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Ω-RESULT ON COEFFICIENTS OF AUTOMORPHIC L-FUNCTIONS OVER SPARSE SEQUENCES

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Abstract. Let $\lambda_f(n)$ denote the n-th normalized Fourier coefficient of a primitive holomorphic form f for the full modular group $\Gamma = SL_2(\mathbb{Z})$. In this paper, we are concerned with Ω-result on the summatory function $\sum_{n\leqslant x}\lambda_f^2(n^2)$, and establish the following result

$$\sum_{n\leqslant x}\lambda_f^2(n^2) = c_1x + \Omega(x^{\frac{4}{9}}),$$

where c_1 is a suitable constant.

1. Introduction and main results

According to the Langlands program, there are many hidden structures underlying the Fourier coefficients of an automorphic form. Thus it is very important and essential to investigate its summatory function over a certain sequence.

Let H_k^* be the set of all normalized Hecke eigencuspforms of even integral weight k for the full modular group $\Gamma = SL_2(\mathbb{Z})$. For $f(z) \in H_k^*$, $f(z)$ has the following Fourier expansion at the cusp ∞

$$f(z) = \sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e^{2\pi inz},$$

where $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)}\lambda_f\left(\frac{mn}{d^2}\right) \tag{1.1}$$

for any integers $m \geq 1$ and $n \geq 1$.

The size and oscillations of $\lambda_f(n)$ deserve deep research. In 1974, Deligne

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[1] proved the Ramanujan-Peterson conjecture

$$|\lambda_f(n)| \le d(n), \tag{1.2}$$

where $d(n)$ is the Dirichlet divisor function.

Hafner and Ivić [4] obtained an O-estimate and $\Omega_\pm$-results for $\sum_{n\le x} \lambda_f(n)$. The second moment $\sum_{n\le x} |\lambda_f(n)|^2$ was treated in Rankin [16] and Selberg [21]. Subsequently, Rankin [17, 18, 19] initiated the theme of lower and upper estimates for the power moments $\sum_{n\le x} |\lambda_f(n)|^{2\beta}$ for $\beta > 0$. Lau, Lü and Wu [12] studied the summation $\sum_{n\le x} \lambda_f^j(n)$, where $j = 3, 4, 5, 6, 7, 8$, and showed that

$$\sum_{n\le x} \lambda_f^j(n) = xP_j(\log x) + O_{f,\varepsilon}(x^{\theta_j+\varepsilon}), \tag{1.3}$$

where the constants θ_j are given in Theorem 1 of Lau, Lü and Wu [12]. Denoting by $\Delta_j(f;x)$ the error term in (1.3), they also obtained the lower bound of $\Delta_j(f;x)$ using the Omega Theorem of Kühleitner and Nowak [9].

On the other hand, the sum over squares $\sum_{n\le x} \lambda_f(n^2)$ was considered in Ivić [5], Fomenko [2] and Sankaranarayanan [20]. Lü [14] obtained the bound of $\sum_{n\le x} \lambda_f(n^j)$ $(j = 3, 4)$. Lao and Sankaranarayanan [10] established the asymptotic formula of the sum $\sum_{n\le x} \lambda_f^2(n^j)$, where $j = 2, 3, 4$. By using the properties of symmetric power L-functions and their Rankin-Selberg L-function, which have been established in the references [3, 7, 8, 13, 22], they showed that

$$\sum_{n\le x} \lambda_f^2(n^j) = c_{j-1}x + O_{f,\varepsilon}(x^{1-\frac{2}{(j+1)^2+2}+\varepsilon}),$$

where $j = 2, 3, 4$.

In this paper, we are concerned with Ω-result on the error term of the asymptotic formula of $\sum_{n\le x} \lambda_f^2(n^2)$. Let

$$E(f,x) = \sum_{n\leqslant x} \lambda_f^2(n^2) - c_1 x.$$

Based on the Omega Theorem of Kühleitner and Nowak [9] (see Lemma 2.1 in Section 2), we establish the following result.

Theorem 1.1. *Let $f(z) \in H_k^*$, and $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then we have*

$$E(f,x) = \Omega(x^{\frac{4}{9}}).$$

Remark. It seems that one can consider similar omega-problems for sums over cubes or 4th powers by similar arguments. However, for the sum $\sum_{n\leqslant x} \lambda_f^2(n^3)$, the condition (C) in Lemma 2.1 is not satisfied, and for the sum $\sum_{n\leqslant x} \lambda_f^2(n^4)$, the corresponding generating function has a factor $L(sym^6 f \times sym^6 f, s)$, whose analytic properties are not clear (since the automorphy of the jth symmetric power lift of an automorphic cuspidal representation over $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is only known for $j \le 4$, see e.g. [3, 7, 8, 22]).

2. Some lemmas

In this section we recall or establish some results, which we shall use in the proof of our main result.

Lemma 2.1. *Let $a(n)$ be an arithmetic function which possesses a generating Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = F(s) = \frac{f_1(m_1 s) \cdots f_K(m_K s)}{g_1(n_1 s) \cdots g_J(n_J s)} h(s)$$

in a suitable half-plane of convergence, where

(A) *for $K \in N$ and $J \in N_0$, $m_1 \le \cdots \le m_K$ and $n_1 \le \cdots \le n_J$ are positive integers.*

(B) *for each $k = 1, \ldots, K$, $f_k(s)$ is a meromorphic function in a certain half-plane $\operatorname{Re} s > \sigma_k^*$ with at most finitely many poles, which possesses a representation as a Dirichlet series*

$$f_k(s) = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s}$$

for Re $s > 1$, *with $a_k(1) \neq 0$, $a_k(n) \ll n^{\varepsilon}$ as $n \to \infty$, for each $\varepsilon > 0$. For every $\sigma' > \sigma_k^*$,*

$$f_k(\sigma + it) \ll |t|^C \quad (as\ |t| \to \infty)$$

uniformly in $\sigma \ge \sigma'$, with an appropriate constant C depending on σ'.

Furthermore, there exist positive real numbers κ_k, $k = 1, \ldots, K$, with the property that $\sum_{k=1}^{K} \kappa_k > 1$, such that

$$\left| \frac{f_k(\sigma + it)}{f_k(1 - \sigma + it)} \right| \gg |t|^{\kappa_k (\frac{1}{2} - \sigma)} \quad (as\ |t| \to \infty)$$

on the vertical line $\sigma = m_k \alpha$, where

$$\alpha := \frac{\sum_{k=1}^{K} \kappa_k - 1}{2 \sum_{k=1}^{K} m_k \kappa_k}.$$

(C) *for $j = 1, \ldots J$ (if $J > 0$), $g_j(s)$ is a meromorphic function with at most finitely many poles in a half-plane $\operatorname{Re} s > \sigma_*^j$. For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that $g_j(s)$ has at most $O(T^{1-\delta})$ zeros in the domain $\operatorname{Re} s \ge \sigma_*^j + \varepsilon$, $0 \le \operatorname{Im} s \le T$. In a certain half-plane of convergence,*

$$g_j(s) = \sum_{n=1}^{\infty} \frac{b_j(n)}{n^s}$$

with $b_j(n) \in \mathbb{C}$, $b_j(1) \neq 0$. The inverse arithmetic function with respect to Dirichlet multiplication (denoted by $b_j^*(n)$) satisfies $b_j^*(n) \ll n^{\varepsilon}$ as $n \to \infty$, for each $\varepsilon > 0$. For every $\sigma' > \sigma_*^j$,

$$g_j(\sigma + it) \ll |t|^C \quad (as\ |t| \to \infty)$$

uniformly in $\sigma \geq \sigma^{'}$*, with an appropriate constant* C *depending on* $\sigma^{'}$.

(D) *for some* $\sigma_0 < \alpha$, $h(s)$ *is a meromorphic function on* $\{s \in \mathbb{C} : \operatorname{Re} s \geq \sigma_0\}$ *with at most finitely many poles, and satisfies*

$$h(\sigma + it) \ll |t|^C \quad (as\ |t| \to \infty)$$

uniformly in $\sigma \geq \sigma_0$*, with an appropriate constant* C*. At least in the half-plane* $\operatorname{Re} s > 1$, $h(s)$ *has a representation as a Dirichlet series*

$$h(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

with $c(1) \neq 0$, $c(n) \ll n^{\varepsilon}$ *as* $n \to \infty$*, for every* $\varepsilon > 0$.

(E) *for* $j = 1, \dots J$ *(if* $J > 0$*),* $n_j \alpha > \sigma_*^j$*, and for* $k = 1, \dots, K$,

$$\frac{\sigma_k^*}{m_k} < \alpha < \frac{1 - \sigma_k^*}{m_k}.$$

(F) $H(x)$ *is an arbitrary expression of the form*

$$H(x) = \sum_{i=1}^{I} x^{\beta_i} P_i(\log x)$$

where $\beta_i \in \mathbb{C}$, $\alpha < \operatorname{Re}\, \beta_i \leq 1$*, and* P_i *are polynomials* $(i = 1, \dots, I)$.

Under the general conditions (A)-(F)*, we have, for* $x \to \infty$,

$$E(x) := \sum_{n \leq x} a(n) - H(x) = \Omega(x^{\alpha}).$$

Proof. This is Theorem 2 in Kühleitner and Nowak [9]. □

Lemma 2.2. *Let* $f(z) \in H_k^*$*, and* $\lambda_f(n)$ *denote its* n*-th normalized Fourier coefficient. For* $j = 2, 3, 4$*, we introduce*

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)}{n^s}, \quad \operatorname{Re} s > 1. \tag{2.1}$$

Let $L(sym^j f, s)$ *be the* j*-th symmetric power* L*-function associated with* f*, and* $L(sym^j f \times sym^j f, s)$ *be the Rankin-Selberg* L*-function of* $sym^j f$ *and* $sym^j f$*. Then, we have that for* $\operatorname{Re} s > 1$,

$$L_j(s) = L(sym^j f \times sym^j f) V_j(s), \tag{2.2}$$

where $V_j(s)$ *converges uniformly and absolutely in the half-plane* $\operatorname{Re} s \geq \frac{1}{2} + \varepsilon$ *for any* $\varepsilon > 0$.

Proof. See Lemma 1.1 in Lao and Sankaranarayanan [10]. □

Lemma 2.3. *Let* $f(z) \in H_k^*$*, define* $F_j(s) = \sum_{n \geq 1} \lambda_f^j(n) n^{-s}$*, where* $\operatorname{Re} s > 1$*. Then*

$$F_j(s) = G_j(s) H_j(s) \text{ for } j = 0, 1, 2, 3, 4, 5, 6, 7, 8,$$

where

$$
\begin{aligned}
&G_0(s) = \zeta(s),\ G_1(s) = L(f,s),\ G_2(s) = \zeta(s)L(sym^2 f, s),\\
&G_3(s) = L(f,s)^2 L(sym^3 f, s),\\
&G_4(s) = \zeta(s)^2 L(sym^2 f, s)^3 L(sym^4 f, s),\\
&G_5(s) = L(f,s)^5 L(sym^3 f, s)^3 L(sym^4 f \times f, s),\\
&G_6(s) = \zeta(s)^5 L(sym^2 f, s)^8 L(sym^4 f, s)^4 L(sym^4 f \times sym^2 f, s),\\
&G_7(s) = L(f,s)^{13} L(sym^3 f, s)^8 L(sym^4 f \times f, s)^5 L(sym^4 f \times sym^3 f, s),\\
&G_8(s) = \zeta(s)^{13} L(sym^2 f, s)^{21} L(sym^4 f, s)^{13} L(sym^4 f \times sym^2 f, s)^6\\
&\qquad L(sym^4 f \times sym^4 f, s),
\end{aligned}
$$

and the function $H_j(s)$ admits a Dirichlet series convergent absolutely in $\operatorname{Re} s > \frac{1}{2}$ *and $H_j(s) \neq 0$ for* $\operatorname{Re} s = 1$.

Proof. This is Lemma 2.1 in Lau, Lü and Wu [12]. □

Lemma 2.4. *The Dirichlet series $L_2(s)$ admits the factorization*

$$L_2(s) = G(s)\psi(2s)\gamma(s),$$

where $G(s) = L(sym^2 f \times sym^2 f, s)$, $\psi(s) = \frac{G_2^8(s)}{G_0^4(s)G_4^3(s)}$, and $\gamma(s)$ is defined by a Dirichlet series that is absolutely convergent in $\operatorname{Re} s > \frac{1}{3}$. *Moreover, the meromorphic function $\psi(s)$ has no pole on the line* $\operatorname{Re} s = 1$.

Proof. In view of the multiplicity of $\lambda_f^2(n^2)$, we can write $L_2(s)$ in (2.1) as an Euler product

$$L_2(s) = \prod_p \Big(1 + \sum_{v \geq 1} \frac{\lambda_f^2(p^{2v})}{p^{vs}}\Big). \tag{2.3}$$

Calculating the logarithm of both sides in (2.3)

$$\log L_2(s) = \sum_p \log\Big(1 + \sum_{v \geq 1} \frac{\lambda_f^2(p^{2v})}{p^{vs}}\Big). \tag{2.4}$$

Applying Taylor-type formula on the right-hand side of (2.4), we learn the p-local factor of $\log L_2(s)$ is

$$\frac{\lambda_f^2(p^2)}{p^s} + \frac{\lambda_f^2(p^4) - \frac{1}{2}\lambda_f^4(p^2)}{p^{2s}} + O(p^{-3s}). \tag{2.5}$$

From [1], we learn that the Hecke L-function attached to $f(z) \in H_k^*$

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \alpha_f(p)p^{-s})^{-1}(1 - \beta_f(p)p^{-s})^{-1},$$

where $\alpha_f(p)$ and $\beta_f(p)$ satisfy

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2.6}$$

Let $f(z) \in H_k^*$, the Rankin-Selberg L-function attached to $sym^i f$ and $sym^j f$ is defined as

$$L(sym^i f \times sym^j f, s) := \prod_p \prod_{m=0}^{i} \prod_{\mu=0}^{j} (1-\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_f(p)^{j-\mu}\beta_f(p)^{\mu} p^{-s})^{-1}$$

$$= \prod_p \prod_{m=0}^{i} \prod_{\mu=0}^{j} (1-\alpha_f(p)^{(i+j)-2(m+\mu)} p^{-s})^{-1}, \quad \operatorname{Re} s > 1. \tag{2.7}$$

The product over primes also gives a Dirichlet series representation for $L(sym^i f \times sym^j f, s)$, for $\operatorname{Re} s > 1$,

$$L(sym^i f \times sym^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^i f \times sym^j f}(n)}{n^s},$$

where $\lambda_{sym^i f \times sym^j f}(n)$ is a multiplicative function. Then we have that for $\operatorname{Re} s > 1$,

$$L(sym^i f \times sym^j f, s) = \prod_p (1 + \sum_k \frac{\lambda_{sym^i f \times sym^j f}(p^k)}{p^{ks}}).$$

Taking $i = j = 2$ in (2.7), we have

$$L(sym^2 f \times sym^2 f, s) = \prod_p (1-\alpha_f(p)^4 p^{-s})^{-1}(1-\alpha_f(p)^2 p^{-s})^{-2}(1-p^{-s})^{-3}$$

$$(1-\alpha_f(p)^{-2} p^{-s})^{-2}(1-\alpha_f(p)^{-4} p^{-s})^{-1}. \tag{2.8}$$

Calculating the logarithm of both sides in (2.8), we have

$$\log L(sym^2 f \times sym^2 f, s) = \sum_p (-\log(1-\alpha_f(p)^4 p^{-s}) - 2\log(1-\alpha_f(p)^2 p^{-s})$$

$$- 3\log(1-p^{-s}) - 2\log(1-\alpha_f(p)^{-2} p^{-s})$$

$$- \log(1-\alpha_f(p)^{-4} p^{-s})). \tag{2.9}$$

Applying Taylor-type formula $\log(1-x) = \sum_{\vartheta=1}^{\infty} \frac{x^\vartheta}{\vartheta}$ on the right-hand side in (2.9), we learn p-local factor of $\log L(sym^2 f \times sym^2 f, s)$ is

$$\sum_{\vartheta=1}^{\infty} \frac{\alpha_f^{4\vartheta}(p) + 2\alpha_f^{2\vartheta}(p) + 3 + 2\alpha_f^{-2\vartheta}(p) + \alpha_f^{-4\vartheta}(p)}{\vartheta p^{\vartheta s}}$$

$$= \sum_{\vartheta=1}^{\infty} \frac{(\alpha_f^{\vartheta}(p) + \alpha_f^{-\vartheta}(p))^4 - 2(\alpha_f^{\vartheta}(p) + \alpha_f^{-\vartheta}(p))^2 + 1}{\vartheta p^{\vartheta s}}. \tag{2.10}$$

Writing $\alpha_f(p) = e^{i\theta} = \cos\theta + i\sin\theta$, (2.10) becomes

$$\sum_{\vartheta=1}^{\infty} \frac{(2\cos(\vartheta\theta))^4 - 2(2\cos(\vartheta\theta)^2) + 1}{\vartheta p^{\vartheta s}}.$$

Let $U_n(x)$ be the n-th Chebyshev polynomial of the second kind, then

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

In particular, $U_1(\cos(\vartheta\theta)) = 2\cos(\vartheta\theta)$. Thus p-local factor of $\log L(sym^2 f \times sym^2 f, s)$ is

$$\begin{aligned}&\sum_{\vartheta=1}^{\infty} \frac{U_1^4(\cos(\vartheta\theta)) - 2U_1^2(\cos(\vartheta\theta)) + 1}{\vartheta p^{\vartheta s}} \\ &= \frac{U_1^4(\cos\theta) - 2U_1^2(\cos\theta) + 1}{p^s} + \frac{\frac{1}{2} - U_1^2(\cos(2\theta)) + \frac{1}{2}U_1^4(\cos(2\theta))}{p^{2s}} + O(p^{-3s}).\end{aligned}$$

Hence, the difference between the local factors of $\log L_2(s)$ and $\log L(sym^2 f \times sym^2 f, s)$ equals

$$\begin{aligned}\Delta P &= \frac{\lambda_f^2(p^2) - (U_1^4(\cos\theta) - 2U_1^2(\cos\theta) + 1)}{p^s} \\ &\quad + \frac{\lambda_f^2(p^4) - \frac{1}{2}\lambda_f^4(p^2) - (\frac{1}{2} - U_1^2(\cos 2\theta) + \frac{1}{2}U_1^4(\cos 2\theta))}{p^{2s}} + O(p^{-3s}) \\ &=: \frac{\Delta P_1}{p^s} + \frac{\Delta P_2}{p^{2s}} + O(p^{-3s}).\end{aligned}$$

From the theory of Hecke operators, we have the following recursive relation

$$\lambda_f(p^j) = \lambda_f(p^{j-1})\lambda_f(p) - \lambda_f(p^{j-2}). \tag{2.11}$$

In view of (2.6), we know $\lambda_f(p) = 2\cos\theta = U_1(\cos\theta)$. Thus we have $\Delta P_1 = 0$. By the recursive relation (2.11), we get

$$\lambda_f^2(p^4) - \frac{1}{2}\lambda_f^4(p^2) = \frac{1}{2}\lambda_f^8(p) - 4\lambda_f^6(p) + 8\lambda_f^4(p) - 4\lambda_f^2(p) + \frac{1}{2}.$$

Observing that $U_1(\cos 2\theta) = U_1^2(\cos\theta) - 2$ and $\lambda_f(p) = U_1(\cos\theta)$, we obtain

$$\Delta P_2 = -3U_1^4(\cos\theta) + 8U_1^2(\cos\theta) - 4.$$

The local factor of $\log G_l(s)$ is

$$\sum_{v\geq 1} \frac{U_1(\cos(v\theta))^l}{vp^{vs}}, \tag{2.12}$$

which is (4.3) in [12]. From Lemma 2.3 and (2.12), we have

$$L_2(s) = \frac{L(sym^2 f \times sym^2 f, s)G_2^8(2s)}{G_0^4(2s)G_4^3(2s)}\gamma(s),$$

where $\gamma(s)$ is a Dirichlet series that is absolutely convergent in $\mathrm{Re}\, s > \frac{1}{3}$. From Lemma 7.1 in [11], we learn that $G_{2j}(s)$ has a pole of order $g_{2j} = \frac{(2j)!}{j!(j+1)!}$ at $s = 1$, i.e., $g_0 = 1,\ g_2 = 1,\ g_4 = 2,\ g_6 = 5,\ g_8 = 14$. Let

$$\psi(s) =: \frac{G_2^8(s)}{G_0^4(s)G_4^3(s)}.$$

Hence the order of $\psi(s)$ at $s=1$ is equal to -2, which shows that $\psi(s)$ has no pole on the line Re $s=1$. This completes the proof. □

3. Proof of Theorem 1.1

We apply Lemma 2.1 to the sum $\sum_{n\leqslant x}\lambda_f^2(n^2)$ and establish its Ω-result on the error term of the asymptotic formula. According to Lemma 2.4, we write

$$L_2(s)=\frac{f_1(s)}{g_1(2s)}h(s),$$

where

$$f_1(s)=L(sym^2f\times sym^2f,s),\quad g_1(s)=G_0^4(s)G_4^3(s),\quad h(s)=G_2^8(2s)\gamma(s).$$

The conditions (A)-(E) required in Lemma 2.1 will be verified with the following choice of parameters in Lemma 2.1:

$$\begin{cases} J=1,\ n_1=2,\ \sigma_*^1=2\alpha-10^{-4}\\ K=1,\ m_1=1,\ K_1=9,\ \sigma_1^*=0\\ \alpha=\frac{\sum_{k=1}^K K_k-1}{2\sum_{k=1}^K m_kK_k}=\frac{4}{9}>\frac{1}{3}.\end{cases}$$

Apparently $f_1(s)$, $g_1(s)$ and $h(s)$ are absolutely convergent Dirichlet series for Re $s>1$:

$$f_1(s)=\sum_{n\geq1}a_1(n)n^{-s},\quad g_1(s)=\sum_{n\geq1}b_1(n)n^{-s},\quad h(s)=\sum_{n\geq1}c(n)n^{-s},$$

with $a_1(1)=b_1(1)=c(1)=1$, and $a_1(n),\ b_1(n),\ b_1^*(n)\ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$ and all $n\geq1$, thanks to the Deligne inequality (1.2). Note that $b_1^*(n)$ is the inverse arithmetic function of $b_1^*(n)$ with respect to Dirichlet convolution. Conditions (A), (B) and (D) in Lemma 2.1 are quite obviously valid, for instance,

$$\left|\frac{f_1(\sigma+i\tau)}{f_1(1-\sigma+i\tau)}\right|\gg|\tau|^{9(\frac{1}{2}-\sigma)},$$

for $\sigma=\alpha$ and $|\tau|\geq1$, as the degree of $f_1(s)$ is 9.

The crucial condition (C) concerns the zero density of $g_1(s)$. Denote by $N_L(\sigma_0,T)$ the number of zeros of a generic L-function $L(s)$ in $\sigma\geq\sigma_0$ and $0\leq t\leq T$. Condition (C) will holds if $N_{g_1}(\sigma,T)\ll T^{1-\frac{1}{10}}$ when $\sigma=\sigma_*^1=2\alpha-10^{-4}$, where g_1 is a meromorphic function with at most finitely many poles in half-plane Re $s>\sigma_*^1$. To this end, we invoke [15, Theorem 1]: if $L(s)$ is in the Selberg class and of degree d, then

$$N_L(\sigma,T)\ll T^{d(1-\sigma)+\varepsilon},\ \frac{2}{d}\leq\sigma<1.$$

From Lemma 2.3, we learn that

$$G_4(s)=\zeta(s)^2L(sym^2f,s)^3L(sym^4f,s),$$

where $L(sym^4f,s)$ is in the Selberg class and has degree $d=5$, and

$$d(1-\sigma)=5(1-2\alpha+10^{-4})\ll0.9,$$

where $L(sym^2 f, s)$ is in the Selberg class and has degree $d = 3$, and

$$d(1-\sigma) = 3(1 - 2\alpha + 10^{-4}) \ll 0.9.$$

For $L(s) = \zeta(s)$, $N_L(\sigma, T) \ll T^{0.9}$. Condition (C) is hence satisfied. Condition (E) is also valid for our choice of parameters. As Lemma 2.1 is applicable, our proof of Theorem 1.1 is complete.

References

[1] P. Deligne, *La Conjecture de Weil*, Inst. Hautes Etudes Sci. Pul. Math. **43** (1974), 29–39.

[2] O. M. Fomenko, *Identities involving coefficients of automorphic L-functions*, J. Math. Sci. **133** (2006), 1749–1755.

[3] S. Gelbart and H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm. Sup. **11** (1978), no. 4, 471–552.

[4] J. L. Hafner and A. Ivić, *On sums of Fourier coefficients of cusp forms*, Enseign. Math. **35** (1989), no. 3-4, 375–382.

[5] A. Ivić, *On sums of Fourier coefficients of cusp form*, IV International Conference "Modern Problems of Number Theory and its Applications": current problems, part II(Russia) (Tula, 2001), 92–97, Mosk. Gos. Univ. im. Lomonosoua, Mekh. Mat. Fak., Moscow, 2002.

[6] H. Iwaniec, *Topics in Classical Automorphic Forms*, American Mathematical Society, Providence, RI, 1997.

[7] H. Kim, *Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2*, Appendix 1 by D. Ramakrishnan, Appendix 2 by H. Kim and P. Sarnak, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183.

[8] H. Kim and F. Shahidi, *Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2*, With an appendix by C. J. Bushnell and G. Henniart, Ann. of Math. **155** (2002), no. 3, 837–893.

[9] M. Kühleitner and W. G. Nowak, *An omega theorem for a class of arithmetic functions*, Math. Nachr. **165** (1994), 79–98.

[10] H. X. Lao and A. Sankaranarayanan, *The average behavior of Fourier coefficients of cusp forms over sparse sequences*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2557–2565.

[11] Y.-K. Lau and G. S. Lü, *Sums of Fourier coefficients of cusp forms*, Quart. J. Math. Oxford Ser. **62** (2011), no. 3, 687–716.

[12] Y.-K. Lau, G. S. Lü, and J. Wu, *Integral power sums of Hecke eigenvalues*, Acta Arith. **150** (2011), no. 2, 193–207.

[13] Y.-K. Lau and J. Wu, *A density theorem on automorphic L-functions and some applications*, Trans. Amer. Math. Soc. **359** (2006), no. 1, 441–472.

[14] G. S. Lü, *On an open problem of Sankaranarayanan*, Sci. China Math. **53** (2010), no. 5, 1319–1324.

[15] A. Mukhopadhyay and K. Srinivas, *A zero density estimate for the Selberg class*, Int. J. Number Theory **3** (2007), no. 2, 263–273.

[16] R. A. Rankin, *Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithemtical functions I. The zeros of the function $\sum_{n=1}^{\infty} \tau(n)/n^s$ on the line $\Re s = 13/2$. II. The order of the Fourier coefficients of the integral modular forms*, Proc. Cambridge Phil. Soc. **35** (1939), 351–372.

[17] ———, *Sums of powers of cusp form coefficients*, Math. Ann. **63** (1983), no. 2, 227–236.

[18] ———, *Sums of powers of cusp form coefficients II*, Math. Ann. **272** (1985), no. 4, 593–600.

[19] ———, *Sums of cusp form coefficients*, Automorphic forms and analytic number theory (Montreal, PQ, 1989), 115–121, Univ. Montreal, Montreal, QC, 1990.

[20] A. Sankaranarayanan, *On a sum involving Fourier coefficients of cusp forms*, Lithuanian Math. J. **46** (2006), no. 4, 459–474.

[21] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid **43** (1940), 47–50.

[22] F. Shahidi, *Third symmetric power L-functions for GL*(2), Compos. Math. **70** (1989), no. 3, 245–273.

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