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GENERATION OF CLASS FIELDS BY SIEGEL-RAMACHANDRA INVARIANTS

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Abstract. We show in many cases that the Siegel-Ramachandra invariants generate the ray class fields over imaginary quadratic fields. As its application we revisit the class number one problem done by Heegner and Stark, and present a new proof by making use of inequality argument together with Shimura's reciprocity law.

1. Introduction

Let K be an imaginary quadratic field with the ring of integers $\mathcal{O}_K$. For a nontrivial ideal $\mathfrak{f}$ of $\mathcal{O}_K$, we denote by $\mathrm{Cl}(\mathfrak{f})$ the ray class group modulo $\mathfrak{f}$ and write C_0 for its identity class. By class field theory there exists a unique abelian extension $K_\mathfrak{f}$ of K, called the *ray class field* modulo $\mathfrak{f}$, whose Galois group is isomorphic to $\mathrm{Cl}(\mathfrak{f})$ via the Artin reciprocity map [10, Chapter V]. In particular, the ray class field modulo $\mathcal{O}_K$ is called the *Hilbert class field* of K and is simply written by H_K.

For a rational pair $\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, the *Siegel function* $g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau)$ on the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is defined by

$$(1)\qquad g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau) = -q^{(1/2)\mathbf{B}_2(r_1)} e^{\pi i r_2(r_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q^n q_z)(1-q^n q_z^{-1}),$$

where $\mathbf{B}_2(X) = X^2 - X + 1/6$ is the second Bernoulli polynomial, $q = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. It has neither zeros nor poles on $\mathbb{H}$. If $\mathfrak{f} \neq \mathcal{O}_K$ and $C \in \mathrm{Cl}(\mathfrak{f})$, then we take any integral ideal $\mathfrak{c}$ in C and $z_1, z_2 \in \mathbb{C}$ such that $\mathfrak{f}\mathfrak{c}^{-1} = \mathbb{Z}z_1 + \mathbb{Z}z_2$ and $z = z_1/z_2 \in \mathbb{H}$. We then define the *Siegel-Ramachandra*

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invariant modulo $\mathfrak{f}$ at C by

$$g_{\mathfrak{f}}(C) = g_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(z)^{12N}, \tag{2}$$

where N is the smallest positive integer in $\mathfrak{f}$ and a, b are integers such that $1 = (a/N)z_1 + (b/N)z_2$. This value depends only on the class C [12, Chapter 2, Remark to Theorem 1.2], and lies in $K_{\mathfrak{f}}$ [12, Chapter 2, Proposition 1.3 and Chapter 11, Theorem 1.1]. Furthermore, it satisfies the transformation formula

$$g_{\mathfrak{f}}(C_1)^{\sigma(C_2)} = g_{\mathfrak{f}}(C_1C_2) \quad (C_1, C_2 \in \mathrm{Cl}(\mathfrak{f})), \tag{3}$$

where σ is the Artin reciprocity map [12, pp. 235–236].

In 1964 Ramachandra [17, Theorem 10] first constructed a primitive generator of $K_{\mathfrak{f}}$ over K for any $\mathfrak{f} \neq \mathcal{O}_K$, however, his invariant involves overly complicated product of Siegel-Ramachandra invariants and the singular values of the modular Δ-function. Thus, Lang [15, p. 292] and Schertz [20, p. 386] conjectured that the simplest invariant $g_{\mathfrak{f}}(C_0)$ would be a primitive generator of $K_{\mathfrak{f}}$ over K (or, over H_K), and Schertz gave a conditional proof [20, Theorems 3 and 4].

In this paper we shall first show in §3 that when $\mathfrak{f} = (N)$ for an integer N (≥ 2), $g_{\mathfrak{f}}(C_0)$ generates $K_{(N)}$ over H_K for almost all imaginary quadratic fields K (Theorem 3.3). We shall further develop a simple criterion for $g_{\mathfrak{f}}(C_0)$ to be a primitive generator of $K_{\mathfrak{f}}$ over K when $\mathfrak{f}$ is just a nontrivial ideal of $\mathcal{O}_K$ (Theorem 3.6 and Remark 3.7) by adopting Schertz's idea. In §4 we shall investigate some properties of Siegel-Ramachandra invariants modulo 2.

Gauss' class number one problem for imaginary quadratic fields was first solved by Heegner [9] in 1952. There was a gap in his proof which heavily relies on the singular values of the Weber functions, however, few years later complete proofs were found independently by Baker [1] and Stark [25]. Moreover, Stark [25] finally filled up the supposed gap in Heegner's proof. In §5 as an application we shall introduce a new proof (Theorems 4.8 and 5.2) by using Siegel functions and Stevenhagen's explicit description of Shimura's reciprocity law [26, §3, 6].

2. Preliminaries

First, we shall briefly review necessary basic properties of Siegel functions and Shimura's reciprocity law.

For a positive integer N let $\zeta_N = e^{2\pi i/N}$ be a primitive N-th root of unity and

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2 \pmod{N}\}$$

be the principal congruence subgroup of level N of $\mathrm{SL}_2(\mathbb{Z})$. Then its corresponding modular curve of level N is denoted by $X(N) = \Gamma(N)\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$. Furthermore, we let $\mathcal{F}_N$ be the field of meromorphic functions on $X(N)$ defined over the N-th cyclotomic field $\mathbb{Q}(\zeta_N)$. We know that $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$, where

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots \tag{4}$$

is the elliptic modular j-function, and $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}, \tag{5}$$

whose action is given as follows: For an element $\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ we decompose it into

$$\alpha = \alpha_1 \cdot \alpha_2 \text{ for some } \alpha_1 \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \alpha_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right] \text{ with } d \in (\mathbb{Z}/N\mathbb{Z})^*.$$

Then, the action of α_1 is given by a fractional linear transformation. And, α_2 acts by the rule

$$\sum_{n \gg -\infty} c_n q^{n/N} \mapsto \sum_{n \gg -\infty} c_n^{\sigma_d} q^{n/N},$$

where $\sum_{n \gg -\infty} c_n q^{n/N}$ is the Fourier expansion of a function in $\mathcal{F}_N$ and σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ defined by $\zeta_N^{\sigma_d} = \zeta_N^d$ [15, Chapter 6, §3]. Here, for later use, we observe that

$$\begin{aligned}[\mathcal{F}_N : \mathcal{F}_1] &= \#\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \\ &= \begin{cases} 6 & \text{if } N = 2, \\ (N^4/2)\prod_{p|N}(1-p^{-1})(1-p^{-2}) & \text{if } N \geq 3 \end{cases}\end{aligned} \tag{6}$$

[23, pp. 21–22].

Proposition 2.1. *For a given integer N (≥ 2) let $\{m(\mathbf{r})\}_{\mathbf{r} \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2}$ be a family of integers such that $m(\mathbf{r}) = 0$ except finitely many $\mathbf{r}$. A product of Siegel functions*

$$g(\tau) = \zeta \prod_{\mathbf{r} = \left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]} g_{\mathbf{r}}(\tau)^{m(\mathbf{r})}$$

belongs to $\mathcal{F}_N$, where $\zeta = \prod_{\mathbf{r}} e^{\pi i r_2(1-r_1)m(\mathbf{r})}$, if

$$\begin{aligned}&\sum_{\mathbf{r}} m(\mathbf{r})(Nr_1)^2 \equiv \sum_{\mathbf{r}} m(\mathbf{r})(Nr_2)^2 \equiv 0 \pmod{\gcd(2,N)\cdot N}, \\ &\sum_{\mathbf{r}} m(\mathbf{r})(Nr_1)(Nr_2) \equiv 0 \pmod{N}, \\ &\gcd(12,N)\cdot\sum_{\mathbf{r}} m(\mathbf{r}) \equiv 0 \pmod{12}.\end{aligned}$$

Proof. See [12, Chapter 3, Theorems 5.2 and 5.3]. □

Remark 2.2. Let $g(\tau)$ be an element of $\mathcal{F}_N$ for some integer N (≥ 2). If both $g(\tau)$ and $g(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$, then $g(\tau)$ is called a *modular unit* (of level N). As is well-known, $g(\tau)$ is a modular unit if and only if it has neither zeros nor poles on $\mathbb{H}$ ([12, p. 36] or [11, Theorem 2.2]). Hence any product of

Siegel functions becomes a modular unit. In particular, $g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau)^{12N/\gcd(6,N)}$ is a modular unit of level N for any $\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$.

For a real number x we denote by $\langle x \rangle$ the fractional part of x in the interval $[0, 1)$.

Proposition 2.3. *Let* $\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$ *for an integer* N (≥ 2).

(i) *We have the q-order formula*

$$\mathrm{ord}_q\ g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau) = \frac{1}{2}\mathbf{B}_2(\langle r_1 \rangle).$$

(ii) *For* $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ *with* $c > 0$ *we get the transformation formula*

$$g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau) \circ \gamma = -ie^{(\pi i/6)(a/c + d/c - 12\sum_{k=1}^{c-1}(k/c - 1/2)(\langle kd/c \rangle - 1/2))} g_{\left[\begin{smallmatrix} r_1 a + r_2 c \\ r_1 b + r_2 d \end{smallmatrix}\right]}(\tau).$$

(iii) *For* $s = \left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right] \in \mathbb{Z}^2$ *we have*

$$g_{\left[\begin{smallmatrix} r_1 + s_1 \\ r_2 + s_2 \end{smallmatrix}\right]}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 r_2 - s_2 r_1)} g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau).$$

(iv) $g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau)^{12N/\gcd(6,N)}$ *is determined only by* $\pm \left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \pmod{\mathbb{Z}^2}$.

(v) *An element* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ $(\simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1))$ *acts on it by*

$$\left(g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau)^{12N/\gcd(6,N)}\right)^{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = g_{\left[\begin{smallmatrix} r_1 a + r_2 c \\ r_1 b + r_2 d \end{smallmatrix}\right]}(\tau)^{12N/\gcd(6,N)}.$$

(vi) $g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau)$ *is integral over* $\mathbb{Z}[j(\tau)]$.

Proof. (i) See [12, p. 31].

(ii) See [12, p. 27, K1 and p. 29] and [14, Chapter IX].

(iii) See [12, p. 28, K2 and p. 29].

(iv) One can easily check this relation by the definition (1) and (iii).

(v) See [12, Chapter 2, Proposition 1.3] and (iv).

(vi) See [11, §3]. □

For an imaginary quadratic field K of discriminant d_K we let

$$\tau_K = \begin{cases} \sqrt{d_K}/2 & \text{if } d_K \equiv 0 \pmod 4, \\ (3 + \sqrt{d_K})/2 & \text{if } d_K \equiv 1 \pmod 4, \end{cases} \tag{7}$$

which generates the ring of integers $\mathcal{O}_K$ of K over $\mathbb{Z}$. Then we have

$$\min(\tau_K, \mathbb{Q}) = X^2 + BX + C = \begin{cases} X^2 - d_K/4 & \text{if } d_K \equiv 0 \pmod 4, \\ X^2 - 3X + (9 - d_K)/4 & \text{if } d_K \equiv 1 \pmod 4. \end{cases}$$

For each positive integer N we define the matrix group

$$W_{N,\tau_K} = \left\{ \begin{bmatrix} t - Bs & -Cs \\ s & t \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \mid t, s \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Proposition 2.4. *For a positive integer N we have*

$$K_{(N)} = K(h(\tau_K) \mid h \in \mathcal{F}_N \text{ is defined and finite at } \tau_K).$$

Proof. See [15, Chapter 10, Corollary to Theorem 2] or [23, Proposition 6.33]. □

Proposition 2.5 (Shimura's reciprocity law)**.** *Let K be an imaginary quadratic field. For each positive integer N, the matrix group W_{N,τ_K} gives rise to the surjection*

$$\begin{array}{ccl} W_{N,\tau_K} & \longrightarrow & \mathrm{Gal}(K_{(N)}/H_K) \\ \alpha & \mapsto & (h(\tau_K) \mapsto h^\alpha(\tau_K) \mid h(\tau) \in \mathcal{F}_N \text{ is defined and finite at } \tau_K), \end{array}$$

whose kernel is

$$\mathrm{Ker}_{N,\tau_K} = \begin{cases} \left\{ \pm\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \pm\begin{bmatrix}-1 & -3\\1 & -2\end{bmatrix}, \pm\begin{bmatrix}-2 & 3\\-1 & 1\end{bmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \pm\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \right\} & \text{otherwise.} \end{cases}$$

Proof. See [26, §3] or [7, pp. 50–51]. □

For an imaginary quadratic field K of discriminant d_K, let

$$\mathrm{C}(d_K) = \{aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y] \mid \gcd(a,b,c) = 1,\ b^2 - 4ac = d_K, \\ (-a < b \leq a < c \text{ or } 0 \leq b \leq a = c)\}$$

be the form class group of reduced quadratic forms of discriminant d_K, whose identity element is

$$\begin{cases} X^2 - (d_K/4)Y^2 & \text{if } d_K \equiv 0 \pmod 4, \\ X^2 + XY + ((1-d_K)/4)Y^2 & \text{if } d_K \equiv 1 \pmod 4 \end{cases}$$

[4, Theorems 2.8 and 3.9]. Note that if $aX^2 + bXY + cY^2 \in \mathrm{C}(d_K)$, then

$$a \leq \sqrt{|d_K|/3}$$

[4, p. 29] and the group $\mathrm{C}(d_K)$ is isomorphic to the ideal class group of K, and hence to $\mathrm{Gal}(H_K/K)$ [4, Theorem 7.7]. Thus, in particular, the class number of K is the same as the order of the group $\mathrm{C}(d_K)$, namely $[H_K : K]$. We denote it by h_K.

Proposition 2.6 (Shimura's reciprocity law)**.** *Let K be an imaginary quadratic field of discriminant d_K, and p be a prime. For each $Q = aX^2 + bXY + cY^2 \in \mathrm{C}(d_K)$ let*

$$\tau_Q = (-b + \sqrt{d_K})/2a \quad (\in \mathbb{H})$$

and u_Q be an element of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm I_2\}$ given as follows:

Case 1. $d_K \equiv 0 \pmod 4$

$$u_Q = \begin{cases} \begin{bmatrix} a & b/2 \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} -b/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a-b/2 & -c-b/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c, \end{cases}$$

Case 2. $d_K \equiv 1 \pmod 4$

$$u_Q = \begin{cases} \begin{bmatrix} a & (3+b)/2 \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} (3-b)/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a+(3-b)/2 & -c-(3+b)/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases}$$

If $h(\tau) \in \mathcal{F}_p$ is defined and finite at τ_K and $h(\tau_K) \in H_K$, then the conjugates of $h(\tau_K)$ via the action of $\mathrm{Gal}(H_K/K)$ *are given by*

$$h^{u_Q}(\tau_Q) \quad (Q \in \mathrm{C}(d_K))$$

possibly with some multiplicity.

Proof. See [26, §6] or [7, Lemma 20]. □

3. Generators of ray class fields

Let K be an imaginary quadratic field. For an integer N (≥ 2) we get

$$g_{(N)}(C_0) = g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N}$$

by the definition (2). In this section we shall show that it plays a role of primitive generator of $K_{(N)}$ over H_K (or, even over K).

Lemma 3.1. *Let* $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right] \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2$ *for an integer* $N(\geq 2)$. *If* $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right] \not\equiv \pm \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \pmod N$, *then* $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12N} \neq g_{\left[\begin{smallmatrix} s/N \\ t/N \end{smallmatrix}\right]}(\tau)^{12N}$.

Proof. Assume on the contrary that $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12N} = g_{\left[\begin{smallmatrix} s/N \\ t/N \end{smallmatrix}\right]}(\tau)^{12N}$. Since

$$\mathrm{ord}_q\ g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12N} = 6N\mathbf{B}_2(0) = \mathrm{ord}_q\ g_{\left[\begin{smallmatrix} s/N \\ t/N \end{smallmatrix}\right]}(\tau)^{12N} = 6N\mathbf{B}_2(\langle s/N \rangle)$$

by Proposition 2.3(i), we must have $s \equiv 0 \pmod N$ by the graph of $\mathbf{B}_2(X) = X^2 - X + 1/6$. And, since

$$\mathrm{ord}_q\ (g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12N})^{\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]} = \mathrm{ord}_q\ g_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{12N} = 6N\mathbf{B}_2(1/N)$$

$$= \operatorname{ord}_q \left(g_{\left[\begin{smallmatrix}0\\ t/N\end{smallmatrix}\right]}(\tau)^{12N}\right)^{\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]} = \operatorname{ord}_q g_{\left[\begin{smallmatrix}t/N\\ 0\end{smallmatrix}\right]}(\tau)^{12N} = 6N\mathbf{B}_2(\langle t/N\rangle)$$

by Proposition 2.3(ii) and (i), it follows that $t \equiv \pm 1 \pmod N$. This proves the lemma. □

Lemma 3.2. (i) *$j(\tau)$ induces a bijective map $j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$.*

(ii) *If K_1 and K_2 are distinct imaginary quadratic fields, then τ_{K_1} and τ_{K_2} are not equivalent under the action of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. (i) See [15, Chapter 3, Theorem 4].

(ii) See [15, Chapter 3, Theorem 1]. □

For a real number x we denote by $[x]$ the greatest integer that is less than or equal to x.

Theorem 3.3. *For a given integer N (≥ 2) we have*

$$\#\{\text{imaginary quadratic fields } K \mid g_{\left[\begin{smallmatrix}0\\ 1/N\end{smallmatrix}\right]}(\tau_K)^{12N} \text{ does not generate } K_{(N)} \text{ over } H_K\}$$
$$\leq \begin{cases} 12 & \text{if } N = 2,\\ ((N+1)[N/2]-1)(N^5/4)\prod_{p\mid N}(1-p^{-1})(1-p^{-2}) & \text{if } N \geq 3.\end{cases}$$

Proof. Let

$$S = \left\{\begin{bmatrix}s\\ t\end{bmatrix} \in \mathbb{Z}^2 \mid (s=0,\ 2\leq t\leq [N/2]) \text{ or } (1\leq s\leq [N/2],\ 0\leq t\leq N-1)\right\},$$

which consists of $(N+1)[N/2]-1$ elements. For each $\left[\begin{smallmatrix}s\\ t\end{smallmatrix}\right] \in S$ we consider the function

$$g(\tau) = g_{\left[\begin{smallmatrix}0\\ 1/N\end{smallmatrix}\right]}(\tau)^{12N} - g_{\left[\begin{smallmatrix}s/N\\ t/N\end{smallmatrix}\right]}(\tau)^{12N} \quad (\in \mathcal{F}_N),$$

which is nonzero by Lemma 3.1. Since $g(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$ by Proposition 2.3(vi), we have

$$\mathbf{N}_{\mathcal{F}_N/\mathcal{F}_1}(g(\tau)) = g(\tau)\prod_{\sigma\neq \mathrm{id}} g(\tau)^\sigma = P(j(\tau)) \tag{8}$$

for some nonzero polynomial $P(X) \in \mathbb{Z}[X]$.

Note by Proposition 2.3(v) that any conjugate of $g(\tau)$ under the action of $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ is of the form

$$g_{\left[\begin{smallmatrix}a/N\\ b/N\end{smallmatrix}\right]}(\tau)^{12N} - g_{\left[\begin{smallmatrix}c/N\\ d/N\end{smallmatrix}\right]}(\tau)^{12N} \quad \text{for some } \begin{bmatrix}a\\ b\end{bmatrix}, \begin{bmatrix}c\\ d\end{bmatrix} \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2,$$

which is holomorphic on $\mathbb{H}$. Now, let

$$Z_{\left[\begin{smallmatrix}s\\ t\end{smallmatrix}\right]} = \{\text{imaginary quadratic fields } K \mid g(\tau_K) = 0\}.$$

If $K \in Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]}$, then (8) gives $P(j(\tau_K)) = 0$, from which we obtain by Lemma 3.2

$$\#Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]} \le \deg P(X).$$

On the other hand, since

$$\begin{aligned} &\mathrm{ord}_q\ (g_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau)^{12N} - g_{\left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right]}(\tau)^{12N}) \\ \ge&\ \min\left\{6N\mathbf{B}_2(\langle a/N\rangle), 6N\mathbf{B}_2(\langle c/N\rangle)\right\} \quad \text{by Proposition 2.3(i)} \\ \ge&\ 6N\mathbf{B}_2(1/2) \quad \text{by the graph of } \mathbf{B}_2(X) = X^2 - X + 1/6 \\ =&\ -N/2, \end{aligned}$$

we deduce that

$$\begin{aligned} \mathrm{ord}_q\ P(j(\tau)) &= \mathrm{ord}_q\ \mathbf{N}_{\mathcal{F}_N/\mathcal{F}_1}(g(\tau)) \\ &\ge -(N/2)\cdot[\mathcal{F}_N : \mathcal{F}_1] \\ &= -(N/2)\cdot \#\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \quad \text{by (5)} \\ &= \begin{cases} -6 & \text{if } N = 2, \\ -(N^5/4)\prod_{p\mid N}(1-p^{-1})(1-p^{-2}) & \text{if } N \ge 3 \end{cases} \quad \text{by (6)}. \end{aligned}$$

Thus we get from the fact $\mathrm{ord}_q\ j(\tau) = -1$ that

$$\#Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]} \le \deg P(X) \le \begin{cases} 6 & \text{if } N = 2, \\ (N^5/4)\prod_{p\mid N}(1-p^{-1})(1-p^{-2}) & \text{if } N \ge 3. \end{cases}$$

And, if we let

$$Z = \bigcup_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]\in S} Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]},$$

then

$$\begin{aligned} \#Z &\le \sum_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]\in S} \#Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]} \le \#S \cdot \max_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]\in S}\{\#Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]}\} \\ &\le \begin{cases} 12 & \text{if } N = 2, \\ ((N+1)[N/2]-1)(N^5/4)\prod_{p\mid N}(1-p^{-1})(1-p^{-2}) & \text{if } N \ge 3. \end{cases} \end{aligned}$$

Now, let K be an imaginary quadratic field lying outside Z. Then the singular value $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N}$ generates $K_{(N)}$ over H_K. Indeed, suppose that it does not generate $K_{(N)}$ over H_K. Then there exists a non-identity element $\alpha = \left[\begin{smallmatrix} t-Bs & -Cs \\ s & t \end{smallmatrix}\right]$ of $W_{N,\tau_K}/\mathrm{Ker}_{N,\tau_K}$ ($\simeq \mathrm{Gal}(K_{(N)}/H_K)$) in Proposition 2.5 which fixes $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N}$. Here we may assume that $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ belongs to S because $W_{N,\tau_K}/\mathrm{Ker}_{N,\tau_K}$ is a subgroup or a quotient of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. We derive that

$$0 = g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N} - (g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N})^{\alpha}$$

$$= g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N} - (g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12N})^\alpha(\tau_K) \quad \text{by Proposition 2.5}$$
$$= g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N} - g_{\left[\begin{smallmatrix} s/N \\ t/N \end{smallmatrix}\right]}(\tau_K)^{12N} \quad \text{by Proposition 2.3(iv) and (v).}$$

But this implies that K belongs to $Z_{\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]}$ ($\subseteq Z$), which yields a contradiction. Therefore we conclude that

$$\{\text{imaginary quadratic fields } K \mid g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12N} \text{ does not generate } K_{(N)} \text{ over } H_K\} \subseteq Z.$$

This completes the proof. □

Let K be an imaginary quadratic field and $\mathfrak{f}$ be a nontrivial ideal of $\mathcal{O}_K$. For a character χ of $\mathrm{Cl}(\mathfrak{f})$ we let $\mathfrak{f}_\chi$ be the conductor of χ and χ_0 be the proper character of $\mathrm{Cl}(\mathfrak{f}_\chi)$ corresponding to χ. If $\mathfrak{f} \neq \mathcal{O}_K$ and χ is also a nontrivial character of $\mathrm{Cl}(\mathfrak{f})$, then we define the *Stickelberger element*

$$S_\mathfrak{f}(\chi, g_\mathfrak{f}) = \sum_{C \in \mathrm{Cl}(\mathfrak{f})} \chi(C) \log |g_\mathfrak{f}(C)|,$$

and the *L-function*

$$L_\mathfrak{f}(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi([\mathfrak{a}])}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})^s} \quad (s \in \mathbb{C}),$$

where $\mathfrak{a}$ runs over all nontrivial ideals of $\mathcal{O}_K$ relatively prime to $\mathfrak{f}$ and $[\mathfrak{a}]$ is the class containing $\mathfrak{a}$.

Proposition 3.4 (The second Kronecker limit formula). *If $\mathfrak{f}_\chi \neq \mathcal{O}_K$, then we have*

$$L_{\mathfrak{f}_\chi}(1, \chi_0) \prod_{\mathfrak{p} \mid \mathfrak{f},\ \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \overline{\chi}_0([\mathfrak{p}])) = -\frac{\pi \chi_0([\gamma \mathfrak{d}_K \mathfrak{f}_\chi])}{3N(\mathfrak{f}_\chi)\sqrt{-d_K}\,\omega(\mathfrak{f}_\chi) T_\gamma(\overline{\chi}_0)}\ S_\mathfrak{f}(\overline{\chi}, g_\mathfrak{f}),$$

where $\mathfrak{d}_K$ is the different of $K/\mathbb{Q}$, γ is a nonzero element of K so that $\gamma \mathfrak{d}_K \mathfrak{f}_\chi$ becomes an ideal of $\mathcal{O}_K$ relatively prime to $\mathfrak{f}_\chi$, $N(\mathfrak{f}_\chi)$ is the smallest positive integer in $\mathfrak{f}_\chi$, $\omega(\mathfrak{f}_\chi) = |\{\zeta \in \mathcal{O}_K^\times \mid \zeta \equiv 1 \pmod{\mathfrak{f}_\chi}\}|$ and

$$T_\gamma(\overline{\chi}_0) = \sum_{x + \mathfrak{f}_\chi \in \pi_{\mathfrak{f}_\chi}(\mathcal{O}_K)^\times} \overline{\chi}_0([x\mathcal{O}_K]) e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(x\gamma)}.$$

Proof. See [15, Chapter 22, Theorems 1 and 2] and [12, Chapter 11, Theorem 2.1]. □

Remark 3.5. (i) The Euler factor $\prod_{\mathfrak{p} \mid \mathfrak{f},\ \mathfrak{p} \nmid \mathfrak{f}_\chi}(1 - \overline{\chi}_0([\mathfrak{p}]))$ is understood to be 1 if there is no prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \mid \mathfrak{f}$ and $\mathfrak{p} \nmid \mathfrak{f}_\chi$.

(ii) As is well-known, $L_{\mathfrak{f}_\chi}(1, \chi_0) \neq 0$ [10, Chapter IV, Proposition 5.7].

Theorem 3.6. *Let K be an imaginary quadratic field and $\mathfrak{f}$ be a nontrivial proper ideal of $\mathcal{O}_K$ whose prime ideal factorization is given by*

$$\mathfrak{f} = \prod_{k=1}^{n} \mathfrak{p}_k^{e_k}.$$

Assume that

$$[K_\mathfrak{f} : K] > 2\sum_{k=1}^{n}[K_{\mathfrak{f}\mathfrak{p}_k^{-e_k}} : K]. \tag{9}$$

Then $g_\mathfrak{f}(C_0)$ generates $K_\mathfrak{f}$ over K.

Proof. Set $F = K(g_\mathfrak{f}(C_0))$. We then derive that

$$\begin{aligned} &\#\{\text{characters } \chi \text{ of } \mathrm{Gal}(K_\mathfrak{f}/K) \mid \chi|_{\mathrm{Gal}(K_\mathfrak{f}/F)} \neq 1\} \\ &= \#\{\text{characters } \chi \text{ of } \mathrm{Gal}(K_\mathfrak{f}/K)\} - \#\{\text{characters } \chi \text{ of } \mathrm{Gal}(F/K)\} \\ &= [K_\mathfrak{f} : K] - [F : K]. \end{aligned} \tag{10}$$

Furthermore, we have

$$\begin{aligned} &\#\{\text{characters } \chi \text{ of } \mathrm{Gal}(K_\mathfrak{f}/K) \mid \mathfrak{p}_k \nmid \mathfrak{f}_\chi \text{ for some } k\} \\ &= \#\{\text{characters } \chi \text{ of } \mathrm{Gal}(K_\mathfrak{f}/K) \mid \mathfrak{f}_\chi \,|\, \mathfrak{f}\mathfrak{p}_k^{-e_k} \text{ for some } k\} \\ &\leq \sum_{k=1}^{n} \#\{\text{characters } \chi \text{ of } \mathrm{Gal}(K_{\mathfrak{f}\mathfrak{p}_k^{-e_k}}/K)\} \\ &= \sum_{k=1}^{n}[K_{\mathfrak{f}\mathfrak{p}_k^{-e_k}} : K]. \end{aligned} \tag{11}$$

Now, suppose that F is properly contained in $K_\mathfrak{f}$. Then we get by the assumption (9) that

$$\begin{aligned} [K_\mathfrak{f} : K] - [F : K] &= [K_\mathfrak{f} : K](1 - 1/[K_\mathfrak{f} : F]) \\ &> 2\sum_{k=1}^{n}[K_{\mathfrak{f}\mathfrak{p}_k^{-e_k}} : K](1 - 1/2) \\ &= \sum_{k=1}^{n}[K_{\mathfrak{f}\mathfrak{p}_k^{-e_k}} : K]. \end{aligned}$$

This, together with (10) and (11), implies that there exists a character χ of $\mathrm{Gal}(K_\mathfrak{f}/K)$ such that

$$\chi|_{\mathrm{Gal}(K_\mathfrak{f}/F)} \neq 1, \tag{12}$$

$$\mathfrak{p}_k \,|\, \mathfrak{f}_\chi \quad \text{for all } k = 1, \ldots, n. \tag{13}$$

Identifying $\mathrm{Cl}(\mathfrak{f})$ and $\mathrm{Gal}(K_\mathfrak{f}/K)$ via the Artin reciprocity map, we obtain from Proposition 3.4 and (13) that

$$0 \neq L_{\mathfrak{f}_\chi}(1,\chi_0) = -\frac{\pi\chi_0([\gamma\mathfrak{d}_K\mathfrak{f}_\chi])}{3N(\mathfrak{f}_\chi)\sqrt{-d_K}\omega(\mathfrak{f}_\chi)T_\gamma(\overline{\chi}_0)}\ S_\mathfrak{f}(\overline{\chi},g_\mathfrak{f}). \tag{14}$$

On the other hand, we achieve that

$$\begin{aligned}
S_\mathfrak{f}(\overline{\chi},g_\mathfrak{f}) &= \sum_{C\in\mathrm{Cl}(\mathfrak{f})}\overline{\chi}(C)\log|g_\mathfrak{f}(C_0)^C| \quad \text{by (3)}\\
&= \sum_{\substack{C_1\in\mathrm{Gal}(K_\mathfrak{f}/K)\\ C_1 \ (\mathrm{mod}\ \mathrm{Gal}(K_\mathfrak{f}/F))}}\ \sum_{C_2\in\mathrm{Gal}(K_\mathfrak{f}/F)}\overline{\chi}(C_1C_2)\log|g_\mathfrak{f}(C_0)^{C_1C_2}|\\
&= \sum_{C_1}\sum_{C_2}\overline{\chi}(C_1)\overline{\chi}(C_2)\log|(g_\mathfrak{f}(C_0)^{C_2})^{C_1}|\\
&= \sum_{C_1}\overline{\chi}(C_1)\log|g_\mathfrak{f}(C_0)^{C_1}|(\sum_{C_2}\overline{\chi}(C_2)) \quad \text{by the fact } g_\mathfrak{f}(C_0)\in F\\
&= 0 \quad \text{by (12)},
\end{aligned}$$

which contradicts (14). Therefore, we conclude $F = K_\mathfrak{f}$ as desired. □

Remark 3.7. (i) For a nontrivial integral ideal $\mathfrak{f}$ of an imaginary quadratic field K, we have a degree formula

$$[K_\mathfrak{f} : K] = \frac{h_K\varphi(\mathfrak{f})\omega(\mathfrak{f})}{\omega_K}, \tag{15}$$

where φ is the (multiplicative) Euler function for ideals, namely

$$\varphi(\mathfrak{p}^n) = (\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})-1)\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})^{n-1}$$

for a prime ideal power $\mathfrak{p}^n$ $(n \geq 1)$, $\omega(\mathfrak{f})$ is the number of roots of unity in K which are $\equiv 1 \pmod{\mathfrak{f}}$ and ω_K is the number of roots of unity in K [16, Chapter VI, Theorem 1].

Let N (≥ 2) be an integer whose prime factorization is given by

$$N = \prod_{a=1}^{A}p_a^{u_a}\prod_{b=1}^{B}q_b^{v_b}\prod_{c=1}^{C}r_c^{w_c} \quad (A,B,C,u_a,v_b,w_c \geq 0),$$

where each p_a (respectively, q_b and r_c) splits (respectively, is inert and ramified) in K. One can then verify that the condition

$$4\sum_{a=1}^{A}\frac{1}{(p_a-1)p_a^{u_a-1}} + 2\sum_{b=1}^{B}\frac{1}{(q_b^2-1)q_b^{2(v_b-1)}} + 2\sum_{c=1}^{C}\frac{1}{(r_c-1)r_c^{2w_c-1}} < \frac{\omega((N))}{\omega_K}$$

implies the assumption (9) when $\mathfrak{f} = (N)$.

(ii) Let d_k $(k = 1,\ldots,n)$ be the exponent of the group $(\mathcal{O}_K/\mathfrak{p}_k^{e_k})^\times$. Schertz [20, Theorem 3] proved that if the conductor of the extension $K_\mathfrak{f}/K$ is exactly

$\mathfrak{f}$, then $g_{\mathfrak{f}}(C_0)$ is a primitive generator of $K_{\mathfrak{f}}$ over K in either case when $n=1$ or

$$d_k \nmid 2 \ (k=1,\ldots,n-1),\ d_n \nmid 2\omega_K \text{ and } \mathfrak{p}_n^{e_n} \nmid \gcd(6,\omega_K). \tag{16}$$

Note that we do not require any condition on the conductor of the extension $K_{\mathfrak{f}}/K$.

4. Siegel-Ramachandra invariants of conductor 2

Throughout this section we let K be an imaginary quadratic field. We shall examine certain properties of the singular value $g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)$ which is a 24-th root of $g_{(2)}(C_0)$. Although most of the results here are classical and known, we will present relatively short and new proofs purely in terms of Siegel functions.

By the definition (1) we have

$$\begin{aligned} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau) &= 2\zeta_4 q^{1/12}\prod_{n=1}^{\infty}(1+q^n)^2,\\ g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau) &= -q^{-1/24}\prod_{n=1}^{\infty}(1-q^{n-1/2})^2,\\ g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau) &= \zeta_8^3 q^{-1/24}\prod_{n=1}^{\infty}(1+q^{n-1/2})^2. \end{aligned} \tag{17}$$

Let $\gamma_2(\tau)$ be the cube root of $j(\tau)$ whose Fourier expansion begins with the term $q^{-1/3}$.

Lemma 4.1. (i) *We have the identity*

$$g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau) = 2\zeta_8.$$

(ii) *We have the relations*

$$\gamma_2(\tau) = \frac{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}+16}{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^4} = \frac{g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^{12}+16}{g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^4} = \frac{g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau)^{12}+16}{g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau)^4}.$$

Proof. (i) We obtain by (17)

$$\begin{aligned} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau) &= 2\zeta_8\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n-1})^2\\ &= 2\zeta_8\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)^2}\cdot\frac{(1-q^n)^2}{(1-q^{2n})^2} = 2\zeta_8. \end{aligned}$$

(ii) Since $g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12} \in \mathcal{F}_2$ by Proposition 2.1 and

$$\mathrm{Gal}(\mathcal{F}_2/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})/\{\pm I_2\}$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

by (5), we derive that

$$\begin{aligned}
&\prod_{\sigma \in \mathrm{Gal}(\mathcal{F}_2/\mathcal{F}_1)} (X - (g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12})^\sigma) \\
&= (X - g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12})^2 (X - g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)^{12})^2 (X - g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12})^2 \\
&\qquad \text{by Proposition 2.3(iv) and (v)} \\
&= (X^3 + 48X^2 + (-q^{-1} + 24 - 196884q + \cdots)X + 4096)^2 \quad \text{by (17)} \\
&= (X^3 + 48X^2 + (-j(\tau) + 768)X + 4096)^2 \quad \text{by (4)} \\
&= ((X+16)^3 - j(\tau)X)^2 \\
&= ((X+16)^3 - \gamma_2(\tau)^3 X)^2.
\end{aligned}$$

Hence we get

$$\gamma_2(\tau) = \xi_1 \frac{g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12} + 16}{g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^4} = \xi_2 \frac{g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)^{12} + 16}{g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)^4} = \xi_3 \frac{g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12} + 16}{g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau)^4}$$

for some cube roots of unity ξ_k $(k = 1, 2, 3)$. Comparing the leading terms of Fourier expansions we conclude $\xi_1 = \xi_2 = \xi_3 = 1$. □

Remark 4.2. Let

$$\eta(\tau) = \sqrt{2\pi}\zeta_8 q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

be the Dedekind eta function, and

$$\mathfrak{f}(\tau) = \zeta_{48}^{-1} \frac{\eta((\tau+1)/2)}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}$$

be the Weber functions. Then one can deduce the following identities

$$\mathfrak{f}(\tau)^2 = \zeta_8^5 g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau), \quad \mathfrak{f}_1(\tau)^2 = -g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau), \quad \mathfrak{f}_2(\tau)^2 = \zeta_4^3 g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau), \tag{18}$$

and hence Lemma 4.1(ii) can be reformulated in terms of the Weber functions as in the classical case [4, Theorem 12.17].

Lemma 4.3. *If x is a real algebraic integer, then* $\min(x, K)$ *has integer coefficients.*

Proof. Since $x \in \mathbb{R}$, we get

$$[K(x) : K] = \frac{[K(x) : \mathbb{Q}(x)] \cdot [\mathbb{Q}(x) : \mathbb{Q}]}{[K : \mathbb{Q}]} = [\mathbb{Q}(x) : \mathbb{Q}],$$

from which it follows that $\min(x, K) = \min(x, \mathbb{Q})$. Furthermore, $\min(x, K)$ has integer coefficients, because x is an algebraic integer. □

Proposition 4.4. *Let K be an imaginary quadratic field of discriminant d_K.*

(i) *$j(\tau_K)$ is a real algebraic integer which generates H_K over K.*

(ii) *If p is a prime dividing the discriminant of $\min(j(\tau_K), K)$, then $(\frac{d_K}{p}) \neq 1$ and $p \leq |d_K|$.*

Proof. (i) See [15, Chapter 5, Theorem 4 and Chapter 10, Theorem 1].
(ii) See [8], [6] or [4, Theorem 13.28]. □

Remark 4.5. For any $\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, $g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau_K)$ is an algebraic integer by Propositions 2.3(vi) and 4.4(i).

Theorem 4.6. *Let $K \neq \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1})$ and set $x = \mathbf{N}_{K_{(2)}/H_K}(g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12})$. Assume that 2 is not inert in K (equivalently, $d_K \equiv 0 \pmod 4$ or $d_K \equiv 1 \pmod 8$).*

(i) *x generates H_K over K.*

(ii) *x is a real algebraic integer dividing 2^{12} whose minimal polynomial $\min(x, K)$ has integer coefficients.*

(iii) *If p is an odd prime dividing the discriminant of $\min(x, K)$, then $(\frac{d_K}{p}) \neq 1$ and $p \leq |d_K|$.*

Proof. (i) Since $g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12} \in \mathcal{F}_2$, $g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12}$ lies in $K_{(2)}$ by Proposition 2.4. Moreover, we have

$$[K_{(2)} : H_K] = \begin{cases} 2 & \text{if } d_K \equiv 0 \pmod 4, \\ 1 & \text{if } d_K \equiv 1 \pmod 8) \end{cases}$$

by the degree formula (15), and

$$\begin{aligned} \mathrm{Gal}(K_{(2)}/H_K) &\simeq W_{2,\tau_K}/\mathrm{Ker}_{2,\tau_K} \\ &= \begin{cases} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\} & \text{if } d_K \equiv 0 \pmod 8, \\ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} & \text{if } d_K \equiv 4 \pmod 8, \\ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} & \text{if } d_K \equiv 1 \pmod 8, \end{cases} \end{aligned}$$

by Proposition 2.5. So we obtain

$$\begin{aligned} x &= \mathbf{N}_{K_{(2)}/H_K}(g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12}) \\ &= \begin{cases} g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} & \text{if } d_K \equiv 0 \pmod 8, \\ g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau_K)^{12} & \text{if } d_K \equiv 4 \pmod 8, \\ g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} & \text{if } d_K \equiv 1 \pmod 8) \end{cases} \end{aligned} \tag{19}$$

by Propositions 2.5 and 2.3(iv), (v); and hence

$$j(\tau_K) = \begin{cases} (256-x)^3/x^2 & \text{if } d_K \equiv 0 \pmod 4, \\ (x+16)^3/x & \text{if } d_K \equiv 1 \pmod 8 \end{cases} \tag{20}$$

by Lemma 4.1. Therefore x generates H_K over K by Proposition 4.4(i).

(ii) We see that $x \in \mathbb{R}$ by the definition (7), (17) and (19). Furthermore, since x is an algebraic integer by Remark 4.5, $\min(x, K)$ has integer coefficients by Lemma 4.3. And, x divides 2^{12} by (19) and Lemma 4.1(i).

(iii) If $h_K = 1$, there is nothing to prove. So we assume $h_K > 1$. If σ_1 and σ_2 are distinct elements of $\mathrm{Gal}(H_K/K)$, then we derive from (20) that

$$\begin{aligned} & j(\tau_K)^{\sigma_1} - j(\tau_K)^{\sigma_2} \\ &= \begin{cases} (x_1-x_2)(-x_1^2x_2^2 + 196608x_1x_2 - 16777216x_1 - 16777216x_2)/x_1^2x_2^2 \\ \quad \text{if } d_K \equiv 0 \pmod 4, \\ (x_1-x_2)(x_1^2x_2 + x_1x_2^2 + 48x_1x_2 - 4096)/x_1x_2 \\ \quad \text{if } d_K \equiv 1 \pmod 8, \end{cases} \end{aligned}$$

where $x_1 = x^{\sigma_1}$ and $x_2 = x^{\sigma_2}$. Observe from (ii) that there is no prime ideal $\mathfrak{p}$ of H_K which contains x_1x_2 and lies above an odd prime. Therefore, if p is an odd prime dividing the discriminant of $\min(x, K)$, then $(\frac{d_K}{p}) \neq 1$ and $p \leq |d_K|$ by Proposition 4.4(ii). □

Remark 4.7. Let $K \neq \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1})$. Since $g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{24}$ generates $K_{(2)}$ over K by Theorem 3.6 and Remark 3.7, so does $g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12}$. If 2 is inert in K, then

$$\mathrm{Gal}(K_{(2)}/H_K) \simeq W_{2,\tau_K}/\mathrm{Ker}_{2,\tau_K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

by Proposition 2.5. And, we derive from Proposition 2.3(iv), (v) and Lemma 4.1(i) that

$$\mathbf{N}_{K_{(2)}/H_K}(g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12}) = g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12} = -2^{12}.$$

Therefore, in this case one cannot develop a theory like Theorem 4.6 with $\mathbf{N}_{K_{(2)}/H_K}(g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)^{12})$.

Theorem 4.8. *Let K be an imaginary quadratic field of discriminant d_K. Assume that 2 is inert and 3 is not ramified in K (equivalently, $d_K \equiv 5 \pmod 8$ and $d_K \not\equiv 0 \pmod 3$).*

(i) *The real algebraic integer $\zeta_8 g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)$ generates $K_{(2)}$ over H_K.*

(ii) *The real algebraic integer $\gamma_2(\tau_K)$ generates H_K over K.*

Proof. (i) Let $\alpha = g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)$. It is an algebraic integer by Remark 4.5, and $\zeta_8\alpha \in \mathbb{R}$ by the definition (7) and (17). Since α^4 is a real cube root of α^{12}, we get from Remark 4.7 that

$$\begin{aligned}[K_{(2)}(\alpha^4) : K_{(2)}] = [K(\alpha^4) : K(\alpha^{12})] &= \frac{[K(\alpha^4) : \mathbb{Q}(\alpha^4)][\mathbb{Q}(\alpha^4) : \mathbb{Q}(\alpha^{12})]}{[K(\alpha^{12}) : \mathbb{Q}(\alpha^{12})]} \\ &= [\mathbb{Q}(\alpha^4) : \mathbb{Q}(\alpha^{12})] = 1 \text{ or } 3.\end{aligned}$$

Furthermore, since $g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^4 \in \mathcal{F}_6$ by Proposition 2.1, we get $\alpha^4 \in K_{(6)}$ by Proposition 2.4, from which it follows that $[K_{(2)}(\alpha^4) : K_{(2)}]$ divides

$$[K_{(6)} : K_{(2)}] = \begin{cases} 2 & \text{if } 3 \text{ splits in } K, \\ 4 & \text{if } 3 \text{ is inert in } K \end{cases}$$

by the degree formula (15). Hence $[K_{(2)}(\alpha^4) : K_{(2)}] = 1$, which implies $\alpha^4 \in K_{(2)}$.

On the other hand, since $\zeta_8^{-1} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^3 \in \mathcal{F}_8$ by Proposition 2.1, we obtain $\zeta_8^{-1}\alpha^3 \in K_{(8)}$ by Proposition 2.4. One can then readily check by Proposition 2.5 that

$$\begin{aligned}&\mathrm{Gal}(K_{(8)}/K_{(2)}) \\ &\simeq \left\langle \begin{bmatrix} 5 & 4 \\ 4 & 1 \end{bmatrix} \right\rangle \times \begin{cases} \left\langle \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} \right\rangle & \text{if } d_K \equiv 5 \pmod{16}, \\ \left\langle \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix} \right\rangle & \text{if } d_K \equiv 13 \pmod{16} \end{cases} \quad (\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}).\end{aligned}$$

Decomposing $\left[\begin{smallmatrix}5&4\\4&1\end{smallmatrix}\right] = \left[\begin{smallmatrix}5&12\\12&29\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]$, we deduce that

$$\begin{aligned}(\zeta_8^{-1}\alpha^3)^{\left[\begin{smallmatrix}5&4\\4&1\end{smallmatrix}\right]} &= (\zeta_8^{-1} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^3)^{\left[\begin{smallmatrix}5&12\\12&29\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]}(\tau_K) \quad \text{by Proposition 2.5} \\ &= (\zeta_8^{-1} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^3 \circ \left[\begin{smallmatrix}5&12\\12&29\end{smallmatrix}\right])^{\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]}(\tau_K) \\ &= (\zeta_8^{-1} g_{\left[\begin{smallmatrix}6\\29/2\end{smallmatrix}\right]}(\tau)^3)^{\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]}(\tau_K) \quad \text{by Proposition 2.3(ii)} \\ &= (\zeta_8^{3} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^3)^{\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]}(\tau_K) \quad \text{by Proposition 2.3(iii)} \\ &= (8\zeta_8 q^{1/4} \prod_{n=1}^{\infty}(1+q^n)^6)^{\left[\begin{smallmatrix}1&0\\0&5\end{smallmatrix}\right]}(\tau_K) \quad \text{by (17)} \\ &= (8\zeta_8^5 q^{1/4} \prod_{n=1}^{\infty}(1+q^n)^6)(\tau_K) \\ &= \zeta_8^{-1} g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)^3 \\ &= \zeta_8^{-1}\alpha^3.\end{aligned}$$

In a similar way, one can verify that $\zeta_8^{-1}\alpha^3$ is invariant under the actions of

$$\begin{bmatrix}7 & 6\\ 2 & 1\end{bmatrix} = \begin{bmatrix}7 & 2\\ 10 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix} \text{ and } \begin{bmatrix}7 & 2\\ 2 & 1\end{bmatrix} = \begin{bmatrix}23 & 14\\ 18 & 11\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}.$$

Thus, $\zeta_8^{-1}\alpha^3$ lies in $K_{(2)}$, so does $\alpha^4/\zeta_8^{-1}\alpha^3 = \zeta_8\alpha$. Lastly, since α^{12} generates $K_{(2)}$ over K, so does $\zeta_8\alpha$. Therefore, we are done.

(ii) Let $\alpha = g_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K)$. Since $\gamma_2(\tau_K) = (\alpha^{12}+16)/\alpha^4$ by Lemma 4.1(ii) and $\alpha^4 \in K_{(2)} \cap \mathbb{R}$ by (i), we have $\gamma_2(\tau_K) \in K_{(2)} \cap \mathbb{R}$. Note from Remark 4.7 that $g_{\left[\begin{smallmatrix}1/2\\ 0\end{smallmatrix}\right]}(\tau_K)^{12}$ and $g_{\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right]}(\tau_K)^{12}$ are the two conjugates of α^{12} over H_K. In particular, $g_{\left[\begin{smallmatrix}1/2\\ 0\end{smallmatrix}\right]}(\tau_K)^4$ and $g_{\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right]}(\tau_K)^4$ belong to $K_{(2)}$ by Lemma 4.1(ii). Hence, the other two conjugates of α^4 over H_K are $\xi_1 g_{\left[\begin{smallmatrix}1/2\\ 0\end{smallmatrix}\right]}(\tau_K)^4$ and $\xi_2 g_{\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right]}(\tau_K)^4$ for some cube roots of unity ξ_1, ξ_2. If ζ_3 lies in $K_{(2)}$, then 3 ramifies in $K_{(2)}$ (but not in K by hypothesis), which contradicts the fact that all prime ideals of K which are ramified in $K_{(2)}$ must divide (2). So we get $\xi_1 = \xi_2 = 1$. And, Lemma 4.1(ii) shows that $\gamma_2(\tau_K)$ is invariant under the action of $\mathrm{Gal}(K_{(2)}/H_K)$; and hence $\gamma_2(\tau_K) \in H_K$. Therefore, we conclude that $\gamma_2(\tau_K)$ is a real algebraic integer which generates H_K over K by the fact $j(\tau_K) = \gamma_2(\tau_K)^3$ and Proposition 4.4(i). □

Remark 4.9. Observe that $\zeta_8 g_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K) = \zeta_8^3\mathfrak{f}_2(\tau_K)^2$ by (18). Besides Theorem 4.8 there are several other theorems which assert that the singular values of the Weber functions and $\gamma_2(\tau)$ generate class fields of imaginary quadratic fields ([27, §126–127], [19], [21], [4, §12]) whose proofs are quite classical. However, they are certainly elegant and worthy of considering. On the other hand, Gee [7] applied Shimura's reciprocity law to the Weber functions which satisfy the following transformation properties:

$$\begin{array}{lll}\mathfrak{f}(\tau)\circ T = \zeta_{48}^{-1}\mathfrak{f}_1(\tau), & \mathfrak{f}_1(\tau)\circ T = \zeta_{48}^{-1}\mathfrak{f}(\tau), & \mathfrak{f}_2(\tau)\circ T = \zeta_{24}\mathfrak{f}_2(\tau),\\ \mathfrak{f}(\tau)\circ S = \mathfrak{f}(\tau), & \mathfrak{f}_1(\tau)\circ S = \mathfrak{f}_2(\tau), & \mathfrak{f}_2(\tau)\circ S = \mathfrak{f}_1(\tau),\end{array}$$

where $T = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ are the generators of $\mathrm{SL}_2(\mathbb{Z})$. But, the general transformation formula for

$$f(\tau)\circ\gamma \quad (f(\tau) = \mathfrak{f}(\tau), \mathfrak{f}_1(\tau), \mathfrak{f}_2(\tau),\ \gamma \in \mathrm{SL}_2(\mathbb{Z}))$$

does not seem to be known, which forces her to produce a redundant step to decompose γ into a product of T and S [7, §5]. Thus we would like to point out that the relation (18) and Proposition 2.3(ii), (iii) will give us an explicit formula for $f(\tau)^2\circ\gamma$, from which one can efficiently apply Shimura's reciprocity law.

5. Application to class number one problem

In this section we shall revisit Gauss' class number one problem for imaginary quadratic fields.

Let K be an imaginary quadratic field of discriminant d_K. Since $j(\tau_K)$ is a real algebraic integer lying in H_K by Proposition 4.4(i), it should be an integer when K has class number one. By determining the form class group $\mathrm{C}(d_K)$ we know that there are only nine imaginary quadratic fields K of class number one with $d_K \geq -163$ [4, p. 261]:

Table 1. Imaginary quadratic fields K of class number one with $d_K \geq -163$.

K	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-7})$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-11})$	$\mathbb{Q}(\sqrt{-19})$	$\mathbb{Q}(\sqrt{-43})$	$\mathbb{Q}(\sqrt{-67})$	$\mathbb{Q}(\sqrt{-163})$
d_K	-3	-4	-7	-8	-11	-19	-43	-67	-163
$j(\tau_K)$	0	12^3	-15^3	20^3	-32^3	-96^3	-960^3	-5280^3	-640320^3
$\gamma_2(\tau_K)$	0	12	-15	20	-32	-96	-960	-5280	-640320

We shall show in this section that the above table is the complete one by utilizing Shimura's reciprocity law and Siegel functions.

Lemma 5.1. *Let $\tau_0 \in \mathbb{H}$, and set $A = |e^{2\pi i\tau_0}|$.*

(i) *If $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{Q}^2$ with $0 < a \leq 1/2$, then $|g_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}(\tau_0)| \leq A^{(1/2)\mathbf{B}_2(a)} e^{2A^a/(1-A)}$.*

(ii) *If $b \in \mathbb{Q}$ with $0 < b < 1$, then $|g_{\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]}(\tau_0)| \leq A^{(1/2)\mathbf{B}_2(0)} |1 - e^{2\pi i b}| e^{2A/(1-A)}$.*

Proof. (i) We derive from the definition (1) that

$$\begin{aligned}
|g_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}(\tau_0)| &\leq A^{(1/2)\mathbf{B}_2(a)}(1+A^a)\prod_{n=1}^{\infty}(1+A^{n+a})(1+A^{n-a}) \\
&\leq A^{(1/2)\mathbf{B}_2(a)}\prod_{n=0}^{\infty}(1+A^{n+a})^2 \quad \text{by the facts } A<1 \text{ and } 0<a\leq 1/2 \\
&\leq A^{(1/2)\mathbf{B}_2(a)}\prod_{n=0}^{\infty}e^{2A^{n+a}} \quad \text{by the inequality } 1+X<e^X \text{ for } X>0 \\
&= A^{(1/2)\mathbf{B}_2(a)}e^{2A^a/(1-A)}.
\end{aligned}$$

(ii) In a similar way, we get that

$$\begin{aligned}
|g_{\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]}(\tau_0)| &\leq A^{(1/2)\mathbf{B}_2(0)}|1-e^{2\pi i b}|\prod_{n=1}^{\infty}(1+A^n)^2 \\
&\leq A^{(1/2)\mathbf{B}_2(0)}|1-e^{2\pi i b}|\prod_{n=1}^{\infty}e^{2A^n} \\
&\qquad \text{by the inequality } 1+X<e^X \text{ for } X>0 \\
&= A^{(1/2)\mathbf{B}_2(0)}|1-e^{2\pi i b}|e^{2A/(1-A)}.
\end{aligned}$$

□

Theorem 5.2. *Let K $(\neq \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1}))$ be an imaginary quadratic field of discriminant d_K. Assume that K has class number one (that is, $H_K = K$).*

(i) *If 2 is not inert in K, then $d_K = -7, -8$.*
(ii) *If 2 is inert and 3 is ramified in K, then there is no such K.*
(iii) *If 2 is inert and 3 is not ramified in K, then $d_K = -11, -19, -43, -67, -163$.*

Proof. Since we are assuming that K is neither $\mathbb{Q}(\sqrt{-3})$ nor $\mathbb{Q}(\sqrt{-1})$, we have $d_K \leq -7$. Let $\alpha = g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)$ $(\neq 0)$ and $A = e^{-\pi\sqrt{|d_K|}}$.

(i) If $d_K \leq -31$, then we see that

$$\begin{aligned}
&|\mathbf{N}_{K_{(2)}/K}(\alpha^{12})|^{1/12} \\
&= \begin{cases} |g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K) g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)| & \text{if } d_K \equiv 0 \pmod 8, \\ |g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K) g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau_K)| & \text{if } d_K \equiv 4 \pmod 8, \\ |g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)| & \text{if } d_K \equiv 1 \pmod 8, \end{cases} \quad \text{by (19)} \\
&\leq \begin{cases} 2A^{1/12}e^{2A/(1-A)} \cdot A^{-1/24}e^{2A^{1/2}/(1-A)} & \text{if } d_K \equiv 0 \pmod 4, \\ 2A^{1/12}e^{2A/(1-A)} & \text{if } d_K \equiv 1 \pmod 8, \end{cases} \quad \text{by Lemma 5.1} \\
&< 1 \quad \text{by the fact } A \leq e^{-\pi\sqrt{31}}.
\end{aligned}$$

On the other hand, since $\mathbf{N}_{K_{(2)}/K}(\alpha^{12})$ is a nonzero integer by Theorem 4.6(ii), the above inequality is false; hence $d_K > -31$. And, we get the conclusion by Table 1.

(ii) Since 2 is inert and 3 is ramified in K (equivalently, $d_K \equiv 21 \pmod{24}$), we have

$$\mathrm{Gal}(K_{(3)}/K) \simeq W_{3,\tau_K}/\mathrm{Ker}_{3,\tau_K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \right\}$$

by Proposition 2.5. Let $\beta = g_{\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right]}(\tau_K)$ $(\neq 0)$. Since $g_{\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right]}(\tau)^{12} \in \mathcal{F}_3$ by Proposition 2.1, we have $\beta^{12} \in K_{(3)}$ by Proposition 2.4. Furthermore, since β^{12} is a real algebraic integer by the definition (1), Propositions 2.3(vi) and 4.4(i), $\mathbf{N}_{K_{(3)}/K}(\beta^{12})$ is a nonzero integer by Lemma 4.3. If $d_K \leq -51$, then we derive that

$$\begin{aligned}
&|\mathbf{N}_{K_{(3)}/K}(\beta^{12})|^{1/12} \\
&= |g_{\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right]}(\tau_K) g_{\left[\begin{smallmatrix} 1/3 \\ 1/3 \end{smallmatrix}\right]}(\tau_K) g_{\left[\begin{smallmatrix} 1/3 \\ 2/3 \end{smallmatrix}\right]}(\tau_K)| \quad \text{by Proposition 2.3(iv) and (v)} \\
&\leq A^{1/12}|1-\zeta_3|e^{2A/(1-A)} \cdot (A^{-1/36}e^{2A^{1/3}/(1-A)})^2 \quad \text{by Lemma 5.1} \\
&= \sqrt{3}A^{1/36}e^{(2A+4A^{1/3})/(1-A)} \\
&< 1 \quad \text{by the fact } A \leq e^{-\pi\sqrt{51}}.
\end{aligned}$$

Hence we must have $-51 < d_K \leq -7$. But, there is no such imaginary quadratic field K with $d_K \equiv 21 \pmod{24}$ as desired.

(iii) Let $x = \zeta_8\alpha$. Since $[K_{(2)} : K] = 3$ by the degree formula (15), we have

$$\min(x, K) = X^3 + aX^2 + bX + c \quad \text{for some } a, b, c \in \mathbb{Z}$$

by Theorem 4.8(i) and Lemma 4.3. Furthermore, we get

$$\min(x^4, K) = X^3 - \gamma_2(\tau_K)X - 16 \quad (\in \mathbb{Z}[X])$$

by Lemma 4.1(ii) and Theorem 4.8(ii). Now, by adopting Heegner's idea [9] one can determine the possible values of a, b, c, from which we obtain

$$\gamma_2(\tau_K) = 0, -32, -96, -960, -5280, -640320.$$

Therefore, we can conclude the assertion (iii) by Table 1 and Lemma 3.2, although we omit the details [4, pp. 272–274]. □

Remark 5.3. (i) In 1903 Landau ([13] or [4, Theorem 2.18]) presented a simple and elementary proof of Theorem 5.2(i) by considering the form class group $\mathrm{C}(d_K)$.

(ii) Theorem 4.8(i) is essentially a gap in Heegner's work, which was fulfilled by Stark [25].

(iii) To every imaginary quadratic order $\mathcal{O}$ of class number one there is an associated elliptic curve $E_{\mathcal{O}}$ over $\overline{\mathbb{Q}}$ admitting complex multiplication by $\mathcal{O}$. It can be defined over $\mathbb{Q}$ and is unique up to $\overline{\mathbb{Q}}$-isomorphism. For a positive integer n, let $X_{\mathrm{ns}}^{+}(n)$ be the modular curve associated to the normalizer of the non-split Cartan subgroup of level n which can be defined over $\mathbb{Q}$ [3]. If every prime p dividing n is inert in $\mathcal{O}$, then $E_{\mathcal{O}}$ gives rise to an integral point of $X_{\mathrm{ns}}^{+}(n)$ [22, p. 195]. Here, by integral points we mean the points corresponding to elliptic curves with integral j-invariant. As Serre pointed out [22, p. 197], the solutions by Heegner and Stark can be viewed as the determination of the integral points of $X_{\mathrm{ns}}^{+}(24)$. And, Baran [2] recently gave a geometric solution of the class number one problem by finding an explicit parametrization for the modular curve $X_{\mathrm{ns}}^{+}(9)$ over $\mathbb{Q}$.

We can also apply the arguments in the proof of Theorem 5.2(i) and (ii) to solve a problem concerning imaginary quadratic fields of class number two.

Theorem 5.4. $\mathbb{Q}(\sqrt{-15})$ *is the unique imaginary quadratic field of class number two in which* 2 *splits.*

Proof. Let K be an imaginary quadratic field of discriminant d_K and class number two in which 2 splits (so $d_K \equiv 1 \pmod 8$). Then the form class group $\mathrm{C}(d_K)$ consists of two reduced quadratic forms, that is

$$Q_1 = X^2 + XY + ((1 - d_K)/4)Y^2,$$
$$Q_2 = aX^2 + bXY + cY^2 \quad \text{for some } a, b, c \in \mathbb{Z} \text{ with } 2 \leq a \leq \sqrt{|d_K|/3}.$$

And, we have by Proposition 2.6 with $p = 2$

$$\tau_{Q_1} = (-1+\sqrt{d_K})/2, \quad u_{Q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\tau_{Q_2} = (-b+\sqrt{d_K})/2a, \quad u_{Q_2} = \begin{bmatrix} * & * \\ r & s \end{bmatrix} \text{ for some } \begin{bmatrix} r \\ s \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Let $\alpha = g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau_K)$ $(\neq 0)$. Then $\alpha^{12} \in H_K$ by (19). If $d_K \leq -31$, then we derive that

$$\begin{aligned}
&|\mathbf{N}_{H_K/K}(\alpha^{12})|^{1/12} \\
&= \prod_{Q\in \mathrm{C}(d_K)} |(g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)^{12})^{u_Q}(\tau_Q)|^{1/12} \quad \text{by Proposition 2.6} \\
&= |g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}((-1+\sqrt{d_K})/2)| \times |g_{\left[\begin{smallmatrix} r/2 \\ s/2 \end{smallmatrix}\right]}((-b+\sqrt{d_K})/2a)| \\
&\qquad \text{by Proposition 2.3(iv) and (v)} \\
&\leq 2A^{1/12}e^{2A/(1-A)} \times \begin{cases} 2A^{1/12a}e^{2A^{1/a}/(1-A^{1/a})} & \text{if } r = 0, \\ A^{-1/24a}e^{2A^{1/2a}/(1-A^{1/a})} & \text{if } r = 1 \end{cases} \\
&\qquad \text{with } A = e^{-\pi\sqrt{|d_K|}} \text{ by Lemma 5.1} \\
&\leq \begin{cases} 4A^{1/12}e^{2A/(1-A)-\pi\sqrt{3}/12+2e^{-\pi\sqrt{3}}/(1-e^{-\pi\sqrt{3}})} & \text{if } r = 0, \\ 2A^{1/12-1/48}e^{2A/(1-A)+2e^{-\pi\sqrt{3}/2}/(1-e^{-\pi\sqrt{3}})} & \text{if } r = 1 \end{cases} \\
&\qquad \text{because } A < 1 \text{ and } 2 \leq a \leq \sqrt{|d_K|/3} \\
&< 1 \quad \text{by the fact } A \leq e^{-\pi\sqrt{31}}.
\end{aligned}$$

On the other hand, since α^{12} is a real algebraic integer, $\mathbf{N}_{H_K/K}(\alpha^{12})$ is a nonzero integer by Lemma 4.3. Therefore we should have $d_K > -31$. One can then easily see by the following remark that $\mathbb{Q}(\sqrt{-15})$ is the unique one. □

Remark 5.5. There are exactly eighteen imaginary quadratic fields of class number two whose discriminants are as follows [18, p. 636]:

$$-15, -20, -24, -35, -40, -51, -52, -88, -91, -115,$$
$$-123, -148, -187, -232, -235, -267, -403, -427.$$

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