Bull. Korean Math. Soc. **52** (2015), No. 4, pp. 1383–1399
http://dx.doi.org/10.4134/BKMS.2015.52.4.1383

PRODUCT-TYPE OPERATORS FROM WEIGHTED BERGMAN-ORLICZ SPACES TO WEIGHTED ZYGMUND SPACES

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Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, φ an analytic self-map of $\mathbb{D}$ and ψ an analytic function in $\mathbb{D}$. Let D be the differentiation operator and $W_{\varphi,\psi}$ the weighted composition operator. The boundedness and compactness of the product-type operator $W_{\varphi,\psi}D$ from the weighted Bergman-Orlicz space to the weighted Zygmund space on $\mathbb{D}$ are characterized.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,\psi}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi} f(z) = \psi(z) f(\varphi(z)),\ z \in \mathbb{D}.$$

If $\psi \equiv 1$, the operator is reduced to, so called, the composition operator and is usually denoted by C_φ. If $\varphi(z) = z$, it is reduced to, so called, the multiplication operator and usually denoted by M_ψ. A standard problem is to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator. Weighted composition operators between various spaces of analytic functions on different domains have been studied by numerous authors, see, e.g., [1, 2, 6, 8, 9, 10, 13, 17, 18, 22, 23, 26, 31, 34, 36, 40] and the references therein.

Let D be the differentiation operator on $H(\mathbb{D})$, that is

$$Df(z) = f'(z),\ z \in \mathbb{D}.$$

Operator $C_\varphi D$ has been studied, for example, in [4, 11, 14, 16, 19, 28, 30, 32]. In [21] Sharma has studied the following operators from Bergman spaces to

Received October 28, 2014; Revised November 22, 2014.
2010 *Mathematics Subject Classification.* Primary 47B38; Secondary 47B33, 47B37.
Key words and phrases. weighted Bergman-Orlicz spaces, product-type operators, weighted Zygmund spaces, boundedness, compactness.

Bloch type spaces:

$$M_\psi C_\varphi Df(z) = \psi(z)f'(\varphi(z)),$$
$$M_\psi DC_\varphi f(z) = \psi(z)\varphi'(z)f'(\varphi(z)),$$
$$C_\varphi M_\psi Df(z) = \psi(\varphi(z))f'(\varphi(z)),$$

and

$$C_\varphi DM_\psi f(z) = \psi'(\varphi(z))f(\varphi(z)) + \psi(\varphi(z))f'(\varphi(z))$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. These operators on weighted Bergman spaces were also studied in [37] and [38] by Stević, Sharma and Bhat. If we consider the product-type operator $W_{\varphi,\psi}D$, it is clear that

$$M_\psi C_\varphi D = W_{\varphi,\psi}D, \ \ M_\psi DC_\varphi = W_{\varphi,\psi\cdot\varphi'}D,$$
$$C_\varphi M_\psi D = W_{\varphi,\psi\circ\varphi}D \ \text{ and } C_\varphi DM_\psi = W_{\varphi,\psi'\circ\varphi} + W_{\varphi,\psi\circ\varphi}D.$$

Quite recently, the present author has considered operator $W_{\varphi,\psi}D$ from weighted Bergman spaces to weighted Zygmund spaces in [5]. For some other product-type operators, see, for example [7, 12, 15, 24, 25, 27, 29, 33, 39, 41] and the references therein. This paper is devoted to characterizing the boundedness and compactness of the operator $W_{\varphi,\psi}D$ from weighted Bergman-Orlicz spaces to weighted Zygmund spaces. It can be regarded as a continuation of the investigation of operators from weighted Bergman-Orlicz spaces to other spaces (see, e.g., [20]).

We introduce the needed spaces and facts in [20]. The function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing function from the interval $[0,\infty)$ onto itself. Clearly, these conditions imply that $\Phi(0) = 0$. It is said that the function Φ is of positive upper type (respectively, negative upper type), if there are $q > 0$ (respectively, $q < 0$) and $C > 0$ such that $\Phi(st) \le Ct^q\Phi(s)$ for every $s > 0$ and $t \ge 1$. By $\mathfrak{U}^q$ we denote the family of all growth functions Φ of positive upper type q ($q \ge 1$), such that the function $t \mapsto \Phi(t)/t$ is nondecreasing on $[0,\infty)$. It is said that function Φ is of positive lower type (respectively, negative upper type), if there are $r > 0$ (respectively, $r < 0$) and $C > 0$ such that $\Phi(st) \le Ct^r\Phi(s)$ for every $s > 0$ and $0 < t \le 1$. By $\mathfrak{L}_r$ we denote the family of all growth functions Φ of positive lower type r ($0 < r \le 1$), such that the function $t \mapsto \Phi(t)/t$ is nonincreasing on $[0,\infty)$. If $f \in \mathfrak{U}^q$, we will also assume that it is convex.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. Let $\alpha > -1$ and $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ the weighted Lebesgue measure on $\mathbb{D}$. Let Φ be a growth function. The weighted Bergman-Orlicz space $A^\Phi_\alpha(\mathbb{D}) := A^\Phi_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^\Phi_\alpha} = \int_{\mathbb{D}} \Phi(|f(z)|)dA_\alpha(z) < \infty.$$

On A^Φ_α is defined the following quasi-norm

$$\|f\|^{lux}_{A^\Phi_\alpha} = \inf\Big\{\lambda > 0 : \int_{\mathbb{D}} \Phi\Big(\frac{|f(z)|}{\lambda}\Big)dA_\alpha(z) \le 1\Big\}.$$

If $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_r$, then the quasi-norm on A_α^Φ is finite and called the Luxembourg norm. The classical weighted Bergman space A_α^p, $p > 0$, corresponds to $\Phi(t) = t^p$, consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that for $p \geq 1$ it is a Banach space, while for $0 < p < 1$ it is a translation-invariant metric space with $d(f,g) = \|f-g\|_{A_\alpha^p}^p$. Moreover, if $\Phi \in \mathfrak{U}^s$, then $A_\alpha^{\Phi_p}$, where $\Phi_p(t) = \Phi(t^p)$, is a subspace of A_α^p. This fact will be used later.

For $\beta > 0$, the weighted Zygmund space $\mathcal{Z}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^\beta |f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}_\beta} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)^\beta |f''(z)|.$$

The little weighted Zygmund space $\mathcal{Z}_{\beta,0}$ consists of those functions f in $\mathcal{Z}_\beta$ such that

$$\lim_{|z| \to 1^-} (1-|z|^2)^\beta |f''(z)| = 0,$$

and it is a closed subspace of the weighted Zygmund space. For a good source of such spaces, we refer to [42]. For weighted Zygmund spaces on the unit disk, the upper half plane, the unit ball and some operators on them, see, e.g. [6, 13, 15, 35] and the references therein.

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively. It is said that a linear operator $L : X \to Y$ is metrically bounded if there exists a positive constant K such that

$$d_Y(Lf, 0) \leq K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. Operator $L : X \to Y$ is said to be metrically compact if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the usual definition of compact operators between Banach spaces. When $X = A_\alpha^\Phi$ and Y is a Banach space, the norm of operator L is

$$\|L\|_{A_\alpha^\Phi \to Y} = \sup_{\|f\|_{A_\alpha^\Phi} \leq 1} \|Lf\|_Y$$

and is usually written by $\|L\|$.

Throughout this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). C will be a constant not necessarily the same at each occurrence. The notation $a \lesssim b$ means that $a \leq Cb$ for some positive constant C. When $a \lesssim b$ and $b \lesssim a$, we write $a \simeq b$.

2. Auxiliary results

Our first lemma characterizes the compactness in terms of sequential convergence. Since the proof is standard, it is omitted (see, Proposition 3.11 in [3]).

Lemma 2.1. *Let $p \geq 1$, $\alpha > -1$, $\beta > 0$ and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the bounded operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$ is compact if and only if for every bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $A_\alpha^{\Phi_p}$ such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that*

$$\lim_{n\to\infty} \|W_{\varphi,\psi}Df_n\|_{\mathcal{Z}_\beta} = 0.$$

In order to prove our main results, we need a useful point evaluation estimate contained in the following lemma.

Lemma 2.2. *Let $p \geq 1$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then there are positive constants $C_n = C_n(\alpha, p)$ and $D_n = D_n(\alpha, p)$ independent of $f \in A_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ such that*

$$|f^{(n)}(z)| \leq \frac{C_n}{(1-|z|^2)^n}\Phi_p^{-1}\left(\left(\frac{D_n}{1-|z|^2}\right)^{\alpha+2}\right)\|f\|^{lux}_{A_\alpha^{\Phi_p}}.$$

Proof. Since $A_\alpha^{\Phi_p}$ is a subspace of A_α^p, by the integral representation for functions in A_α^p, we have that for every $f \in A_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ (see Theorem 2.2 in [42])

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\alpha+2}} dA_\alpha(w). \tag{1}$$

Differentiating (1) under the integral sign n times yields

$$f^{(n)}(z) = c_{n,\alpha}\int_{\mathbb{D}} \frac{\overline{w}^n f(w)}{(1-\overline{w}z)^{\alpha+n+2}} dA_\alpha(w).$$

Then

$$\left|f^{(n)}(z)\right| \leq c_{n,\alpha}\int_{\mathbb{D}} \frac{|f(w)|}{|1-\overline{w}z|^{n+\alpha+2}} dA_\alpha(w). \tag{2}$$

By the fact (see, e.g., Theorem 1.12 in [42]) that

$$\frac{1}{(1-|z|^2)^n} \simeq \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\overline{w}z|^{\alpha+n+2}} dA(w), \tag{3}$$

there is a positive constant c_1 such that

$$c_1\frac{(1-|z|^2)^n}{|1-\overline{w}z|^{\alpha+n+2}} dA_\alpha(w)$$

is a probability measure. From (3) and applying Jensen's inequality in (2), we obtain

$$(1-|z|^2)^{np}|f^{(n)}(z)|^p \leq c_2(1-|z|^2)^n\int_{\mathbb{D}} \frac{|f(w)|^p}{|1-\overline{w}z|^{\alpha+n+2}} dA_\alpha(w).$$

From this, we have

$$c_1 c_2^{-1}(1-|z|^2)^{np}|f^{(n)}(z)|^p \leq c_1 \int_{\mathbb{D}} |f(w)|^p \frac{(1-|z|^2)^n}{|1-\overline{w}z|^{\alpha+n+2}} dA_\alpha(w). \tag{4}$$

Therefore, from (4), the monotonicity and convexity of function Φ, we get

$$\begin{aligned}
\Phi\Big(\frac{c_1 c_2^{-1}(1-|z|^2)^{np}|f^{(n)}(z)|^p}{(\|f\|^{lux}_{A^\Phi_\alpha})^p}\Big) &\leq c_1 \int_{\mathbb{D}} \Phi_p\Big(\frac{|f(w)|}{\|f\|^{lux}_{A^\Phi_\alpha}}\Big) \frac{(1-|z|^2)^n}{|1-\overline{w}z|^{\alpha+n+2}} dA_\alpha(w) \\
&\leq 2^n c_1 \Big(\frac{2}{1-|z|^2}\Big)^{\alpha+2} \int_{\mathbb{D}} \Phi_p\Big(\frac{|f(w)|}{\|f\|^{lux}_{A^\Phi_\alpha}}\Big) dA_\alpha(w) \\
&\leq \frac{D_n}{(1-|z|^2)^{\alpha+2}}. \qquad (5)
\end{aligned}$$

From (5), we obtain

$$|f^{(n)}(z)| \leq \frac{C_n}{(1-|z|^2)^n} \Phi_p^{-1}\Bigg(\Big(\frac{D_n}{1-|z|^2}\Big)^{\alpha+2}\Bigg) \|f\|^{lux}_{A^{\Phi_p}_\alpha},$$

where $C_n = c_1^{-\frac{1}{p}} c_2^{\frac{1}{p}}$ and $D_n = \sqrt[\alpha+2]{2^{n+\alpha+2} c_1}$. The proof is finished. □

The following lemma provides a class of useful test functions in space $A^{\Phi_p}_\alpha$.

Lemma 2.3. *Let $p > 0$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $t \geq 0$ and $w \in \mathbb{D}$, the following function is in $A^{\Phi_p}_\alpha$*

$$f_{w,t}(z) = \Phi_p^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\Big(\frac{1-|w|^2}{1-\overline{w}z}\Big)^{\frac{2(\alpha+2)}{p}+t},$$

where C is an arbitrary positive constant.

Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{w,t}\|^{lux}_{A^{\Phi_p}_\alpha} \lesssim 1.$$

Proof. Let

$$g(z) = \Big(\frac{1-|w|^2}{1-\overline{w}z}\Big)^{\frac{2(\alpha+2)}{p}+t}.$$

Since $\Phi_p^{-1}(t) = (\Phi^{-1}(t))^{1/p}$, we have

$$\int_{\mathbb{D}} \Phi(|f_{w,t}(z)|^p) dA_\alpha(z) = \int_{\mathbb{D}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\big|g(z)\big|^p\Big) dA_\alpha(z) = I + J,$$

where

$$I = \int_{\{z\in\mathbb{D}:|g(z)|\leq 1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\big|g(z)\big|^p\Big) dA_\alpha(z)$$

and

$$J = \int_{\{z\in\mathbb{D}:|g(z)|> 1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\big|g(z)\big|^p\Big) dA_\alpha(z).$$

Since $\Phi(t)/t$ is nondecreasing on $[0,\infty)$, it follows that

$$\frac{\Phi(t|g(z)|)}{t|g(z)|} \leq \frac{\Phi(t)}{t}$$

for all $z \in \{z \in \mathbb{D} : |g(z)| \leq 1\}$. This gives

$$\begin{aligned}
I &= \int_{\{z\in\mathbb{D}:|g(z)|\leq 1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\big|g(z)\big|^p\Big) dA_\alpha(z) \\
&\leq \int_{\{z\in\mathbb{D}:|g(z)|\leq 1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\Big)\big|g(z)\big|^p dA_\alpha(z) \\
&\leq C^{\alpha+2} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+pt+2}}{|1-\overline{w}z|^{2(\alpha+2)+pt}} dA_\alpha(z) \\
&\lesssim 1,
\end{aligned}$$

where we have used Theorem 1.12 in [42]. From the definition of positive upper type and the fact $s \geq 1$, we obtain

$$\begin{aligned}
J &= \int_{\{z\in\mathbb{D}:|g(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\big|g(z)\big|^p\Big) dA_\alpha(z) \\
&\lesssim \int_{\{z\in\mathbb{D}:|g(z)|>1\}} \Phi\Big(\Phi^{-1}\Big(\Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2}\Big)\Big)\big|g(z)\big|^{ps} dA_\alpha(z) \\
&\leq \Big(\frac{C}{1-|w|^2}\Big)^{\alpha+2} \int_{\mathbb{D}} \frac{(1-|w|^2)^{2s(\alpha+2+\frac{pt}{2})}}{|1-\overline{w}z|^{2s(\alpha+2+\frac{pt}{2})}} dA_\alpha(z) \\
&\lesssim 1.
\end{aligned}$$

From this the lemma follows. □

In the last result of this section, we construct some suitable linear combinations of the test functions in Lemma 2.3 which will be used the proofs of the main results.

Lemma 2.4. *Let $p > 0$, $\alpha > -1$, $w \in \mathbb{D}$ and $\Phi \in \mathfrak{U}^s$. Then for a fixed $j \in \{1,2,3\}$, there exist constants c_1, c_2 and c_3 such that the function*

$$g_{w,j}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(w),i}(z)$$

satisfies

$$g_{w,j}^{(j)}(\varphi(w)) = c\Phi_p^{-1}\Big(\Big(\frac{C}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(w)}^j}{(1-|\varphi(w)|^2)^j} \quad \text{and} \quad g_{w,j}^{(k)}(\varphi(w)) = 0$$

for each $k \in \{1,2,3\} \setminus \{j\}$, where c is a nonzero constant.

Moreover,

$$\sup_{w\in\mathbb{D}} \|g_{w,j}\|_{A_\alpha^{\Phi_p}}^{lux} \lesssim 1.$$

Proof. We write $a = 2(\alpha+2)/p$. Set

$$g(z) = \sum_{i=0}^{2} x_{i+1} f_{\varphi(w),i}(z), \quad z \in \mathbb{D}.$$

First, considering the case of $j=1$, from $g''(\varphi(w)) = g'''(\varphi(w)) = 0$ we obtain the following linear system:

$$\begin{cases} \sum\limits_{i=0}^{2} (a+i)(a+1+i)x_{i+1} = 0 \\ \sum\limits_{i=0}^{2} (a+i)(a+1+i)(a+2+i)x_{i+1} = 0. \end{cases}$$

If we regard x_3 as a free variable, by solving this linear system we obtain

$$x_1 = \frac{(a+2)(a+3)}{a(a+1)} x_3$$

and

$$x_2 = -\frac{2(a+3)}{(a+1)} x_3.$$

Therefore, by a direct calculation and the expression of $g'(\varphi(w))$ we find that

$$c = \frac{2}{a+1} x_3.$$

Clearly $x_3 \neq 0$ if and only if $c \neq 0$. Hence, let x_3 be a nonzero number. For this x_3, we take $c_3 = x_3$. Then replacing x_3 with c_3 in the above relations between x_1, x_2 and x_3, we obtain c_1 and c_2. For such obtained c_1, c_2 and c_3, we get the needed function

$$g_{w,1}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(w),i}(z).$$

Since we can similarly prove the lemma for the cases of $j=2$ and $j=3$, the proof is omitted here. By Lemma 2.3, the asymptotic estimate

$$\sup_{w \in \mathbb{D}} \|g_{w,j}\|^{lux}_{A^{\Phi_p}_\alpha} \lesssim 1$$

is also obvious. The proof is finished. $\square$

3. The operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$

First we consider the boundedness of operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$. In the boundedness criteria, we assume that $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Under this assumption, $A^{\Phi_p}_\alpha$ is a complete metric space (see, for example, [20]).

Theorem 3.1. *Let $p \geq 1$, $\alpha > -1$, $\beta > 0$ and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent*:

(i) *The operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$ is bounded.*

(ii) *Functions φ and ψ satisfy the following conditions*:

$$M_1 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty,$$

$$M_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\Big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\Big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty,$$

and

$$M_3 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}\big|\psi(z)\big|\big|\varphi'(z)\big|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \infty.$$

Moreover, if the operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$ is bounded, then

$$\big\|W_{\varphi,\psi}D\big\| \simeq 1 + M_1 + 2M_2 + M_3.$$

Proof. (i)⇒(ii). Suppose that (i) holds. We consider the function $f(z) = z$. Since the operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$ is bounded, we have

$$J_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\big| \le \|W_{\varphi,\psi}Dz\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}D\|. \tag{6}$$

For $w \in \mathbb{D}$ and D_1 (the constant in Lemma 2.2), by Lemma 2.4 there exist constants c_1, c_2 and c_3 such that the function

$$g_{w,1}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(w),i}(z)$$

satisfies $\sup_{w\in\mathbb{D}}\|g_{w,1}\|^{lux}_{A_\alpha^{\Phi_p}} \le C$, $g''_{w,1}(\varphi(w)) = g'''_{w,1}(\varphi(w)) = 0$ and

$$g'_{w,1}(\varphi(w)) = c\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2}. \tag{7}$$

From these facts and the boundedness of $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$, we have

$$\begin{aligned}\frac{(1-|w|^2)^\beta|\varphi(w)|}{1-|\varphi(w)|^2}|\psi''(w)|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) &\le \|W_{\varphi,\psi}Dg_{w,1}\|_{\mathcal{Z}_\beta}\\ &\le C\|W_{\varphi,\psi}D\|.\end{aligned}$$

This leads to

$$J_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi(z)|}{1-|\varphi(z)|^2}|\psi''(z)|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \le C\|W_{\varphi,\psi}D\|. \tag{8}$$

Then for the fixed $\delta \in (0,1)$, from (6) it follows that

$$\begin{aligned}&\sup_{\{z:|\varphi(z)|\le\delta\}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)\\ &\quad\le \frac{J_1}{1-\delta^2}\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-\delta^2}\Big)^{\alpha+2}\Big) \le C\|W_{\varphi,\psi}D\|,\end{aligned} \tag{9}$$

and from (8) it follows that

$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$

$$\leq \frac{J_2}{\delta} \leq C\|W_{\varphi,\psi}D\|. \tag{10}$$

Hence, combing (9) and (10), we obtain

$$M_1 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \leq C\|W_{\varphi,\psi}D\| < \infty. \tag{11}$$

Next we prove $M_2 < \infty$. First taking the function $f(z) = z^2$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z) + \psi''(z)\varphi(z)\big| \leq C\|W_{\varphi,\psi}D\|. \tag{12}$$

Since

$$J_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi''(z)| \leq C\|W_{\varphi,\psi}D\|$$

and the fact $\|\varphi\|_\infty \leq 1$ imply

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi''(z)||\varphi(z)| \leq C\|W_{\varphi,\psi}D\|,$$

by (12) we have

$$K_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big| \leq C\|W_{\varphi,\psi}D\|. \tag{13}$$

For $w \in \mathbb{D}$ and D_2, by Lemma 2.4 there exist c_1, c_2 and c_3 such that the function

$$g_{w,2}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(w),i}(z)$$

satisfies $\sup_{w\in\mathbb{D}} \|g_{w,2}\|^{lux}_{A^{\Phi_p}_\alpha} \leq C$, $g'_{w,2}(\varphi(w)) = g'''_{w,2}(\varphi(w)) = 0$ and

$$g''_{w,2}(\varphi(w)) = c\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^2}. \tag{14}$$

From these and the boundedness of $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$, we get

$$\frac{(1-|w|^2)^\beta|\varphi(w)|^2}{(1-|\varphi(w)|^2)^2}\big|\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big) \tag{15}$$

$$\leq \|W_{\varphi,\psi}Dg_{w,2}\|_{\mathcal{Z}_\beta} \leq C\|W_{\varphi,\psi}D\|.$$

Then (15) shows

$$K_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \tag{16}$$

$$\leq C\|W_{\varphi,\psi}D\|.$$

Hence, for the fixed $\delta \in (0,1)$, by (13)

(17)
$$\sup_{\{z:|\varphi(z)|\leq\delta\}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\leq \frac{K_1}{(1-\delta^2)^2}\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-\delta^2}\Big)^{\alpha+2}\Big) \leq C\|W_{\varphi,\psi}D\|,$$

and by (16)

(18)
$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\leq \frac{K_2}{\delta^2} \leq C\|W_{\varphi,\psi}D\|.$$

So, from (17) and (18), we obtain

(19)
$$\begin{aligned} M_2 &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)\\ &\leq C\|W_{\varphi,\psi}D\| < \infty. \end{aligned}$$

Finally, we prove $M_3 < \infty$. Taking the function $f(z) = z^3$, we have

$$\begin{aligned} \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|&\psi''(z)\varphi(z)^2 + 2\psi(z)\varphi''(z)\varphi(z)^2\\ &+ 4\psi'(z)\varphi'(z)\varphi(z) + 2\psi(z)\varphi'(z)^2\Big|\\ \leq \|W_{\varphi,\psi}Dz^3\|_{\mathcal{Z}_\beta} &\leq C\|W_{\varphi,\psi}D\|. \end{aligned} \tag{20}$$

By (6), (13), (20) and the fact that $\|\varphi\|_\infty \leq 1$,

$$L_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2 \leq C\|W_{\varphi,\psi}D\|. \tag{21}$$

For $w \in \mathbb{D}$ and D_3, Lemma 2.4 shows that there exist constants c_1, c_2 and c_3 such that the function

$$g_{w,3}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(w),i}(z)$$

satisfies $\sup_{w\in\mathbb{D}}\|g_{w,3}\|^{lux}_{A^{\Phi_p}_\alpha} \leq C$, $g'_{w,3}(\varphi(w)) = g''_{w,3}(\varphi(w)) = 0$ and

$$g'''_{w,3}(\varphi(w)) = c\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(w)}^3}{(1-|\varphi(w)|^2)^3}. \tag{22}$$

Then

$$\frac{(1-|w|^2)^\beta|\varphi(w)|^3}{(1-|\varphi(w)|^2)^3}|\psi(w)||\varphi'(w)|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(w)|^2}\Big)^{\alpha+2}\Big)$$
$$\le \|W_{\varphi,\psi}Dg_{w,3}\|_{\mathcal{Z}_\beta}\le C\|W_{\varphi,\psi}D\|.$$

From this, we get

$$L_2=\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi(z)|^3}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\le C\|W_{\varphi,\psi}D\|. \tag{23}$$

For the fixed $\delta\in(0,1)$, by (21)

$$\sup_{\{z:|\varphi(z)|\le\delta\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}\big|\psi(z)\big|\big|\varphi'(z)\big|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big)$$
$$\le\frac{L_1}{(1-\delta^2)^3}\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-\delta^2}\Big)^{\alpha+2}\Big)\le C\|W_{\varphi,\psi}D\|, \tag{24}$$

and by (23)

$$\sup_{\{z:|\varphi(z)|>\delta\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}\big|\psi(z)\big|\big|\varphi'(z)\big|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \tag{25}$$
$$\le\frac{L_2}{\delta^3}\le C\|W_{\varphi,\psi}D\|.$$

Therefore, by (24) and (25)

$$M_3=\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}\big|\psi(z)\big|\big|\varphi'(z)\big|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) \tag{26}$$
$$\le C\|W_{\varphi,\psi}D\|<\infty.$$

(ii)⇒(i). By Lemma 2.2, for all $f\in A_\alpha^{\Phi_p}$ we have

$$\begin{aligned}
&\|W_{\varphi,\psi}Df\|_{\mathcal{Z}_\beta} \qquad (27)\\
&=\big|\psi(0)f'(\varphi(0))\big|+\big|(\psi\cdot f'\circ\varphi)'(0)\big|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|(\psi\cdot f'\circ\varphi)''(z)\big|\\
&\le\big|\psi(0)f'(\varphi(0))\big|+\big|(\psi\cdot f'\circ\varphi)'(0)\big|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f'(\varphi(z))\big|\\
&\quad+2\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f''(\varphi(z))\big|\\
&\quad+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f'''(\varphi(z))\big|\\
&\le C\Big(1+M_1+2M_2+M_3\Big)\|f\|^{lux}_{A_\alpha^{\Phi_p}}.
\end{aligned}$$

From condition (ii) and (27), it follows that $W_{\varphi,\psi}D:A_\alpha^{\Phi_p}\to\mathcal{Z}_\beta$ is bounded. From (11), (19), (26), (27) and Lemma 2.2, we also obtain the asymptotic expression of $\|W_{\varphi,\psi}D\|$. The proof is finished. □

We begin to characterize the compactness of operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$.

Theorem 3.2. *Let $p \geq 1$, $\alpha > -1$, $\beta > 0$ and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent*:

(i) *The operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$ is compact.*

(ii) *Functions φ and ψ are such that $\psi \in \mathcal{Z}_\beta$,*

$$J_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big| < \infty,$$

$$J_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \big|\psi(z)\big|\big|\varphi'(z)\big|^2 < \infty,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}\big|\psi(z)\big|\big|\varphi'(z)\big|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) = 0.$$

Proof. (i)$\Rightarrow$(ii). Suppose that (i) holds. Then the operator $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$ is bounded. In the proof of Theorem 3.1, we have proven that $\psi \in \mathcal{Z}_\beta$, $J_1 < \infty$ and $J_2 < \infty$.

Next consider a sequence $\{\varphi(z_n)\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such sequence does not exist, then condition (ii) obviously holds. By Lemma 2.4, we can choose constants c_1, c_2 and c_3 such that the functions

$$g_{n,1}(z) := g_{\varphi(z_n),1}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(z_n),i}(z)$$

satisfy $\sup_{n\in\mathbb{N}} \|g_{n,1}\|^{lux}_{A^{\Phi_p}_\alpha} \leq C$, $g''_{n,1}(\varphi(z_n)) = g'''_{n,1}(\varphi(z_n)) = 0$ and

$$g'_{n,1}(\varphi(z_n)) = c\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}. \tag{28}$$

From the proof of Theorem 3.6 in [20], it follows that the sequence $\{g_{n,1}\}_{n\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$ as $n \to \infty$. Hence, by Lemma 2.1,

$$\lim_{n\to\infty} \|W_{\varphi,\psi}Dg_{n,1}\|_{\mathcal{Z}_\beta} = 0.$$

From these, we have

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{1-|\varphi(z_n)|^2}\big|\psi''(z_n)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

By Lemma 2.4, there are constants c_1, c_2 and c_3 such that the functions

$$g_{n,2}(z) := g_{\varphi(z_n),2}(z) = \sum_{i=0}^{2} c_{i+1} f_{\varphi(z_n),i}(z)$$

satisfy $\sup_{n\in\mathbb{N}} \|g_{n,2}\|^{lux}_{A^{\Phi_p}_\alpha} \le C$, $g'_{n,2}(\varphi(z_n)) = g'''_{n,2}(\varphi(z_n)) = 0$ and

$$g''_{n,2}(\varphi(z_n)) = c\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(z_n)}^2}{(1-|\varphi(z_n)|^2)^2}. \tag{29}$$

From the proof of Theorem 3.6 in [20], the sequence $\{g_{n,2}\}_{n\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$ as $n\to\infty$. Then by Lemma 2.1,

$$\lim_{n\to\infty} \|W_{\varphi,\psi} D g_{n,2}\|_{\mathcal{Z}_\beta} = 0.$$

As a result, we get

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^2}\big|\psi(z_n)\varphi''(z_n) + 2\psi'(z)\varphi'(z_n)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

Finally, we choose the functions

$$g_{n,3}(z) := g_{\varphi(z_n),3} = \sum_{i=0}^{2} c_{i+1} f_{\varphi(z_n),i}(z).$$

Lemma 2.4 shows that all the functions $g_{n,3}$ satisfy $\sup_{n\in\mathbb{N}} \|g_{n,3}\|^{lux}_{A^{\Phi_p}_\alpha} \le C$,

$$g'''_{n,3}(\varphi(z_n)) = c\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big)\frac{\overline{\varphi(z_n)}^3}{(1-|\varphi(z_n)|^2)^3} \tag{30}$$

and

$$g'_{n,3}(\varphi(z_n)) = g''_{n,3}(\varphi(z_n)) = 0. \tag{31}$$

The sequence $\{g_{n,3}\}_{n\in\mathbb{N}}$ also uniformly converges to zero on any compact subset of $\mathbb{D}$ as $n\to\infty$. Then from Lemma 2.1, it follows that

$$\lim_{n\to\infty} \|W_{\varphi,\psi} D g_{n,3}\|_{\mathcal{Z}_\beta} = 0.$$

Consequently, from these facts we obtain

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^3}|\psi(z_n)||\varphi'(z_n)|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z_n)|^2}\Big)^{\alpha+2}\Big) = 0.$$

The proof of the implication is finished.

(ii)$\Rightarrow$(i). We first check that $W_{\varphi,\psi}D : A^{\Phi_p}_\alpha \to \mathcal{Z}_\beta$ is bounded. For this we observe that condition (ii) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$L_1(z) = \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi''(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon, \tag{32}$$

(33)

$$L_2(z) = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon,$$

and

$$L_3(z) = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-|\varphi(z)|^2}\Big)^{\alpha+2}\Big) < \varepsilon \tag{34}$$

for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$.

Since $\psi \in \mathcal{Z}_\beta$, by (32) we have

$$\begin{aligned} M_1 = \sup_{z\in\mathbb{D}} L_1(z) &\le \sup_{z\in\mathbb{D}\setminus K} L_1(z) + \sup_{z\in K} L_1(z) \\ &\le \frac{\|\psi\|_{\mathcal{Z}_\beta}}{1-\eta^2}\Phi_p^{-1}\Big(\Big(\frac{D_1}{1-\eta^2}\Big)^{\alpha+2}\Big)+\varepsilon. \end{aligned}$$

From (33) and $J_1 < \infty$, we obtain

$$\begin{aligned} M_2 = \sup_{z\in\mathbb{D}} L_2(z) &\le \sup_{z\in\mathbb{D}\setminus K} L_2(z) + \sup_{z\in K} L_2(z) \\ &\le \frac{J_1}{(1-\eta^2)^2}\Phi_p^{-1}\Big(\Big(\frac{D_2}{1-\eta^2}\Big)^{\alpha+2}\Big)+\varepsilon. \end{aligned}$$

From (34) and $J_2 < \infty$, we also get

$$\begin{aligned} M_3 = \sup_{z\in\mathbb{D}} L_2(z) &\le \sup_{z\in\mathbb{D}\setminus K} L_3(z) + \sup_{z\in K} L_3(z) \\ &\le \frac{J_2}{(1-\eta^2)^3}\Phi_p^{-1}\Big(\Big(\frac{D_3}{1-\eta^2}\Big)^{\alpha+2}\Big)+\varepsilon. \end{aligned}$$

So, by Theorem 3.1, the operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$ is bounded.

To prove that the operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p} \to \mathcal{Z}_\beta$ is compact, by Lemma 2.1, we just need to prove that, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $A_\alpha^{\Phi_p}$ such that $\|f_n\|^{lux}_{A_\alpha^{\Phi_p}} \le M$ and $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $n \to \infty$, then

$$\lim_{n\to\infty}\|W_{\varphi,\psi}Df_n\|_{\mathcal{Z}_\beta} = 0.$$

For any $\varepsilon > 0$ and the above η, we have, by using again the condition (ii) and Lemma 2.2,

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|W_{\varphi,\psi}Df_n(z)\big| \\ =\ &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|(\psi\cdot f_n'\circ\varphi)''(z)\big| \\ \le\ &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big| \\ &+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n''(\varphi(z))\big| \\ &+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big| \end{aligned}$$

$$
\begin{aligned}
&\le \sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big| + \sup_{z\in K}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big|\\
&\quad + 2\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n''(\varphi(z))\big|\\
&\quad + 2\sup_{z\in K}(1-|z|^2)^\beta\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\big|f_n''(\varphi(z))\big|\\
&\quad + \sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big|\\
&\quad + \sup_{z\in K}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big|\\
&\le \|\psi\|_{\mathcal{Z}_\beta}\sup_{\{z:|z|\le\eta\}}\big|f_n'(z)\big| + M\sup_{z\in K}L_1(z) + J_1\sup_{\{z:|z|\le\eta\}}\big|f_n''(z)\big| + M\sup_{z\in K}L_2(z)\\
&\quad + J_2\sup_{\{z:|z|\le\eta\}}\big|f_n'''(z)\big| + M\sup_{z\in K}L_3(z)\\
&\le \|\psi\|_{\mathcal{Z}_\beta}\sup_{\{z:|z|\le\eta\}}\big|f_n'(z)\big| + J_1\sup_{\{z:|z|\le\eta\}}\big|f_n''(z)\big| + J_2\sup_{\{z:|z|\le\eta\}}\big|f_n'''(z)\big| + 3M\varepsilon.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(35)\qquad &\|W_{\varphi,\psi}Df_n\|_{\mathcal{Z}_\beta}\\
&\le \|\psi\|_{\mathcal{Z}_\beta}\sup_{\{z:|z|\le\eta\}}\big|f_n'(z)\big| + J_1\sup_{\{z:|z|\le\eta\}}\big|f_n''(z)\big| + J_2\sup_{\{z:|z|\le\eta\}}\big|f_n'''(z)\big|\\
&\quad + 3M\varepsilon + \big|(\psi\cdot f_n'\circ\varphi)(0)\big| + \big|(\psi\cdot f_n'\circ\varphi)'(0)\big|\\
&= \|\psi\|_{\mathcal{Z}_\beta}\sup_{\{z:|z|\le\eta\}}\big|f_n'(z)\big| + J_1\sup_{\{z:|z|\le\eta\}}\big|f_n''(z)\big| + J_2\sup_{\{z:|z|\le\eta\}}\big|f_n'''(z)\big|\\
&\quad + 3M\varepsilon + \big|\psi(0)f_n'(\varphi(0))\big| + \big|\psi'(0)f_n'(\varphi(0)) + \psi(0)\varphi'(0)f_n''(\varphi(0))\big|.
\end{aligned}
$$

It is easy to see that, if $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$, then $\{f_n'\}_{n\in\mathbb{N}}$, $\{f_n''\}_{n\in\mathbb{N}}$ and $\{f_n'''\}_{n\in\mathbb{N}}$ also do as $n\to\infty$. Since $\{z:|z|\le\eta\}$ and $\{\varphi(0)\}$ are compact subsets of $\mathbb{D}$, letting $n\to\infty$ in (35) gives

$$\lim_{n\to\infty}\|W_{\varphi,\psi}Df_n\|_{\mathcal{Z}_\beta}=0.$$

From Lemma 2.1, it follows that the operator $W_{\varphi,\psi}D : A_\alpha^{\Phi_p}\to\mathcal{Z}_\beta$ is compact. The proof is finished. □

Acknowledgments. The author wishes to express his sincere thanks to the anonymous referee for a thorough review and insightful suggestions. This work was partially supported by the National Natural Science Foundation of China (No.11201323), the Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2013QZJ01) and the Introduction of Talent Project of SUSE (No.2014RC04).

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