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A new class of life distributions based on unknown age

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Abstract. Based on increasing concave ordering a new class of life distribution is introduced. The new class of life distribution is named used better than aged in increasing concave ordering and is denoted by UBAC(2). The implication of our proposed class of life distribution with other classes is given. The properties of UBAC(2) under convolution, discrete mixture and formation of a coherent system are studied. Finally a characterization of the proposed class of life distributions by Laplace transform is discussed.

Key Words: *Age-smooth, convolution, increasing concave ordering, Laplace-Stieltjes transform, series system, survival functions, UBA class of life distributions*

1. INTRODUCTION

It is well known that the concept of positive aging describes the adverse effects of age on the lifetimes of units, and it has been found very useful to classify life distributions by stochastic orderings, as can be observed in the recent literature. For definitions of several classes of life distributions, e.g., IFR, DMRL, UBA, UBAE, DVRL see Bryson and Siddiqui (1969), Barlow and Proschan (1975), Abu-Yossef (2004) and Ahmad (2004). Many applications in reliability theory involve the modeling of life time data. In these applications, for the purchasing used items with unknown age, it may be realistic for the buyer to assume that those items have been used for a long period of time. Hence, it would be of a great importance to have some criteria to compute the remaining life of the purchased item with its performance under the true age. As a criterion for comparing ages of, for instance, electrical equipments, computers, radios or alike, Bhattacharjee (1986) discussed the tail behavior of age smooth failure distributions. Cline (1987) studied the connection between the class of age smooth distributions and the classes of life distributions with subexponential tails which have many applications in queuing theory,

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random walk and infinite divisibility.
Let X be a non-negative continuous random variable with distribution function F; survival function $\bar{F}$ = 1 $-F$ and finite mean μ = E[X] = $\int_0^\infty \bar{F}(x)dx$. At age t, the random residual life is defined by X_t with survival function $\bar{F}_t = \frac{\bar{F}(t+x)}{\bar{F}(t)}$, $x, t \geq 0$. The mean residual life of X_t is given by

$$\mu(t) = E[X_t] = \int_t^\infty \bar{F}(u)du/\bar{F}(t)\ , t \geq 0, \bar{F}(t) > 0$$

The distribution function F belongs to the increasing failure rate (IFR) class iff X_t is decreasing in $t \geq 0$ in stochastic ordering and F belongs to the new better than used (NBU) class iff X_t is smaller than X in stochastic ordering for every $t \geq 0$. We refer the reader to Barlow and Proschan (1981) and Ross (1983).
The main object in this paper is to introduce a new class of life distribution based on the notation of increasing concave ordering. We call this class used better than aged in increasing concave ordering (UBAC(2)) class.
The paper is organized as follows: In section 2, definitions and relationships are given. In section 3 we study the UBAC(2) property under convolution, discrete mixture and formation of a coherent system. In section 4 characterization of UBAC(2) by Laplace transform is discussed.

2. DEFINITIONS AND RELATIONSHIPS

Definition 1. (Bhattacharjee (1986)) F is called age-smooth if for every support point x of F

$$\lim_{t\to\infty} \bar{F}_t(x) \in [0, 1]$$

It is important to note That IFR and DFR classes are contained in the age-smooth class of life distribution.

Properties of age-smooth class

(i) If F is age-smooth and if F is non-lattice then

$$\lim_{t\to\infty} \bar{F}_t(x) = e^{-rx} \text{ for some } 0 \leq \gamma \leq \infty, \text{and for every } x \geq 0$$

(see Bhattacharjee (1986) and Alzaid (1994))

(ii) If F has a failure rate function $r(t)$, then $\lim_{t\to\infty} r(t) = \gamma$ (see Bhattacharjee (1986))

(iii) If F has finite mean, then F is age-smooth iff

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} = \int_0^\infty \bar{F}(x)dx = \gamma^{-1}$$

(see Bhattacharjee (1986))

(iv) The finitely age-smooth distribution are closed under convolution. This implies that if

$$\lim_{t\to\infty} \frac{\bar{F}_i(x+y+t)}{F_i(t)} = e^{-\gamma_i(x+y)} \quad x, y \geq 0, i = 1, 2$$

for some $0 \leq \gamma_1 \leq \gamma_2 \leq \infty$ then

$$\lim_{t\to\infty} \frac{\overline{F_1 * F_2}(x+y+t)}{\overline{F_1 * F_2}} = e^{-\gamma_i(x+y)} \qquad x, y \geq 0, i = 1, 2$$

where $F_1 * F_2$ is the convolution of F_1 with F_2 (see Embrechts and Goldie (1980)).

Two classes were discussed by Ahmed (2004) who called used better than aged (UBA) and used better than aged in expectation (UBAE).

Definition 2. The distribution function F is said to be used better than aged (UBA) if for all x, t ≥ 0

$$\bar{F}(x+t) \geq \bar{F}(t)e^{-\gamma x} \tag{1}$$

Definition 3. The distribution function F is said to be used better than aged in expectation (UBAE) if for all x, t ≥ 0

$$\int_t^\infty \bar{F}\,(u)du \;\geq\; \frac{\bar{F}(t)}{\gamma} \quad \text{or} \quad \nu(t) \;\geq \frac{\bar{F}(t)}{\gamma} \tag{2}$$

where $\nu(t) = \int_t^\infty \bar{F}\,(u)du$

Using the increasing concave ordering we define the following new class of life distributions

Definition 4. If F has finite mean, then F is called used better than aged in increasing concave ordering (denoted by UBAC(2)) if it is finitely and positively smooth and

$$\int_0^x \bar{F}\,(X)dx \;\geq\; \gamma^{-1}(1-e^{-\gamma x}) \tag{3}$$

It is clear that

$$\text{IFR} \Rightarrow \text{UBA} \Rightarrow \text{UBAC(2)}$$

It was shown that the UBA class is a subclass of UBAE and the IHR (increasing hazard rate) is contained in the UBA class. Similar implications between UBAE, NBUE (new better than used) and HNBUE (harmonic new better than in expectation) were given by Di Crescenzo (1999). Also, Willmot and Cai (2000) showed that the UBA class includes the DMRL class (decreasing mean residual life) while the UBAE includes the DVRL (decreasing variance residual life). Then we have

$$\begin{array}{c} \text{IHR} \subset \text{DMRL} \subset \text{UBA} \subset \text{UBAC(2)} \subset \text{UBAE} \\ \qquad\qquad\qquad\qquad\qquad \cup \\ \qquad\qquad\qquad\qquad\qquad \text{DVRL} \end{array}$$

For definition and properties of these classes we refer the readers to the surveys by Deshpande et al (1986) and Deshpande and Purohit (2005).

3. CLOSURE PROPERTIES

In this section we discuss the closure properties of the UBAC(2) under convolution and discrete mixing and formation of coherent system.

Theorem 1. UBAC(2) class of life distribution is closed under the convolution.

Proof. Let F1 and F2 be UBAC(2) and assume without loss of generality that $\gamma_1 \leq \gamma_2$. Then from the de_nition of UBAC(2) class we have

$$\int_0^x \bar{F}(\omega + t)\, d\omega = \int_0^x \int_0^t \bar{F}_1(\omega + t - u)\, dF_2(u) d\omega + \int_0^x \int_t^\infty \bar{F}_1(\omega + t - u)\, dF_2(u) d\omega$$
$$= I + II$$

Then

$$\int_0^x \int_0^t \bar{F}_1(\omega + t - u)\, dF_2(u) d\omega$$

since $\bar{F}_1$ is UBAC(2), then

$$I \geq \int_0^t \gamma_1{}^{-1} \bar{F}_1(t-u)(1 - e^{-r_1 x}) dF_2(u)$$

Since

$$\int_0^t \bar{F}_1(t-u) dF_2(u) = \bar{F}(t) - \overline{F_2}(t)$$

then

$$I \geq \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})(\bar{F}(t) - \bar{F}_2(t))$$

Also, for u $> t \Rightarrow t - u < 0,$ then

$$\bar{F}_1(t - u + \omega) \geq \bar{F}_1(\omega)$$

then from the assumption that $\bar{F}_1$ is UBAC(2), we have

$$\int_0^x \bar{F}_1(\omega) d\omega \geq \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})$$

Then

$$\text{II} \geq \int_0^x \gamma_1{}^{-1}(1 - e^{-r_1 x}) d\bar{F}(u)$$
$$= \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})\bar{F}_2(t)$$

Therefore

$$\int_0^x \bar{F}(\omega + t) d\omega = I + II \geq \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})(\bar{F}(\mathrm{t}) - \bar{F}_2(t) + \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})\bar{F}_2(t)$$
$$= \gamma_1{}^{-1}(1 - e^{-\gamma_1 x})\bar{F}(t)$$

The $\bar{F} = \overline{F_1 * F_2}$ is UBAC(2). This completes the proof.

Theorem 2. Let F_i, i = 1; 2…, n be UBAC(2) life distribution with

$$\lim_{t \to \infty} \frac{\bar{F}_i(x+y+t)}{F_i(t)} = e^{-\alpha_i(x+y)},$$

$x, y \geq 0$ *and* $0 \leq \alpha_i \leq 1$ *such that* $\sum_{i=1}^n \alpha_i = 1,$ *then* $F(x) = \sum_{i=1}^n \alpha_i F_i(x)$ is UBAC(2).

Proof. Without loss of generality we take n = 2 and $\alpha_1 \leq \alpha_2$. Let F_1 and F_2 are UBAC(2), then we have for $\epsilon \geq 0$

$$-\epsilon \bar{F}_i(x) \leq \int_0^x (u+t) du - \alpha_1{}^{-1}(1 - e^{-\alpha_1 x})\, \bar{F}_i(t) \leq \epsilon \bar{F}_i(t)$$

for all x and sufficiently large t. This implies that

$$\left|\frac{\alpha_1\int_0^x \bar{F}_1(u+t)du+\alpha_2\int_0^x \bar{F}_2(u+t)du}{\alpha_1 F_1(t)+\alpha_2 F_2(t)}-\alpha_1{}^{-1}(1-e^{-\alpha_1 x})\right|\le \epsilon$$

and hence

$$\lim_{t\to\infty}\frac{\alpha_1\int_0^x \bar{F}_1(u+t)du+\alpha_2\int_0^x \bar{F}_2(u+t)du}{\alpha_1 F_1(t)+\alpha_2 F_2(t)}=\alpha_1{}^{-1}(1-e^{-\alpha_1 x})$$

This result holds for every finite n. This completes the proof of the theorem.

Theorem 3. A series system of n independent UBAC(2) components is UBAC(2).

Proof. Let $X_1, X_2\ldots, X_n$ be independent UBAC(2) components. Then

$$\int_0^x \frac{P(\min(X_1 \ldots., X_n \ge u+t)}{P(\min(X_1 \ldots, X_n \ge t)}du = \int_0^x II_{i=1}^n \frac{P(X_i \ge u+t)}{P(X_i \ge t)}du$$

$$= \int_0^x II_{i=1}^n \frac{\overline{F_i}(u+t)}{F_i(t)}du$$

$$\ge \int_0^x II_{i=1}^n e^{-\gamma_i u}du \qquad \text{since } \overline{F_i} \text{ is UBA}$$

$$= \int_0^x e^{-u\sum_{i=1}^n \gamma_i}du$$

$$= \left(\sum_{i=1}^n \gamma_i\right)^{-1}\left(1-e^{-x\sum_{i=1}^n r_i}\right)$$

This implies that a series system $X_1, X_2 \ldots, X_n$ is UBAC(2).

4. CHARACTERIZATION USING LAPLACE TRANSFORM

In this section we present necessary and sufficient conditions for a life distribution to have the UBAC(2) property by using the Laplace transformation. Let F be a distribution function such that F(0) = 0. The Laplace transform of F is defined by

$$\phi(s)=\int_0^\infty e^{-sx}\,dF(u)\,, s>0$$

Further define

$$\alpha_n(s)=\frac{(-1)^n}{n!}\frac{d^n}{ds^n}\left(\frac{1-\phi(s)}{s}\right)\ ,n\ge 0, s\ge 0 \qquad (4)$$

Let $\alpha_{n+1}(s) = s^{n+1}\alpha_n(s)$ for $n \ge 0$ and $s \ge 0$ with starting value $\alpha_0(s) = 1$. The transform $\alpha_n(s)$ and $\alpha_{n+1}(s)$ can be written in the forms

$$\alpha_n(s)=\int_0^\infty \frac{u^n}{n!}e^{-su}\bar{F}(u)du \qquad (5)$$

And

$$\alpha_{n+1}(s) = \int_0^\infty \frac{s(su)^n}{n!} e^{-su}\bar{F}(u) \quad (6)$$

Vinogradov (1973) has used the Laplace transform to give necessary and sufficient conditions for a distribution to have increasing failure rate. Since then, many authors have derived similar characterizations for other families of life distribution. Blockk and Savits (1980) have obtained characterizations for IFRA, DMRL, NBU, and NBUE classes. Klefsjo (1982) studied characterization of HNBUE class in terms of $\alpha_n(s)$ sequence.

Theorem 4. Let F be a life distribution with F(0) = 0, then F has the UBAC(2) property if and only if

$$\sum_{J=1}^{k+l+1} \alpha_j(s) = \gamma^{-1} s\alpha_k(s)\left(1-\left(1+\frac{\gamma}{s}\right)^{l-1}\right) \qquad for \quad l>0, x>0 \quad (7)$$

Proof. Assume that F is UBAC(2), then by using the form (6) we have

$$\sum_1^{k+l+1} \alpha_j(s) = \sum_1^{K+L+1} s\int_0^\infty \frac{(su)^{j-1}}{(j-1)!} \quad e^{-su}\bar{F}(u)du$$

$$= s\int_0^\infty \left(\sum_1^{K+L+1} \frac{(su)^{j-1}}{(j-1)!}\right) e^{-su}\bar{F}(u)du$$

$$= \int_0^\infty \left(\int_u^\infty \frac{s(su)^{k+l-1}}{(k+l)!}\right) e^{-su}\bar{F}(u)du$$

$$= s^3 \int_0^\infty e^{-su} \frac{(su)^{k-1}}{(k)!} \int_0^\infty e^{-sx} \frac{(sx)^{l-1}}{(l)!} \int_0^{x+u} F(v)dvdxdu$$

since F is UBAC(2), then

$$\sum_1^{k+l+1} \alpha_j(s) \geq s^3 \int_0^\infty e^{-su} \frac{(su)^{k-1}}{(l)!} \int_0^\infty e^{-sx} \frac{(sx)^{l-1}}{(l)!} \left(\gamma^{-1}(1-e^{-\gamma x})\bar{F}(u)\right)dxdu$$

$$= s\int_0^\infty su^{-su} \frac{(su)^{k-1}}{(k)!} \bar{F}(u)du \int_0^\infty su^{-sx} \frac{(sx)^{l-1}}{(l)!} \gamma^{-1}(1-e^{-\gamma x})dx$$

$$= s\gamma^{-1}\alpha_k(s)(\int_0^\infty se^{-sx} \frac{(sx)^{l-1}}{(l)!} dx - \int_0^\infty se^{-(s+\gamma)x} \frac{(sx)^{l-1}}{(l)!})dx$$

$$= s\gamma^{-1}\alpha_k(s)\left(1-\left(1+\frac{\gamma}{s}\right)^{l-1}\right) \qquad for\ l>0, x>0$$

This completes the proof of the necessary part.

Now, to prove the sufficient part, note that condition (7) can be written in the following form

$$\int_0^\infty \int_0^\infty \int_0^{x+y} \bar{F}(\omega)d\omega dG_{k-1}(y)dG_{k-1}(v) \geq \int_0^\infty \gamma^{-1}\bar{F}(y)dG(y)(1-(1+\frac{\gamma}{s})^{l-1}) \quad (8)$$

where $G_n(y) = s\sum_{j=n+1}^\infty \frac{(sy)^j}{j!} e^{-sy}$.

It is obvious that

$$G_n(\mathrm{y}) = \int_0^\infty s\frac{(sy)^n}{\Gamma(n+1)} e^{-sy}\mathrm{dy} = P(\sum_{i=1}^{n+1} Y_i \leq y)$$

where Y_1, Y_2,…,Y_{n+1} are n mutually independent and exponential with parameter s. Hence $G_n(y)$ represent a gamma distribution with characteristic function

$$\phi_{n+1}(\mathrm{v}) = (1 - \frac{iv}{s})^{-(n+1)}$$

It can be shown that if $\frac{n+1}{s} \to \infty$ then $\lim_{n\to\infty} \phi_{n+1}(v) = e^{(ivx)}$. Then the function $G_n(y)$ converges to the degenerate distribution, that is

$$G_n(\mathrm{y}) \to \begin{cases} 0 & for\ y < x \\ 1 & for\ y \geqq x \end{cases}$$

Taking the limits for both sides in (8) as $n \to \infty$ we have

$$\int_0^{x+y} \bar{F}\,(\omega) d\omega \ \geq \gamma^{-1} \bar{F}(\mathrm{y})(1 + e^{-\gamma x})$$

which gives

$$\int_0^{x} \bar{F}\,(\omega + \mathrm{y}) d\omega \ \geq \ \gamma^{-1}(1 + e^{-\gamma x})$$

That is, the distribution F is UBAC(2). This completes the proof.

5. CONCLUSION

In this paper we presented a new class of life distributions based on the increasing concave ordering. The class is denoted by UBAC(2). We studied the UBAC(2) property under convolution, discrete mixture and formation of a coherent system. Also, we discussed characterization of this class by Laplace transform. Further properties of this class need to be discussed in a future work.

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