

INTEGRAL POINTS ON THE CHEBYSHEV DYNAMICAL SYSTEMS

SU-ION IH

*Dedicated to Joseph H. Silverman respectfully and gratefully
on the occasion of his 60th birthday*

ABSTRACT. Let K be a number field and let S be a finite set of primes of K containing all the infinite ones. Let $\alpha_0 \in \mathbb{A}^1(K) \subset \mathbb{P}^1(K)$ and let Γ_0 be the set of the images of α_0 under especially all Chebyshev morphisms. Then for any $\alpha \in \mathbb{A}^1(K)$, we show that there are only a finite number of elements in Γ_0 which are S -integral on \mathbb{P}^1 relative to (α) . In the light of a theorem of Silverman we also propose a conjecture on the finiteness of integral points on an arbitrary dynamical system on \mathbb{P}^1 , which generalizes the above finiteness result for Chebyshev morphisms.

1. Introduction

In [5] Silverman proved, among other things, a result on the finiteness of S -integral points on \mathbb{P}^1 relative to a nonexceptional point on any orbit under a fixed rational function of degree ≥ 2 . In the current article we will consider a slight variation of this theorem — a conjecture in the case of arbitrary morphisms and a theorem in the case of Chebyshev morphisms.

• **Integral points.** Let K be a number field with algebraic closure \overline{K} , let S be a finite set of primes of K containing the infinite ones, and let $\alpha, \beta \in \overline{K}$. We say that β is S -integral on \mathbb{P}^1 relative to (α) if no K -conjugate of β meets any K -conjugate of α at any primes of K lying outside S . More precisely, this means that for any prime $v \notin S$ of K and any K -embeddings $\sigma : K(\beta) \rightarrow \overline{K}_v$ and $\tau : K(\alpha) \rightarrow \overline{K}_v$, we have

$$(1.1) \quad \begin{cases} |\sigma(\beta) - \tau(\alpha)|_v \geq 1 & \text{if } |\tau(\alpha)|_v \leq 1; \text{ and} \\ |\sigma(\beta)|_v \leq 1 & \text{if } |\tau(\alpha)|_v > 1. \end{cases}$$

Here \overline{K}_v is an algebraic closure of a completion K_v of K with respect to a v -adic absolute value on K and is assumed to contain K for convenience; and

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a K -embedding means a field embedding whose restriction to K is equal to the identity.

Note that this definition extends naturally to the case where α is the point at infinity. We say that β is S -integral on \mathbb{P}^1 relative to the point at infinity if $|\sigma(\beta)|_v \leq 1$ for all primes $v \notin S$ of K and all K -embeddings $\sigma : K(\beta) \rightarrow \overline{K}_v$. Thus our S -integral points coincide with the usual S -integers when α is the point at infinity. Since the notion of β being S -integral on \mathbb{P}^1 relative to (α) is symmetric with respect to α and β (easy to see this symmetry directly from (1.1), cf. the last sentence of this discussion of integral points below), we have the notion of ∞ being S -integral on \mathbb{P}^1 relative to (α) , too. Further, this definition extends to an arbitrary effective divisor D on \mathbb{P}^1 in a natural way. More precisely, β is defined to be S -integral on \mathbb{P}^1 relative to D if β is S -integral on \mathbb{P}^1 relative to (α) for any $\alpha \in \text{Supp}(D)$.

For any effective divisor D on \mathbb{P}^1 , we write

$$\begin{aligned} \mathbb{P}_D^1(\overline{\mathcal{O}}_{K,S}) &= \{P \in \mathbb{P}^1(\overline{K}) : P \text{ is } S\text{-integral on } \mathbb{P}^1 \text{ relative to } D\}; \text{ and} \\ \mathbb{P}_D^1(\mathcal{O}_{K,S}) &= \{P \in \mathbb{P}^1(K) : P \text{ is } S\text{-integral on } \mathbb{P}^1 \text{ relative to } D\} \\ &= \mathbb{P}_D^1(\overline{\mathcal{O}}_{K,S}) \cap \mathbb{P}^1(K). \end{aligned}$$

Further, (via $\mathbb{G}_m(\overline{K}) \subset \mathbb{P}^1(\overline{K})$) we write

$$\begin{aligned} \mathbb{G}_{m,D}(\overline{\mathcal{O}}_{K,S}) &= \mathbb{P}_{(0)+(\infty)+D}^1(\overline{\mathcal{O}}_{K,S}); \text{ and} \\ \mathbb{G}_{m,D}(\mathcal{O}_{K,S}) &= \mathbb{P}_{(0)+(\infty)+D}^1(\mathcal{O}_{K,S}) \\ &= \mathbb{G}_{m,D}(\overline{\mathcal{O}}_{K,S}) \cap \mathbb{G}_m(K). \end{aligned}$$

Their elements are called S -integral points on \mathbb{G}_m relative to D . For more details of an intrinsic and geometric definition of integral points, see [1, pp. 2011–2012]. In particular in the case of \mathbb{P}^1 , this geometric description or the equivalent definition in terms of the well-known v -adic chordal metric helps us more directly see that (1.1) is symmetric with respect to α and β .

• **Arithmetical dynamical systems.** Refer to [1, §4]. Keep K and S as above. Fix the following notation:

$$\begin{array}{ll} \varphi & \text{a } K\text{-morphism } \mathbb{P}^1 \rightarrow \mathbb{P}^1, \text{ of finite degree } \geq 2, \\ Q_0 & \text{a point in } \mathbb{P}^1(K). \end{array}$$

Now we introduce the two dynamical objects Γ_0 and Γ associated to φ (more precisely, $[\varphi]$ below) and Q_0 — the former reminiscent of a finitely generated subgroup of $\mathbb{G}_m(\overline{K})$ and the latter reminiscent of a division group in $\mathbb{G}_m(\overline{K})$, cf. see §2.1 below or [1, p. 2012].

For any morphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and any subset Y of $\mathbb{P}^1(\overline{K})$, $\phi^+(Y)$ and $\phi^-(Y)$ denote respectively the forward and backward orbits of Y under ϕ , that is,

$$\phi^0 := \text{identity};$$

$$\begin{aligned} \phi^n &:= \phi \circ \cdots \circ \phi \quad (n \geq 1 \text{ times}); \\ \phi^{-n}(Y) &:= (\phi^n)^{-1}(Y); \\ \phi^+(Y) &:= \bigcup_{n \geq 0} \phi^n(Y); \quad \text{and} \\ \phi^-(Y) &:= \bigcup_{n \geq 0} \phi^{-n}(Y). \end{aligned}$$

If $Y = \{Q_0\}$ is a singleton, then we simply write $\phi^+(Q_0)$ for $\phi^+(\{Q_0\})$. And $\mathbb{P}^1(\overline{K})_{\varphi\text{-preper}}$ denotes the set of all φ -preperiodic points on $\mathbb{P}^1(\overline{K})$, that is, of those points whose forward orbits under φ are finite sets.

Definition. We define the following:

$$(1.2) \quad [\varphi] = \left\{ \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \mid \begin{array}{l} \phi \text{ is a } K\text{-morphism of finite degree } \geq 2 \\ \text{such that } \phi \circ \varphi = \varphi \circ \phi \end{array} \right\};$$

$$(1.3) \quad \Gamma_0 = \bigcup_{\phi \in [\varphi]} \phi^+(Q_0); \quad \text{and}$$

$$(1.4) \quad \Gamma = \left(\bigcup_{\phi \in [\varphi]} \phi^-(\Gamma_0) \right) \cup \mathbb{P}^1(\overline{K})_{\varphi\text{-preper}}.$$

For background materials on arithmetical dynamical systems, see [6].

• **The Chebyshev dynamical systems.** For convenience, we give a brief review of Chebyshev polynomials.

Definition. Recursively,

$$\begin{aligned} P_1(z) &:= z, \quad P_2(z) := z^2 - 2; \quad \text{and} \\ P_{m+1}(z) + P_{m-1}(z) &= zP_m(z) \quad \text{for all } m \geq 2. \end{aligned}$$

Then a *Chebyshev polynomial* is defined to be any of the P_m ($m \geq 2$).

These polynomials satisfy the following properties (see [4, §7] and [6, Chapter 1 and Theorems 6.9/6.79] for details).

- (1) For any $m \geq 1$, $P_m(\omega + \omega^{-1}) = \omega^m + \omega^{-m}$, equivalently $P_m(2 \cos \theta) = 2 \cos(m\theta)$, where $\omega \in \mathbb{C}^\times$ and $\theta \in \mathbb{R}$.
- (2) For any $\ell, m \geq 1$, $P_\ell \circ P_m = P_{\ell m}$.
- (3) For any $m \geq 3$, P_m has $m - 1$ distinct critical points in the finite plane, but only two critical values, i.e., ± 2 .
- (4) Let $m \geq 2$ be an integer and let $\phi(z) \in K(z)$ be a rational map of degree ≥ 2 . Then $\phi \circ P_m = P_m \circ \phi$ if and only if for some integer $n \geq 2$, $\phi = P_n$ in case of m being even and $\phi = \pm P_n$ in case of m being odd.

Definition. The morphism induced by any P_m ($m \geq 2$) on \mathbb{P}^1 (or \mathbb{A}^1) is called a *Chebyshev morphism* and the resulting dynamical system is called a *Chebyshev dynamical system*.

It will be clearly understood from context which space is meant, \mathbb{P}^1 or \mathbb{A}^1 . A crucial property of Chebyshev morphisms we need to keep in mind is the existence of the following commutative diagram:

$$(*) \quad \begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1 \end{array}$$

where ψ is the morphism extending the morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m, z \mapsto z^m$ ($m \geq 2$), φ is the Chebyshev morphism induced by P_m , i.e., extending $\mathbb{A}^1 \rightarrow \mathbb{A}^1, z \mapsto P_m(z)$, and π is the morphism extending $\mathbb{G}_m \rightarrow \mathbb{A}^1, z \mapsto z + z^{-1}$.

In addition, if $m \geq 2$ is an odd integer, then we replace ψ (resp. φ) with the one induced by $z \mapsto -z^m$ (resp. $-P_m$). Then the resulting diagram is commutative, too.

• **The main conjecture and the main theorem.** Keeping Silverman’s result in [5, Theorem A] in mind, we propose the following slight variant as a conjecture.

Conjecture 1.1. *Let K be a number field and let S be a finite set of primes of K containing all the infinite ones. Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be an arbitrary K -morphism of finite degree ≥ 2 and form $[\varphi]$ as in (1.2). Let $Q_0 \in \mathbb{P}^1(K)$ and form Γ_0 as in (1.3). Suppose that $Q \in \mathbb{P}^1(K)$ is not exceptional for any $\phi \in [\varphi]$. Then we have*

$$\#(\mathbb{P}^1_{(Q)}(\mathcal{O}_{K,S}) \cap \Gamma_0) < \infty,$$

i.e., there are only a finite number of elements in Γ_0 which are S -integral on \mathbb{P}^1 relative to (Q) .

Remarks. (1) For the details of the definition of *exceptional point*, see [5, p. 807 (and p. 798)]. Note that $Q \in \mathbb{P}^1(K)$ is not exceptional for any $\phi \in [\varphi]$ if and only if Q is not a totally ramified fixed point of any $\phi \in [\varphi]$, since $\phi \in [\varphi] \Rightarrow \phi^n \in [\varphi]$ for any integer $n \geq 1$. Indeed, $[\varphi]$ is closed under composition.

(2) For the variations of the definition of $[\varphi]$, see [1, §4.2]. All the exact same comments (including the possible extensions and restrictions of the definitions of Γ_0 and Γ) as in [1, §4.2] apply to this context, but we do not repeat them here, cf. see the second remark in §2.3 below. One thing to add though is that, e.g., in case of using the ones in [1, pp. 2032–2033 (i)–(ii)], we replace the part “any $\phi \in [\varphi]$ ” by “ $\phi_n \circ \dots \circ \phi_1$ with any $\phi_1, \dots, \phi_n \in [\varphi]$ ” in Conjecture 1.1 if $[\varphi]$ need not be closed under composition. In this way we have the variants of Conjecture 1.1 corresponding to the variants of the choice for $[\varphi]$.

(3) In Conjecture 1.1 note that all the iterates φ^n ($n \geq 1$) belong to $[\varphi]$ and that for any $Q \in \mathbb{P}^1(K)$, the following equivalences hold: Q is not exceptional for φ if and only if Q is not exceptional for some φ^n ($n \geq 1$) if and only if Q is not exceptional for any φ^n ($n \geq 1$). Now look at (2) above and choose the variant $[\varphi] := \{\varphi^n : n = 1, 2, \dots\}$ (closed under composition) or $\{\varphi\}$. Then the resulting variant of Conjecture 1.1 with this new choice of $[\varphi]$ according to (2)

is nothing but Silverman’s result in [5, Theorem A]. Thus Conjecture 1.1 with the variations of the choice for $[\varphi]$ in (2) is a slight generalization of the result of Silverman.

(4) Further, keep φ as in Conjecture 1.1 and fix a finite number of any K -morphisms $\varphi_1, \dots, \varphi_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of finite degree ≥ 2 . As an imitation of the constructions in [5, Theorem B], define Φ to be the monoid generated by all the elements of $[\varphi]$ (= possibly any of the variants in (2) above) and $\varphi_1, \dots, \varphi_n$ under composition and $\Gamma_{0,\Phi}$ to be the set $\Phi(Q_0) := \{\phi(Q_0) \in \mathbb{P}^1(K) : \phi \in \Phi\}$. Finally, assume that Q is not exceptional for any $\phi \in \Phi - \{\text{id}\}$. Then it may be tempting to try to prove that $\#(\mathbb{P}_{(Q)}^1(\mathcal{O}_{K,S}) \cap \Gamma_{0,\Phi}) < \infty$, too. This would be a mixture of Conjecture 1.1 (with the variations of the choice for $[\varphi]$ in (2)) and another result of Silverman in [5, Theorem B], cf. as a reference, an object Γ_Φ corresponding to $\Gamma_{0,\Phi}$ is accordingly defined, but it is not needed here, so we do not go into its detail. In addition, it may also be tempting to go to a larger set Φ defined to be the monoid generated by all the elements of $[\varphi]$ and $[\varphi_1], \dots, [\varphi_n]$.

The main goal of this article is to affirm Conjecture 1.1 for the Chebyshev morphisms:

Theorem 1.2. *Keep the notation of Conjecture 1.1. If φ is the Chebyshev morphism induced by any P_m ($m \geq 2$), then Conjecture 1.1 holds.*

It is clear that the idea similar to the proof of Theorem 1.2 gives the corresponding result for the variant of the Chebyshev morphisms coming from the variant of Chebyshev polynomials in [3, §4.2].

2. Integral division points on \mathbb{G}_m and the proof of Theorem 1.2

In order to prove Theorem 1.2 we will mainly use two recent results on integral division points on \mathbb{G}_m , one in [1] and the other in [2]. Notice that the former does not explicitly involve heights while the latter does. Both will play a role in establishing the finiteness result in Theorem 1.2 in what follows.

2.1. Finiteness properties of integral division points on \mathbb{G}_m

We start by introducing the notion of “division group” in the context of \mathbb{G}_m . See also [1, p. 2012].

Definition. Let K be a number field and let Γ_0 be a finitely generated subgroup of $\mathbb{G}_m(\overline{K})$. Then the group

$$\Gamma := \{\xi \in \mathbb{G}_m(\overline{K}) : \xi^n \in \Gamma_0 \text{ for some integer } n \geq 1\}$$

is called the *division group* (above, over, or) attached to Γ_0 in $\mathbb{G}_m(\overline{K})$. The elements of Γ are called the *division points* above (or over) Γ_0 in $\mathbb{G}_m(\overline{K})$, or simply *division points* (in $\mathbb{G}_m(\overline{K})$), if the choice of Γ_0 is clear from context. An arbitrary subgroup of $\mathbb{G}_m(\overline{K})$ is called a *division group* in $\mathbb{G}_m(\overline{K})$ if it is the division group attached to some finitely generated subgroup of $\mathbb{G}_m(\overline{K})$.

We cite some results on integral division points on \mathbb{G}_m that we will make use of below.

Theorem 2.1. *Let K be a number field with algebraic closure \overline{K} , S a finite set of primes of K containing all the infinite ones, Γ a division group in $\mathbb{G}_m(\overline{K})$, and D an effective divisor on \mathbb{G}_m . Let h be the standard absolute logarithmic Weil height on $\mathbb{G}_m(\overline{K})$ (or on $\mathbb{P}^1(\overline{K})$) and let μ_∞ be the set of all roots of unity in \overline{K} . Then all the following statements hold:*

- (i) ([1, Theorem 2.5 (i)]). *If $\text{Supp}(D)$ contains at least two points whose quotient is not a root of unity, then the set*

$$\begin{aligned} \mathbb{G}_{m,D}(\overline{\mathcal{O}}_{k,S})_\Gamma &:= \mathbb{G}_{m,D}(\overline{\mathcal{O}}_{K,S}) \cap \Gamma \\ &= \{\gamma \in \Gamma : \gamma \text{ is } S\text{-integral on } \mathbb{G}_m \text{ relative to } D\} \end{aligned}$$

is finite

- (ii) ([2, Theorem 1.3 (ii)]). *If $\alpha \in \mathbb{G}_m(\overline{K})$ and $(\gamma_n)_{n \geq 1}$ is a sequence of distinct elements of $\mathbb{G}_{m,(\alpha)}(\overline{\mathcal{O}}_{K,S}) \cap \Gamma$, then we have*

$$\lim_{n \rightarrow \infty} h(\gamma_n) = h(\alpha).$$

- (iii) (A consequence of (ii)). *If $\alpha \in \mu_\infty$, then the set*

$$\mathbb{G}_{m,(\alpha)}(\overline{\mathcal{O}}_{K,S})_{\Gamma, \geq \epsilon} := \mathbb{G}_{m,(\alpha)}(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : h(\gamma) \geq \epsilon\}$$

is finite for any real number $\epsilon > 0$.

For future reference, note that in (i)–(iii) we can replace \mathbb{G}_m with \mathbb{P}^1 , i.e., that (i) $\#(\mathbb{P}^1_D(\overline{\mathcal{O}}_{k,S}) \cap \Gamma) < \infty$; (ii) the same convergence is true of any $\alpha \in \mathbb{G}_m(\overline{K})$ and any sequence $(\gamma_n)_{n \geq 1}$ of distinct elements of $\mathbb{P}^1_{(\alpha)}(\overline{\mathcal{O}}_{K,S}) \cap \Gamma$; and (iii) for any $\alpha \in \mu_\infty$ and any real number $\epsilon > 0$, $\#(\mathbb{P}^1_{(\alpha)}(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : h(\gamma) \geq \epsilon\}) < \infty$. See [1, p. 2012, Remark (ii)] (and also [2, Theorem 6.3 and Remark (2) of §7.1] whose remark is essentially the same as the former one) and (iii)' in §2.3 below.

2.2. The proof of Theorem 1.2

Note that the Chebyshev polynomials P_m ($m \geq 2$) and $-P_m$ have only one exceptional point ∞ and for convenience we rephrase Theorem 1.2 to prove.

Theorem 2.2 (= Rephrasing of Theorem 1.2). *Let K be a number field and let S be a finite set of primes of K containing all the infinite ones. Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be an arbitrary Chebyshev morphism and get the resulting $[\varphi]$ as in (1.2). Let $\alpha_0 \in \mathbb{A}^1(K) \subset \mathbb{P}^1(K)$ and get the resulting Γ_0 as in (1.3). Then for any $\alpha \in \mathbb{A}^1(K)$, we have*

$$\#(\mathbb{P}^1_{(\alpha)}(\mathcal{O}_{K,S}) \cap \Gamma_0) < \infty,$$

i.e., there are only a finite number of elements in Γ_0 which are S -integral on \mathbb{P}^1 relative to (α) .

In order to prove this theorem, we first introduce the following proposition, cf. V. Sookdeo's recent work in [7].

Proposition 2.3. *Keep the notation and hypotheses of Theorem 2.2. Use Γ_0 to form Γ as in (1.4).*

(i) *Suppose that α is not φ -preperiodic. Then we have*

$$\#(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \Gamma) < \infty,$$

i.e., there are only a finite number of elements in Γ which are S -integral on \mathbb{P}^1 relative to (α) .

(ii) *Suppose that α is φ -preperiodic and that \widehat{h} is the canonical height on $\mathbb{P}^1(\overline{K})$ associated to φ and an effective divisor of degree one on \mathbb{P}^1 . Then for any real number $\epsilon > 0$, we have*

$$\#(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : \widehat{h}(\gamma) \geq \epsilon\}) < \infty,$$

i.e., there are only a finite number of elements in Γ which have height $\geq \epsilon$ and which are S -integral on \mathbb{P}^1 relative to (α) .

Proof. Assume that φ is induced by the Chebyshev polynomial P_m ($m \geq 2$) and recall the commutative diagram (*):

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1 \end{array}$$

For convenience, we use affine coordinate for $\mathbb{G}_m(\overline{K}) \subset \mathbb{P}^1(\overline{K})$ and $\mathbb{A}^1(\overline{K}) \subset \mathbb{P}^1(\overline{K})$. Write $\pi^{-1}(\alpha_0) = \{a_0, a_0^{-1}\}$ and $\pi^{-1}(\alpha) = \{a, a^{-1}\}$. (Indeed, we have $a_0^2 - \alpha_0 a_0 + 1 = a^2 - \alpha a + 1 = 0$.) Let

$$\Gamma'_0 = \{a_0^n \in \overline{K}^\times : n \in \mathbb{Z}\},$$

i.e., the cyclic subgroup of $\mathbb{G}_m(\overline{K}) = \overline{K}^\times$ generated by a_0 , and let

$$\Gamma' = \{\xi \in \overline{K}^\times : \xi^n \in \Gamma'_0 \text{ for some integer } n \geq 1\},$$

i.e., the division group above Γ'_0 in $\mathbb{G}_m(\overline{K})$. Note that π induces an (at-most 2-1) map

$$\mathbb{G}_m(\overline{K}) \twoheadrightarrow \mathbb{A}^1(\overline{K})$$

such that

$$\pi^{-1}(\Gamma - \{\infty\}) \subset \Gamma',$$

which easily follows from the commutativity of the above diagram (*) where now ψ and φ run over the at-least-second-power morphisms and over the Chebyshev morphisms respectively and in case of $m \geq 2$ being odd and ψ being any odd-power morphism, the morphisms ψ and φ are also allowed to be replaced by $-\psi$ and $-\varphi$ respectively.

(i) Since α is not φ -preperiodic by hypothesis, it follows that $a \notin \mu_\infty$ and hence that $a/a^{-1} = a^2 \notin \mu_\infty$. Thus Theorem 2.1(i) and its subsequent remark imply that

$$\#(\mathbb{P}_{(a)+(a^{-1})}^1(\overline{\mathcal{O}}_{K,S}) \cap \Gamma') < \infty.$$

Since $\pi^{-1}(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \mathbb{A}^1(\overline{K})) \subset \mathbb{P}_{(a)+(a^{-1})}(\overline{\mathcal{O}}_{K,S})$, we get

$$\#(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \Gamma) < \infty,$$

as desired.

(ii) In this case we note that $a \in \mu_\infty$ in contrast to case (i). Recall from Theorem 2.1 that h is the Weil height on $\mathbb{P}^1(\overline{K})$ and let $\epsilon > 0$ be arbitrary. Then Theorem 2.1(iii) and its subsequent remark imply that

$$\#(\mathbb{P}_{(a)}^1(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma' : h(\gamma) \geq \epsilon/2\}) < \infty$$

and hence also that

$$\#(\mathbb{P}_{(a)+(a^{-1})}^1(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma' : h(\gamma) \geq \epsilon/2\}) < \infty.$$

Note that $2h = \widehat{h} \circ \pi$. Since $\pi^{-1}(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \mathbb{A}^1(\overline{K})) \subset \mathbb{P}_{(a)+(a^{-1})}^1(\overline{\mathcal{O}}_{K,S})$ again, it then follows that

$$\#(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : \widehat{h}(\gamma) \geq \epsilon\}) < \infty,$$

as desired. □

Now we are ready to finish the proof of Theorem 2.2 (= Theorem 1.2).

Back to the proof of Theorem 2.2. First, if α is not φ -preperiodic, then the desired finiteness property is immediate from Proposition 2.3(i), since

$$\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \Gamma_0 \subset \mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \Gamma.$$

Second, suppose that α is φ -preperiodic. For any real number $\epsilon > 0$, it is obvious that Proposition 2.3(ii) implies that

$$\#(\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \{\gamma \in \Gamma_0 : \widehat{h}(\gamma) \geq \epsilon\}) < \infty.$$

Since Northcott's theorem yields

$$\#(\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \{\gamma \in \Gamma_0 : \widehat{h}(\gamma) < \epsilon\}) < \infty,$$

we then get $\#(\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \Gamma_0) < \infty$ to complete the whole proof. □

2.3. Closing remarks

We would like to add a few comments. First, it is obvious that Theorem 2.1(ii) actually implies a more general statement than Theorem 2.1(iii), i.e., that under the same hypotheses and notation as in Theorem 2.1, for any $\alpha \in \mathbb{G}_m(\overline{K})$ and any real number $\epsilon > 0$, we have

$$(iii)' \quad \#(\mathbb{G}_{m,(\alpha)}(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : h(\gamma) \geq h(\alpha) + \epsilon\}) < \infty;$$

again, indeed, $\#(\mathbb{P}_{(\alpha)}^1(\overline{\mathcal{O}}_{K,S}) \cap \{\gamma \in \Gamma : h(\gamma) \geq h(\alpha) + \epsilon\}) < \infty$ for any $\alpha \in \mathbb{G}_m(\overline{K})$ and any real number $\epsilon > 0$, cf. see the remark following Theorem 2.1.

Recall the above proof of Theorem 2.2 in the case of α being φ -preperiodic. Then we can similarly get Theorem 2.2: Indeed, for any $\alpha \in \mathbb{A}^1(\overline{K})$ and for any real number $\epsilon > 0$, we have

$$\begin{aligned} \mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \Gamma_0 &= (\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \{\gamma \in \Gamma_0 : \widehat{h}(\gamma) \geq \widehat{h}(\alpha) + \epsilon\}) \cup \\ &\quad (\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \cap \{\gamma \in \Gamma_0 : \widehat{h}(\gamma) < \widehat{h}(\alpha) + \epsilon\}), \end{aligned}$$

the first set of the right-hand side of which is finite from an argument similar to the one using (iii)' and its subsequent comment in the case of α being φ -preperiodic in the above proof of Theorem 2.2 and the second set of the right-hand side of which is also finite from Northcott's theorem again. (Here we notice that $\mathbb{P}_{(\alpha)}^1(\mathcal{O}_{K,S}) \subset \mathbb{P}^1(K)$.)

However, we prefer the earlier proof (of Theorem 2.2) for two reasons; first, it relies on the more elementary prior result, namely Theorem 2.1(i), at least in the case of α being non- φ -preperiodic and second, more interestingly, the finiteness property of Proposition 2.3(i) must be noteworthy in itself. Anyway, in addition, it is easy to see that [2, (i) (or (ii)) \Rightarrow (iii) of Conjecture 7.2] implies Conjecture 1.1 together with Northcott's theorem in general.

Second, recall the commutative diagram (*). Suppose that $m \geq 2$ is even and that φ (resp. π) is replaced by $-\varphi$ (resp. $-\pi$). The resulting new diagram is still commutative. Moreover, it is equal to the original diagram with old φ and old π up to multiplication by $-1 \in \mu_\infty = \mathbb{G}_m(\overline{K})_{\text{tor}}$, which keeps Γ' unchanged in the proofs of Proposition 2.3 and Theorem 2.2. (For any $z \in \overline{K}^\times$, it is obvious that $z \in \Gamma' \Rightarrow -z \in \Gamma'$.) Therefore the argument similar to those proofs shows that Theorem 2.2 (as well as Proposition 2.3) remains true if we throw all the \pm Chebyshev morphisms (of degree even or odd) into $[\varphi]$ and accordingly enlarge Γ_0 (and Γ). Of course, if we take alternatively any Chebyshev morphism φ_0 of odd degree, then we get the set $[\varphi_0]$ in (1.2) already consisting of all the \pm Chebyshev morphisms and accordingly the enlarged Γ_0 (and Γ).

This fits in well with remark (2) following Conjecture 1.1, cf. look at, e.g., the second set of [1, §4.2 (i)] and for any Chebyshev morphism ϕ , note that $-\phi$ commutes with some (indeed, any) Chebyshev morphism ϕ_1 of odd degree and

that of course, ϕ_1 commutes with any given Chebyshev morphism φ . Alternatively, we can also refer to the first set of [1, §4.2 (i)] together with the fact that $(-\phi)^n = -\phi^n$ for any integer $n \geq 1$ if ϕ is a Chebyshev morphism of even degree, which implies that $-\phi$ and ϕ have the exact same set of preperiodic points in $\mathbb{P}^1(\overline{K})$ and hence that $\langle \varphi, -\phi \rangle = 0$ for any Chebyshev morphism φ , cf. see [1, p. 2033] for the notation $\langle \cdot, \cdot \rangle$.

Finally, as a straightforward remark, note that it is obvious that Proposition 2.3 implies both [2, Conjecture 7.3] and [1, p. 2033 (iii)] for the Chebyshev morphisms. Also note that the hypothesis of Proposition 2.3 is that α is not φ -preperiodic. Since $\mathbb{P}^1(\overline{K})_{\varphi\text{-preper}} \subset \Gamma$, this is weaker than the hypothesis that $\alpha \notin \Gamma$. Therefore as a corollary to Proposition 2.3, we also get the truth of [1, Conjecture 4.1] for the Chebyshev morphisms.

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