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REMARKS ON NONSPECIAL LINE BUNDLES ON GENERAL k-GONAL CURVES

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Abstract. In this work we obtain conditions for nonspecial line bundles on general k-gonal curves failing to be normally generated. Let $\mathcal{L}$ be a nonspecial very ample line bundle on a general k-gonal curve X with $k \geq 4$ and $\deg\mathcal{L} \geq \frac{3}{2}g + \frac{g-2}{k} + 1$. If $\mathcal{L}$ fails to be normally generated, then $\mathcal{L}$ is isomorphic to $\mathcal{K}_X - (ng^1_k + B) + R$ for some $n \geq 1$, B and R satisfying (1) $h^0(R) = h^0(B) = 1$, (2) $n+3 \leq \deg R \leq 2n+2$, (3) $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$. Its converse also holds under some additional restrictions. As a corollary, a very ample line bundle $\mathcal{L} \simeq \mathcal{K}_X - g^0_d + \xi^0_e$ is normally generated if $g^0_d \in X^{(d)}$ and $\xi^0_e \in X^{(e)}$ satisfy $d \leq \frac{g}{2} - \frac{g-2}{k} - 3$, $\text{supp}(g^0_d \cap \xi^0_e) = \emptyset$ and $\deg(g^0_d \cap F) \leq k-2$ for any $F \in g^1_k$.

1. Introduction

Let X be a smooth algebraic curve of genus g over an algebraically closed field of characteristic zero. A very ample line bundle $\mathcal{L}$ on X is said to be normally generated if $H^0(\mathbb{P}^N, \mathcal{O}(m)) \to H^0(X, \mathcal{L}^m)$ is surjective for all $m \geq 0$, where $\mathbb{P}^N := \mathbb{P}H^0(\mathcal{L})^*$. Any line bundle on X of degree at least $2g+1$ is normally generated [2, 7, 8] and any very ample line bundle on X of degree $2g$ is normally generated unless X is hyperelliptic ([5]). On the other hand, if X is hyperelliptic, any line bundle of degree $\leq 2g$ is not normally generated (Corollary 3.4 in [5]). If X is a trigonal curve of genus $g > 4$ with $2g - m_X \leq \deg\mathcal{L} \leq 2g$, then by Corollary 1 in [6] a nonspecial very ample line bundle $\mathcal{L}$ fails to be normally generated under the following specific condition:

$$\mathcal{L} \simeq \mathcal{K}_X - \beta g^1_3 + R \quad \text{for some } R \geq 0,$$

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where $\beta = 2g - \deg\mathcal{L}$ and m_X is the Maroni invariant of X.

The aim of this work is to characterize nonspecial very ample line bundles on general k-gonal curves which fail to be normally generated. If a nonspecial very ample line bundle $\mathcal{L}$ on a general k-gonal curve X fails to be normally generated, then by Theorem 3 in [3] and Theorem 2.1 in [4] we obtain an equivalence $\mathcal{L} \simeq \mathcal{K}_X - (rg^1_k + \tilde{B}) + \tilde{R}$ for some $r \geq 1$, $\tilde{B} \geq 0$ and $\tilde{R} \geq 0$ with $\deg\tilde{R} \leq 2r+2$. Here, g^1_k is the unique pencil of degree k on X. In order to figure out sufficient conditions for the failure of normal generation, it is helpful to more specify conditions on those divisors $\tilde{R}$ and $\tilde{B}$ (see the proof of Theorem 3.2). Since one of typical ways to deal with multiple coverings is to exploit properties of their base curves, it is useful to relate $\tilde{R}$ or $\tilde{B}$ to a fiber of the covering morphism. Thus we transform the above equivalence to the following form:

$$\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R \text{ with } \deg(R \cap F) \leq 1 \text{ for any } F \in g^1_k. \tag{1.1}$$

Furthermore we may assume that the transformation (1.1) satisfies $\deg(B \cap F) \leq k-2$ for any $F \in g^1_k$, which guaranties the uniqueness of the presentation (1.1) of $\mathcal{L}$ (Proposition 2.2).

Through this observation we get a result on a general k-gonal curve X of genus g and $4 \leq k \leq \frac{g+1}{2}$ as follows (Theorem 3.2). If a nonspecial very ample line bundle $\mathcal{L}$ on X with $\deg\mathcal{L} \geq \frac{3}{2}g + \frac{g-2}{k} + 1$ fails to be normally generated, then $\mathcal{L}$ is isomorphic to $\mathcal{K}_X - (ng^1_k + B) + R$ for some $n \geq 1$, $R \geq 0$ and $B \geq 0$ satisfying the following:

$$(*(n)) \quad n+3 \leq \deg R \leq 2n+2,\ h^0(R) = 1,\ h^0(B) = 1,\ \operatorname{supp}(R \cap B) = \emptyset,$$

$$(**) \quad \deg(R \cap F) \leq 1 \text{ and } \deg(B \cap F) \leq k-2 \text{ for any } F \in g^1_k.$$

Such a line bundle $\mathcal{L}$ has an equivalence $\mathcal{L} \simeq \mathcal{K}_X - (g^0_{nk-n} + B) + (R - R_n)$ for $R_n \in X^{(n)}$ and $g^0_{nk-n} := ng^1_k - R_n$, and hence there exists a divisor $\tilde{F} \in g^1_k$ such that $\deg(\tilde{F} \cap g^0_{nk-n}) = k-1$ since $n \geq 1$ and $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$. In fact, a line bundle $\mathcal{L} \simeq \mathcal{K}_X - g^0_d + \xi^0_e$ is very ample and normally generated if $g^0_d \in X^{(d)}$ and $\xi^0_e \in X^{(e)}$ satisfy $\operatorname{supp}(g^0_d \cap \xi^0_e) = \emptyset$, $d \leq \frac{g}{2} - \frac{g-2}{k} - 3$, $\xi^0_e \not\leq F$ and $\deg(g^0_d \cap F) \leq k-2$ for any $F \in g^1_k$ (Corollary 3.3). Therefore, in Theorem 3.2, the condition $n \geq 1$ is crucial for failing to be normally generated.

Conversely, a line bundle $\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R$ which satisfies the conditions $(*(n))$, $(**)$ and $2n(k-2) + \deg B \leq \frac{g-4}{2}$ is very ample and fails to be normally generated (Theorem 3.4). In addition, for both hyperelliptic and trigonal cases, nonspecial non-normally generated very ample line bundles which were already classified in [5] and [6] also admit the same type of presentation as (1.1) with conditions $(*(n))$ and $(**)$ (Remark 3.5, Remark 3.6).

Regarding special line bundles on general k-gonal curves, it was shown that if a special very ample line bundle $\mathcal{L}$ of $\deg\mathcal{L} > \frac{3g-1}{2}$ on a general k-gonal curve X with $4 \leq k \leq \frac{g+1}{2}$ fails to be normally generated, then we have $\deg B > \frac{c(k-2)}{2}$, where $c := \deg\mathcal{L} - \frac{3g-1}{2}$ and B is the base locus of $\mathcal{K}_X \otimes \mathcal{L}^{-1}$ ([1]). Note that the special line bundle $\mathcal{L}$ is isomorphic to $\mathcal{K}_X - (ng^1_k + B)$ for some $n \geq 0$ and a divisor B with $\deg(B \cap F) < k-2$ for any $F \in g^1_k$ by the very ampleness of $\mathcal{L}$.

Notations.

(1) $H^i(\mathcal{L}) := H^i(X, \mathcal{L})$, $h^0(D) := h^0(X, \mathcal{O}_X(D))$.
(2) g^0_d: an effective divisor of degree d with $h^0(X, \mathcal{O}_X(g^0_d)) = 1$.
(3) $\mathcal{L} - g^n_d := \mathcal{L}(-D)$ where $g^n_d = |D|$.
(4) $\langle D\rangle_{\mathcal{L}}$: the linear space spanned by the points of $\varphi_{\mathcal{L}}(D)$ for a very ample $\mathcal{L}$.

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2. Preliminaries

The aim of this work is to characterize nonspecial line bundles on a general k-gonal curve X which are very ample and fail to be normally generated. To do this we in advance want to find out conditions such as $(*(n))$ and $(**)$ for a description $\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R$ on X. This can be done through Proposition 2.2 which will be proved by the following theorem.

Theorem 2.1 ([4], Theorem 2.1)**.** *Let X be a general k-gonal curve of genus $g \geq 4$, $k \geq 4$, and let g^1_k be the unique pencil of degree k on X. If X has a line bundle $\mathcal{L}$ with* $\mathrm{Cliff}(\mathcal{L}) \leq \frac{g-4}{2}$ *and* $\deg\mathcal{L} \leq g-1$, *then $|\mathcal{L}|$ is compounded of g^1_k, that is, $|\mathcal{L}| = (\dim|\mathcal{L}|)g^1_k + B$ for some $B \geq 0$.*

The following proposition will be used for the proof of Theorem 3.2 which demonstrates necessary conditions for nonspecial line bundles on general k-gonal curves not to be normally generated. The condition $\deg(R \cap F) \leq 1$ in $(**)$ which enables us to use properties of the base curve $\mathbb{P}^1$ is essential to show the converse of Theorem 3.2 with some restrictions.

Proposition 2.2. *Let X be a general k-gonal curve of genus g. Assume that a very ample line bundle $\mathcal{L}$ on X is given by $\mathcal{L} \simeq \mathcal{K}_X - (rg^1_k + \tilde{B}) + \tilde{R}$ for some $r \geq 1$, divisors $\tilde{R} \geq 0$ and $\tilde{B} \geq 0$ on X satisfying $r(k-2) + \deg\tilde{B} \leq \frac{g-4}{2}$ and the condition $(*(r))$. Then we have the following*:

(1) *$\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R$ for some $n \geq 1$, divisors $R \geq 0$ and $B \geq 0$ satisfying $n(k-2) + \deg B \leq \frac{g-4}{2}$, $(*(n))$ and $(**)$.*
(2) *Furthermore, the above presentation is unique if $nk + \deg B \leq \frac{g-4}{2}$.*

Proof. (1) Suppose that $\deg(\tilde{R} \cap \tilde{F}) = t \geq 2$ for some $\tilde{F} \in g^1_k$. Set $D_t := \tilde{R} \cap \tilde{F}$. Then the equivalence $\mathcal{L} \simeq \mathcal{K}_X - (rg^1_k + \tilde{B}) + \tilde{R}$ gives

$$|\mathcal{K}_X \otimes \mathcal{L}^{-1}(\tilde{R} - D_t)| = |(r-1)g^1_k + (\tilde{F} - D_t) + \tilde{B})|.$$

By the following inequalities

$$\begin{aligned}\mathrm{Cliff}((r-1)g_k^1+(\tilde{F}-D_t)+\tilde{B}) &\le \deg(rg_k^1+\tilde{B})-t-2(r-1)\\ &= (\deg(rg_k^1+\tilde{B})-2r)-(t-2)\\ &\le r(k-2)+\deg\tilde{B}\le \frac{g-4}{2},\end{aligned}$$

Theorem 2.1 implies that $|(r-1)g_k^1+(\tilde{F}-D_t)+\tilde{B}|$ is compounded of g_k^1, i.e.,

$$|(r-1)g_k^1+(\tilde{F}-D_t)+\tilde{B}| = sg_k^1+\{\text{base locus}\} \quad \text{for some } s\ge r-1.$$

Then we have

$$|(r-1)g_k^1+(\tilde{F}-D_t)+\tilde{B}| = (r-1)g_k^1+(\tilde{F}-D_t+\tilde{B})$$

since the conditions $D_t\le\tilde{R}$ and $\mathrm{supp}(\tilde{B}\cap\tilde{R})=\emptyset$ imply $h^0(\tilde{F}-D_t+\tilde{B})=1$. Thus $\mathcal{L}$ admits the following presentation:

$$\mathcal{L}\simeq\mathcal{K}_X-((r-1)g_k^1+(\tilde{F}-D_t)+\tilde{B})+(\tilde{R}-D_t).$$

From $\mathrm{supp}(\tilde{R}\cap\tilde{B})=\emptyset$ and $\tilde{R}\cap\tilde{F}=D_t$ we get

$$\mathrm{supp}((\tilde{F}-D_t+\tilde{B})\cap(\tilde{R}-D_t))=\emptyset.$$

And we have $(r-1)+3\le\deg(\tilde{R}-D_t)\le 2(r-1)+2$ since $\mathcal{L}$ is very ample and $\deg\tilde{R}\le 2r+2$ and $\deg(\tilde{R}-D_t)\le\deg\tilde{R}-2$. Iterating such processes as many as possible, by $\deg\tilde{R}\le 2r+2$ we finally obtain for some $n'\ge 1$

$$\mathcal{L}\simeq\mathcal{K}_X-(n'g_k^1+B')+R' \quad\text{with}\quad n'(k-2)+\deg B'\le\frac{g-4}{2} \tag{2.1}$$

such that $R'\ge 0$ and $B'\ge 0$ satisfy $(*(n'))$ and $\deg(R'\cap F)\le 1$ for any $F\in g_k^1$.

Assume that there exists a divisor $\tilde{F}\in g_k^1$ such that $\deg(B'\cap\tilde{F})=k-1$. Let $E_{k-1}:=B'\cap\tilde{F}$ and $P:=\tilde{F}-E_{k-1}$. The presentation (2.1) gives

$$|\mathcal{K}_X\otimes\mathcal{L}^{-1}(R'+P)|=|n'g_k^1+(B'-E_{k-1})+(E_{k-1}+P)|=|(n'+1)g_k^1+(B'-E_{k-1})|,$$

whose Clifford index does not exceed $\frac{g-4}{2}$ due to $n'(k-2)+\deg B'\le\frac{g-4}{2}$. Hence Theorem 2.1 implies

$$|\mathcal{K}_X\otimes\mathcal{L}^{-1}(R'+P)|=(n'+1)g_k^1+(B'-E_{k-1}),$$

whence

$$\mathcal{L}\simeq\ \mathcal{K}_X-((n'+1)g_k^1+(B'-E_{k-1}))+(R'+P).$$

Since $P\nleq B'$, $\mathrm{supp}(B'\cap R')=\emptyset$ and $E_{k-1}+P=\tilde{F}$, we have

$$\mathrm{supp}((B'-E_{k-1})\cap(R'+P))=\emptyset,\ \deg((R'+P)\cap\tilde{F})=1.$$

The hypothesis $k\ge 4$ gives $\deg B'\ge k-1\ge 3$ and thus $\mathrm{Cliff}(R'+P)\le 2n'+3\le n'(k-2)+\deg B'\le\frac{g-4}{2}$. According to Theorem 2.1 it turns out $h^0(R'+P)=1$ since $\deg(R'\cap F)\le 1$ for any $F\in g_k^1$.

By iterating the same arguments as the above, we finally get the following equivalence:

$$\mathcal{L} \simeq \mathcal{K}_X - (ng_k^1 + B) + R \;\text{ with } n(k-2) + \deg B \leq \frac{g-4}{2}$$

for some integer $n \geq 1$, $R \geq 0$ and $B \geq 0$ satisfying the conditions $(*(n))$ and $(**)$.

(2) Suppose that the line bundle $\mathcal{L}$ admits another presentation for an integer $m \geq 1$ such as

$$\mathcal{L} \simeq \mathcal{K}_X - (mg_k^1 + \hat{B}) + \hat{R} \;\text{ with } mk + \deg\hat{B} \leq \frac{g-4}{2} \tag{2.2}$$

which satisfies the conditions $(*(m))$ and $(**)$. Then the equivalences

$$\mathcal{L} \simeq \mathcal{K}_X - (ng_k^1 + B) + R \simeq \mathcal{K}_X - (mg_k^1 + \hat{B}) + \hat{R}$$

imply

$$ng_k^1 + B + \hat{R} \simeq mg_k^1 + \hat{B} + R.$$

Set $\mathcal{D} := |ng_k^1 + B + \hat{R}| = |mg_k^1 + \hat{B} + R|$. If $\mathcal{D}$ is not compounded of g_k^1, then $\dim \mathcal{D} \geq \max\{n+1, m+1\}$. If $m \geq n$, then we get $\dim|\mathcal{D}| \geq m+1$, whence the condition $\deg\hat{R} \leq 2m+2$ yields

$$\mathrm{Cliff}(\mathcal{D}) \leq nk + \deg B + \deg\hat{R} - 2(m+1) \leq nk + \deg B \leq \frac{g-4}{2}.$$

It is a contradiction to Theorem 2.1. We have the same conclusion in case $m \leq n$ and hence $\mathcal{D}$ is compounded of g_k^1. Since $\deg(R \cap F) \leq 1$, $\deg(\hat{R} \cap F) \leq 1$, $\deg(B \cap F) \leq k-2$ and $\deg(\hat{B} \cap F) \leq k-2$ for all $F \in g_k^1$, it turns out that $\mathcal{D} = ng_k^1 + B + \hat{R} = mg_k^1 + \hat{B} + R$, whence $m = n$ and $B + \hat{R} = \hat{B} + R$. By $\mathrm{supp}(B \cap R) = \emptyset$ and $\mathrm{supp}(\hat{B} \cap \hat{R}) = \emptyset$ we conclude $B = \hat{B}$ and $R = \hat{R}$. Thus the proof is completed. □

3. Main theorems

This section is devoted to characterizing nonspecial line bundles on general k-gonal curves which fail to be normally generated. The following theorem given by M. Green and R. Lazarsfeld plays a key role to prove our results.

Theorem 3.1 ([3], Theorem 3)**.** *Let X be a smooth curve of genus g and let $\mathcal{L}$ be a nonspecial very ample line bundle on X with $\deg\mathcal{L} > \frac{3g+1}{2}$. If $\mathcal{L}$ fails to be normally generated, then there is a line bundle $\mathcal{A} \simeq \mathcal{L}(-R)$, $R > 0$, which satisfies the following conditions*:

$$\begin{aligned} &\text{(i) } \deg\mathcal{A} \geq \frac{g-1}{2}, \qquad &&\text{(ii) } \mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L}),\\ &\text{(iii) } h^1(\mathcal{A}) \geq h^1(\mathcal{L}) + 2, &&\text{(iv) } h^0(\mathcal{A}) \geq 2. \end{aligned} \tag{3.1}$$

By Theorem 3.1 and Proposition 2.2 we obtain necessary conditions for the failure of the normal generation of nonspecial line bundles on general k-gonal curves.

Theorem 3.2. *Let X be a general k-gonal curve of genus g with $4 \leq k \leq \frac{g+1}{2}$ and let $\mathcal{L}$ be a nonspecial very ample line bundle with* $\deg\mathcal{L} \geq \frac{3}{2}g + \frac{g-2}{k} + 1$. *If $\mathcal{L}$ fails to be normally generated, then $\mathcal{L}$ admits the following presentation*:

$$\mathcal{L} \simeq \mathcal{K}_X - (ng_k^1 + B) + R, \tag{3.2}$$

for some integer $n \geq 1$, divisors R and B satisfying the conditions $(*(n))$ *and* $(**)$. *Furthermore, the presentation* (3.2) *is unique in case* $nk + \deg B \leq \frac{g-4}{2}$.

Proof. Since $\mathcal{L}$ is not normally generated and $\deg\mathcal{L} > \frac{3g+1}{2}$, there is a line bundle $\mathcal{A} \simeq \mathcal{L}(-\tilde{R}), \tilde{R} > 0$ such that $\mathcal{A}$ satisfies the conditions in (3.1). The assumption $4 \leq k \leq \frac{g+1}{2}$ yields the inequalities:

$$\begin{aligned}\frac{k-2}{2k}(g-2) - \frac{g-4}{2} =& \frac{(k-2)(g-2)-(g-4)k}{2k} \\ =& \frac{2k-2g+4}{2k} \leq \frac{-g+5}{2k} < 0,\end{aligned}$$

whence we have $\mathrm{Cliff}(\mathcal{A}) < \frac{g-4}{2}$ by $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L}) = 2g - \deg\mathcal{L} \leq \frac{k-2}{2k}(g-2)$.

Assume $\deg\mathcal{A} \leq g-1$. By Theorem 2.1, we get $|\mathcal{A}| = (\dim|\mathcal{A}|)g_k^1 + \tilde{B}$. Then

$$\dim|\mathcal{A}| \leq \frac{g-2}{2k} - \frac{\deg\tilde{B}}{k-2}$$

by $\mathrm{Cliff}(\mathcal{A}) = (k-2)\dim|\mathcal{A}| + \deg\tilde{B} \leq \mathrm{Cliff}(\mathcal{L}) \leq \frac{(k-2)}{2k}(g-2)$. This yields

$$\frac{g-1}{2} \leq \deg(\mathcal{A}) \leq \frac{g-2}{2} + (1 - \frac{k}{k-2})\deg\tilde{B} < \frac{g-1}{2},$$

which cannot occur. Thus $\deg\mathcal{A} \geq g$, i.e., $\deg\mathcal{K}_X \otimes \mathcal{A}^{-1} \leq g-2$.

According to Theorem 2.1 we have $|\mathcal{K}_X \otimes \mathcal{A}^{-1}| = rg_k^1 + \tilde{B}$, where $r := \dim|\mathcal{K}_X \otimes \mathcal{A}^{-1}|$. Thus the conditions that $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$ and $\mathcal{A} \simeq \mathcal{L}(\tilde{R})$ yield

$$r(k-2)+\deg\tilde{B} = \mathrm{Cliff}(\mathcal{K}_X \otimes \mathcal{A}^{-1}) \leq \mathrm{Cliff}(\mathcal{L}) = 2g - \deg\mathcal{L} = rk+2+\deg\tilde{B}-\deg\tilde{R},$$

whence $\deg\tilde{R} \leq 2r+2$. Since $\mathcal{L}$ is very ample, it turns out $r+3 \leq \deg\tilde{R} \leq 2r+2$ and hence $r \geq 1$. Therefore,

$$\mathcal{L} \simeq \mathcal{K}_X - (rg_k^1 + \tilde{B}) + \tilde{R} \ \text{ with } r \geq 1,\ r+3 \leq \deg\tilde{R} \leq 2r+2.$$

Since $\deg\tilde{R} \leq 2r+2 \leq \mathrm{Cliff}(rg_k^1 + \tilde{B}) \leq \frac{g-4}{2}$ for $k \geq 4$, Theorem 2.1 forces

$$|\tilde{R}| = lg_k^1 + R' \text{ with } l \leq r-1,$$

whence

$$\mathcal{L} \simeq \mathcal{K}_X - ((r-l)g_k^1 + \tilde{B}) + R', \quad R' := \tilde{R} - lg_k^1,$$

which satisfies that $r-l\geq 1,\ h^0(R')=1,\ h^0(\tilde{B})=1$ and $\deg R'\leq 2(r-l)+2$. Here, we may assume $\mathrm{supp}(R'\cap\tilde{B})=\emptyset$. As a result, by Proposition 2.2 the line bundle $\mathcal{L}$ can be expressed by

$$\mathcal{L}\simeq\mathcal{K}_X-(ng^1_k+B)+R$$

for some divisors R and B satisfying $(*(n))$ and $(**)$. The uniqueness of the presentation (3.2) follows from Proposition 2.2. □

From Theorem 3.2 we obtain a result on the normal generation of line bundles on general k-gonal curves as follows.

Corollary 3.3. *Let X be a general k-gonal curve of genus g with $4\leq k\leq\frac{g+1}{2}$. And let $g^0_d\in X^{(d)}$ and $\xi^0_e\in X^{(e)}$ satisfy $\mathrm{supp}(g^0_d\cap\xi^0_e)=\emptyset$, $\xi^0_e\not\leq F$ and $\deg(g^0_d\cap F)\leq k-2$ for any $F\in g^1_k$. If $e\geq 3$ and $d\leq\frac{g}{2}-\frac{g-2}{k}-3$, then $\mathcal{L}\simeq\mathcal{K}_X-g^0_d+\xi^0_e$ is a nonspecial normally generated line bundle.*

Proof. The line bundle $\mathcal{L}$ is trivially nonspecial and the condition $d\leq\frac{g}{2}-\frac{g-2}{k}-3$ is equivalent to $\deg\mathcal{L}\geq\frac{3}{2}g+\frac{g-2}{k}+1+e$. Suppose that $\mathcal{L}\simeq\mathcal{K}_X-g^0_d+\xi^0_e$ is not very ample. Then,

$$\mathcal{K}_X-g^0_d+\xi^0_e\simeq\mathcal{K}_X-h^0_t+\zeta^0_2\ \text{ for some } h^0_t\in X^{(t)},\ \zeta^0_2\in X^{(2)}.$$

This implies that $g^0_d+\zeta^0_2\simeq h^0_t+\xi^0_e$ and $g^0_d+\zeta^0_2\neq h^0_t+\xi^0_e$ as divisors, whence $\dim|g^0_d+\zeta^0_2|\geq 1$. Since $d+2\leq\frac{g-4}{2}$ and $\deg(g^0_d,F)\leq k-2$ for any $F\in g^1_k$, Theorem 2.1 implies

$$|h^0_t+\xi^0_e|=|g^0_d+\zeta^0_2|=g^1_k+B\ \text{ and }\ B\leq g^0_d.$$

Hence the condition $\mathrm{supp}(g^0_d\cap\xi^0_e)=\emptyset$ gives $\xi^0_e\leq\tilde{F}$ for some $\tilde{F}\in g^1_k$. Therefore $\mathcal{L}$ is very ample by the hypothesis $\xi^0_e\not\leq F$ for any $F\in g^1_k$.

Assume that $\mathcal{L}$ fails to be normally generated. Since $\deg\mathcal{L}\geq\frac{3}{2}g+\frac{g-2}{k}+1$, Theorem 3.2 gives $\mathcal{L}\simeq\mathcal{K}_X-(ng^1_k+B)+R$ for some R and B satisfying the conditions $(*(n))$ and $(**)$. The equivalence $\mathcal{L}\simeq\mathcal{K}_X-g^0_d+\xi^0_e$ yields

$$ng^1_k+B+\xi^0_e\simeq g^0_d+R,$$

and

$$\mathrm{Cliff}(g^0_d+R)\leq d+2n+2-2n\leq\frac{g-4}{2}.$$

According to Theorem 2.1, the linear system $|ng^1_k+B+\xi^0_e|=|g^0_d+R|$ is compounded of g^1_k and $\dim|g^0_d+R|\geq n\geq 1$. It can not occur since $\deg(g^0_d\cap F)\leq k-2$ and $\deg(R,F)\leq 1$ for any $F\in g^1_k$. Thus $\mathcal{L}$ is normally generated. □

The converse of Theorem 3.2 holds under the condition (3.3) in Theorem 3.4.

Theorem 3.4. *Let X be a general k-gonal curve of genus $g \geq 12$ with $k \geq 4$ and let R and B be divisors on X satisfying $(*(n))$ for some $n \geq 1$ and $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$. Then $\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R$ is very ample and fails to be normally generated if*

$$2n(k-2) + \deg B \leq \frac{g-4}{2}. \tag{3.3}$$

Proof. Assume that $\mathcal{L}$ is not very ample. Then there exist divisors g^0_d and ξ^0_2 such that

$$\mathcal{L} \simeq \mathcal{K}_X - (ng^1_k + B) + R \simeq \mathcal{K}_X - g^0_d + \xi^0_2,$$

with $\mathrm{supp}(g^0_d \cap \xi^0_2) = \emptyset$. Then we have $ng^1_k + B + \xi^0_2 \simeq g^0_d + R$. Let

$$\mathcal{D} := |ng^1_k + B + \xi^0_2| = |g^0_d + R|.$$

The equation (3.3) gives

$$\mathrm{Cliff}(ng^1_k + B + \xi^0_2) \leq nk + \deg B + 2 - 2n = n(k-2) + \deg B + 2 \leq \frac{g-4}{2},$$

and hence $\mathcal{D} = (\dim|\mathcal{D}|)g^1_k + B'$ by Theorem 2.1 where B' is a base locus. Then

$$\dim|\mathcal{D}| = \dim|ng^1_k + B + \xi^0_2| = \dim|g^0_d + R| \leq n + 2$$

since $\dim|ng^1_k + B| = n$. It gives a contradiction since $\deg R \geq n+3$, $\mathrm{supp}(R \cap B) = \emptyset$, and $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$. Thus $\mathcal{L}$ is very ample.

To show the failure of the normal generation of $\mathcal{L}$, we observe that

$$h^1(\mathcal{I}_{R/X}(2)) = 0$$

which comes from the inequality

$$\deg(\mathcal{K}_X \otimes \mathcal{L}^{-2}(R)) = 2nk + 2\deg B - (2g-2) - \deg R < 0$$

due to the equation (3.3). Therefore, $\mathcal{L}$ is not normally generated if R fails to impose independent conditions on quadrics in $\langle R \rangle_{\mathcal{L}}$ by the exact sequences:

$$\begin{aligned} 0 \to \mathcal{I}_R(2) \to \quad & \mathcal{O}_{\mathbb{P}^N}(2) \quad \to \mathcal{O}_R(2) \to 0, \\ 0 \to \mathcal{I}_{X/\mathbb{P}^N}(2) \to \quad & \mathcal{I}_{R/\mathbb{P}^N}(2) \quad \to \mathcal{I}_{R/X}(2) \to 0, \end{aligned}$$

where $\mathbb{P}^N := \mathbb{P}H^0(\mathcal{L})^*$.

We claim that R fails to impose independent conditions on quadrics in $\langle R \rangle_{\mathcal{L}}$. Set $s := \dim\langle R \rangle_{\mathcal{L}}$. The Riemann-Roch theorem gives $\deg R = n + s + 2$ which forces $s \leq n$ for $\deg R \leq 2n+2$. Theorem 2.1 implies $\dim|sg^1_k| = s$ by $s(k-2) \leq \frac{g-4}{2}$. Since

$$\begin{aligned} h^0(\mathcal{L}(-sg^1_k)) &\geq \deg\mathcal{L} - sk - g + 1 \\ &= 2g - 2 - nk - \deg B + \deg R - sk - g + 1 \\ &= 2g - 2 - nk - \deg B + s + n + 2 - sk - g + 1 \\ &= (g+1) - (k-1)(n+s) - \deg B \\ &\geq (g+1) - (2n(k-1) + \deg B) \geq 1 \quad \text{by (3.3)}, \end{aligned}$$

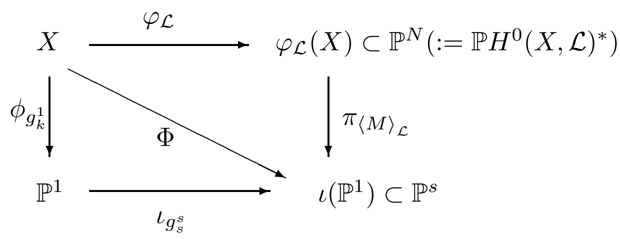

FIGURE 1.

there exists an effective divisor $M \in |\mathcal{L}(-sg^1_k)|$. Note that $|\mathcal{L}(-M)| = sg^1_k$. Accordingly we obtain the commutative diagram in Figure 1, where $\pi_{\langle M \rangle_{\mathcal{L}}}$ is a natural projection from $\langle M \rangle_{\mathcal{L}}$.

Let P be a point of X with $P \leq R$. Then we have $h^0((n+s)g^1_k + B - R) = h^0((n+s)g^1_k + B - (R-P)) = 0$, since $\deg R = n+s+2$, $\mathrm{supp}(B \cap R) = \emptyset$, $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$ and

$$|(n+s)g^1_k + B| = (n+s)g^1_k + B$$

by (3.3). Then,

$$\begin{aligned} & h^0(M) - h^0(M-P) \\ = & \, h^0(\mathcal{L}(-sg^1_k)) - h^0(\mathcal{L}(-sg^1_k - P)) \\ = & \, h^0((n+s)g^1_k + B - R) - h^0((n+s)g^1_k + B - (R-P)) + 1 \\ = & \, 1 \end{aligned}$$

and hence P is not a base point of $|M|$. This implies $\langle M \rangle_{\mathcal{L}} \cap \langle R \rangle_{\mathcal{L}} = \emptyset$ since $\dim \langle M + R \rangle_{\mathcal{L}} = N$, $\dim \langle M \rangle_{\mathcal{L}} = N - s - 1$ and $\dim \langle R \rangle_{\mathcal{L}} = s$. Hence $\pi_{\langle M \rangle_{\mathcal{L}}}$ can be regarded as a projection to $\langle R \rangle_{\mathcal{L}}$. The condition that $\deg(R \cap F) \leq 1$ for any $F \in g^1_k$ implies that $\varphi_{\mathcal{L}}(R)$ lies on a rational normal curve $\iota(\mathbb{P}^1) \subset \mathbb{P}^s$ with $\deg R \geq 2s+2$. Since any $2s+2$ points on a rational normal curve fails to impose independent conditions on quadrics, $\varphi_{\mathcal{L}}(R)$ does fail to impose independent conditions on quadrics in $\langle R \rangle_{\mathcal{L}}$. □

The following two remarks tell that for a non-normally generated line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) = 0$, the presentation (3.2) with $(*(n))$ and $(**)$ also holds in case $k = 2, 3$.

Remark 3.5. Let X be a hyperelliptic curve and let $\mathcal{L}$ be a very ample line bundle of degree $2g - \beta$ on X with $\beta \geq 0$. Since $\mathcal{L}$ is very ample, it is nonspecial and thus there exist divisors $\xi^0_e \in X^{(e)}$ and $g^0_d \in X^{(d)}$ with $(\xi^0_e \cap g^0_d) = \emptyset$ such that

$$\mathcal{L} \simeq \mathcal{K}_X - g^0_d + \xi^0_e, \; d = \beta + e - 2.$$

The very ampleness of $\mathcal{L}$ gives $e \geq 3$. Set $h_d^0 := dg_2^1 - g_d^0$ and $R := \xi_e^0 + h_d^0$. Then we get

$$\mathcal{L} \simeq \mathcal{K}_X - dg_2^1 + R \text{ with } (*(d)) \text{ and } (**),$$

since the conditions that $\text{supp}(\xi_e^0 \cap g_d^0) = \emptyset$ and $g_d^0 = dg_2^1 - h_d^0$ imply $h^0(\xi_e^0 + h_d^0) = 1$.

Remark 3.6. Let X be a trigonal curve of genus $g > 4$ and Maroni-invariant m_X. Let $\mathcal{L}$ be a nonspecial very ample line bundle of degree $2g - \beta$ with $0 \leq \beta \leq m_X$. If $\mathcal{L}$ fails to be normally generated, then Corollary 1 in [6] tells that

$$\mathcal{L} \simeq \mathcal{K}_X - \beta g_3^1 + \tilde{R} \text{ with } \deg\tilde{R} = 2\beta + 2. \tag{3.4}$$

We observe that the form (3.4) is equivalent to the presentation (3.2) in the following. For the presentation (3.4), set

$$\{F_1, \dots, F_f\} := \{\ \tilde{R} \cap F \,|\, \tilde{R} \cap F = F,\ F \in g_3^1\},$$
$$\{D_1, \dots, D_l\} := \{\ \tilde{R} \cap F \,|\, \deg(\tilde{R} \cap F) = 2,\ F \in g_3^1\}.$$

Then, $\deg((\tilde{R} - \sum F_i - \sum D_j) \cap F) \leq 1$ for any $F \in g_k^1$ and

$$\begin{aligned}\mathcal{L} &\simeq \mathcal{K}_X - \beta g_3^1 + \tilde{R} \simeq \mathcal{K}_X - (\beta - f)g_3^1 + (\tilde{R} - fg_3^1)\\ &\simeq \mathcal{K}_X - ((\beta - f - l)g_3^1 + B) + R,\end{aligned}$$

where $B := \sum_{i=1}^{l}(g_3^1 - D_j)$ and $R := \tilde{R} - fg_3^1 - \sum_{i=1}^{l} D_i$. The conditions $h^1(\mathcal{L}) = 0$ and $\deg\tilde{R} \leq 2\beta + 2$ yield $\beta - f - l \geq 1$, whence R and B satisfy conditions $(*(\beta - f - l))$ and $(**)$.

Conversely, any line bundle which has a presentation (3.2) can be written in the form (3.4): Assume that there exist some divisors R and B satisfying $(*(n))$ and $(**)$ for an integer $n \geq 1$ such that $\mathcal{L} \simeq \mathcal{K}_X - (ng_3^1 + B) + R$. Let B' be a divisor such that $B + B' = fg_3^1$ where $f := \deg B$ and let $\gamma := 2n + 2 - \deg R$ and $\beta := n + f + \gamma$. Then we have

$$\begin{aligned}\mathcal{L} &\simeq \mathcal{K}_X - (ng_3^1 + B) + R \simeq \mathcal{K}_X - (n + f)g_3^1 + (R + B')\\ &\simeq \mathcal{K}_X - (n + f + \gamma)g_3^1 + (R + B' + \gamma g_3^1)\\ &\simeq \mathcal{K}_X - \beta g_3^1 + R_{2\beta+2},\end{aligned}$$

where $R_{2\beta+2} := R + B' + \gamma g_3^1$. Therefore two presentations (3.2) and (3.4) are equivalent.

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