang Yoon
School of Mathematical and Statistical Sciences
The University of Texas Rio Grande Valley
Edinburg, Texas 78539, USA
E-mail address: `jasang.yoon@utrgv.edu`