ON THE RETRACTS AND RECODINGS OF CONTINUING CODES

Jisang Yoo

ABSTRACT. We investigate what happens when we try to work with continuing block codes (i.e., left or right continuing factor maps) between shift spaces that may not be shifts of finite type. For example, we demonstrate that continuing block codes on strictly sofic shifts do not behave as well as those on shifts of finite type; a continuing block code on a sofic shift need not have a uniformly bounded retract, unlike one on a shift of finite type. A right eresolving code on a sofic shift can display any behavior arbitrary block codes can have. We also show that a right continuing factor of a shift of finite type is always a shift of finite type.

1. Introduction

There has been many studies on sliding block codes between shift spaces. They form bases for further studies on topics such as hidden Markov chains, cellular automata, and relative thermodynamic formalism. Sliding block codes on shifts of finite type can be divided into two categories: finite-to-one codes and infinite-to-one codes. The simplest class in the first category is the class of right closing codes. Many things are known about right closing codes between shift spaces [2, 6].

Right continuing codes are a natural generalization of right resolving codes and right closing codes. Right continuing codes are infinite-to-one analog of right closing codes.

Right continuing codes are less well understood than right closing codes are, but some nice properties of such codes are known. For example, continuing codes between SFTs work nice with Markov measures; if ϕ is a continuing code from an SFT X onto another SFT Y, each Markov measure on Y lifts to uncountably many Markov measures on X (see [3]); if ϕ is a fiber-mixing code from an SFT X onto another SFT Y, i.e., given $x, \bar{x} \in X$ with $\phi(x) = \phi(\bar{x}) = y \in Y$, there is $z \in \phi^{-1}(y)$ that is left asymptotic to x and right asymptotic to $\bar{x}$, then ϕ projects each Markov measure on X onto a Gibbs measure on Y

with a Hölder continuous potential (see [4, 7]). (Fiber-mixing condition implies that ϕ is left continuing and right continuing.) It is also known that one can always construct a right continuing code from a given irreducible sofic shift onto a given irreducible SFT of lower entropy (see [5]). In [1], it is proved that any factor map from an irreducible SFT to a sofic shift can be lifted to a right continuing factor map between irreducible SFTs via some one-to-one almost everywhere left closing factor maps.

In this paper, we investigate how right continuing codes behave with respect to shift spaces that are not SFTs. We show that right continuing codes defined on general shift spaces don't necessarily have a retract. But when it does have a retract, we show that it can be assumed to be zero via recoding. We show that there is no right continuing codes from an SFT onto a non-SFT. We also learn that the property of right eresolving codes on non-SFTs are not strong enough to tame block codes in any meaningful way, in particular, the property of right eresolving doesn't even imply the property of right continuing.

We hope that this work will be helpful in understanding right continuing codes and infinite-to-one codes on general shift spaces.

2. Preliminary

An *SFT* or a shift of finite type is a shift space determined by a finite set of forbidden words. An N-step SFT is an SFT determined by a set of forbidden words of length $N + 1$. It is known that any SFT is topologically conjugate to a 1-step SFT. A *non-SFT* is a shift space that is not an SFT. The shift map will be denoted as σ.

A (*sliding*) *block code* is a homomorphism between two topological dynamical systems that are shift spaces. A block code ϕ is a 1-block code if $(\phi(x))_0$ is determined by x_0. It is known that any block code can be assumed to be a 1-block code without loss of generality via recoding, i.e., any block code is topologically conjugate to a 1-block code. It is also known that any block code on a SFT can be recoded to be a 1-block code on a 1-step SFT. A block code from X *onto* Y is called a *factor map* between X and Y, and Y is called a *factor* of X. Factors of SFTs are called *sofic shifts*.

Definition 2.1. The language $\mathcal{B}(X)$ of a shift space X is the collection of words appearing in points of X. The set $\mathcal{B}_n(X)$ is defined to be the collection of words of length n appearing in points of X. In other words, $\mathcal{B}_n(X) = \{x_{[0,n-1]} : x \in X\}$.

Definition 2.2. Two points x, x' in a shift space are *left asymptotic* (resp. *right asymptotic*) to each other if there is $n \in \mathbb{Z}$ with $x_{(-\infty,n]} = x'_{(-\infty,n]}$ (resp. $x_{[n,\infty)} = x'_{[n,\infty)}$), or equivalently if $d(\sigma^{-k}x, \sigma^{-k}x') \to 0$ (resp. $d(\sigma^k x, \sigma^k x') \to 0$) as $k \to \infty$.

Definition 2.3. A 1-block code ϕ from a shift space X onto another shift space Y is *right eresolving* if for $a_0 \in \mathcal{B}_1(X)$ and $b_0 b_1 \in \mathcal{B}_2(Y)$ with $\phi(a_0) = b_0$, there is at least one a_1 with $a_0 a_1 \in \mathcal{B}_2(X)$ and $\phi(a_1) = b_1$.

Definition 2.4. A block code ϕ from a shift space X onto another shift space Y is *right continuing* or *u-eresolving* if for each $x \in X$ and $y \in Y$ such that $\phi(x)$ is left-asymptotic to y, there is at least one $\bar{x} \in X$ such that $\bar{x}$ is left asymptotic to x and $\phi(\bar{x}) = y$.

The terms *left eresolving, left continuing* and *s-eresolving* are defined analogously.

Definition 2.5. A block code from a shift space onto another shift space is *continuing* if it is left continuing or right continuing.

Definition 2.6. A right continuing code ϕ from a shift space X onto another shift space Y is said to have a *retract* $n \in \mathbb{N}$ if for all $x \in X$ and $y \in Y$ with $(\phi x)_i = y_i$ for all $i \leq 0$, there is $\bar{x} \in X$ with $\phi\bar{x} = y$ and $\bar{x}_i = x_i$ for all $i \leq -n$.

We mention a result [3] on a relationship between right eresolving codes and right continuing codes on SFTs. (The term *right resolving* in the paper [3] corresponds to the term *right eresolving* in this paper. In [3], the result was stated only for the irreducible case, but the proof does not use the irreducible condition.)

Proposition 2.1. *For a factor map ϕ between SFTs, the following three conditions are equivalent.*

(1) *ϕ is topologically equivalent to a right eresolving 1-block factor map between 1-step SFTs.*
(2) *ϕ is right continuing.*
(3) *ϕ is right continuing and has a retract.*

After we prove that the right continuing factor of an SFT is always an SFT, the requirement of the codomain being an SFT from the above proposition can be safely dropped.

3. A right continuing code without a retract

This example and the proof are also included in [5] by personal communication.[1]

Example 3.1. Let X be a shift space with the alphabet $\{1, \bar{1}, 2, 3\}$ defined by forbidden blocks $\{\bar{1}2^n 3 : n \geq 0\}$ and let Y be the full shift $\{1, 2, 3\}^{\mathbb{Z}}$. Then X is an irreducible sofic shift. Let $\phi : X \to Y$ be the 1-block code defined by $\phi(1) = \phi(\bar{1}) = 1$, $\phi(2) = 2$, $\phi(3) = 3$. Then the map ϕ is a right-continuing factor map without a retract.

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[1]That paper needed the result before this paper's draft were written.

Proof. The map ϕ is clearly onto Y.

Next, we need to check that it is right continuing. Suppose $x \in X$ and $y \in Y$ with $\phi(x_i) = y_i$ for all $i \leq 0$. If $x^{(1)} := x_{(-\infty,-1]} \cdot x_0 y_{[1,\infty)}$ is in X, then we are done because $x^{(1)}$ is left-asymptotic to x and $\phi(x^{(1)}) = y$. Otherwise, the word $\bar{1}2^n3$ for some $n \geq 0$ occurs exactly once in $x^{(1)}$. Let $x^{(2)}$ be the point obtained from $x^{(1)}$ by replacing all occurrences of $\bar{1}2^n3$ with 12^n3. Then the point $x^{(2)}$ is in X, is left-asymptotic to x and satisfies $\phi(x^{(2)}) = y$. This completes the proof of the right continuing property.

Finally, we check that the map ϕ has no retract. This follows from considering, for each $n \geq 0$, the points $x = \bar{1}^\infty 2^n \cdot 22^\infty \in X$ and $y = 1^\infty 2^n \cdot 23^\infty \in Y$. $\square$

4. Right eresolving codes on non-SFTs

Consider an arbitrary 1-block code ϕ from a shift space X onto another shift space Y. One can transform $\phi : X \to Y$ into a right eresolving code $\sqrt{\phi} : \sqrt{X} \to \sqrt{Y}$ without losing too much of the original code's properties.

First, we choose a symbol a disjoint from the alphabets of X and Y. Then define shift spaces $\sqrt{X} = \{\eta(x), \sigma(\eta(x)) : x \in X\}$ and $\sqrt{Y} = \{\eta(y), \sigma(\eta(y)) : y \in Y\}$ where η is defined by $\eta(x)_{2i} = x_i$ and $\eta(x)_{2i+1} = a$ for all $i \in \mathbb{Z}$. Then define a 1-block code $\sqrt{\phi}$ from $\sqrt{X}$ onto $\sqrt{Y}$ by

$$\sqrt{\phi}(c) = \begin{cases} a & \text{if } c = a, \\ \phi(a) & \text{if } c \neq a \end{cases}$$

for $c \in \mathcal{B}_1(\sqrt{X})$.

The map $\sqrt{\phi}$ is easily seen to be right eresolving and shares many properties with the original ϕ. For example, if ϕ is not right continuing, then neither is $\sqrt{\phi}$. Thus right eresolving codes are not necessarily right continuing. This is in contrast with Proposition 2.1 in the case of SFTs.

5. A right continuing factor of an SFT is an SFT

A right eresolving factor of 1-step SFT is easily seen to be a 1-step SFT. Since a right continuing codes between SFTs can be assumed to be right eresolving by recoding (see Proposition 2.1), one might think that this would easily imply that a right continuing factor of an SFT is an SFT. But the proof of Proposition 2.1 uses the assumption of Y being an SFT, which in turns is what we need to prove. Nevertheless, we manage to prove the following result.

Theorem 5.1. *If X is an SFT and ϕ a right continuing code onto a shift space Y, then Y is an SFT.*

Throughout this section, we assume that ϕ is a right continuing 1-block code from a 1-step SFT X onto a shift space Y unless stated otherwise. We cannot assume without loss of generality that ϕ is eresolving to make things easier at this point. We will not assume the retract to be 0 because we want to see how

the memory of Y grows depending on the size of the retract. We prove the theorem through a series of lemmas.

Lemma 5.2. *There is $R \in \mathbb{N}$ such that for all $x \in X$ and $y \in Y$ with $\phi(x_i) = y_i$, $i \le 0$, there exist $x'_{[-R,1]} \in \mathcal{B}(X)$ with $x'_{-R} = x_{-R}$ and $\phi(x'_i) = y_i$, $i \in [-R,1]$.*

Proof. Suppose it is false. Then for all $n \in \mathbb{N}$, there exist $x^{(n)} \in X$ and $y^{(n)} \in Y$ such that $\phi(x_i^{(n)}) = y_i^{(n)}$, $i \le 0$ and that there does not exist a word $x'_{[-n,1]} \in \mathcal{B}(X)$ satisfying

$$(*) \qquad\qquad x_{-n}^{(n)} = x'_{-n}, \phi(x'_{[-n,1]}) = y_{[-n,1]}^{(n)}.$$

By a compactness argument, there is a limit point $(x,y) \in X \times Y$ of the sequence of pairs $(x^{(n)}, y^{(n)})$. We have $\phi(x_i) = y_i$, $i \le 0$ and since ϕ is right continuing, there are $\bar{n} \in \mathbb{N}$ and $\bar{x} \in X$ such that $\phi(\bar{x}) = y$ and $\bar{x}_i = x_i$, $i \le -\bar{n}$. We can choose $n \in \mathbb{N}$ such that $n > \bar{n}$ and $x_i = x_i^{(n)}$, $y_i = y_i^{(n)}$, $i \in [-\bar{n}, 1]$. But the word $x'_{[-n,1]} := x_{[-n,-\bar{n}-1]}^{(n)} \bar{x}_{[-\bar{n},1]}$, which is in $\mathcal{B}(X)$ because X is a 1-step SFT, satisfies $(*)$ and that contradicts our initial assumption. $\square$

Since X is assumed to be a 1-step SFT, it is easily shown by repeated application of the previous lemma that the number R is a retract of ϕ. (It is noted by a referee that the existence of a retract is also mentioned in [5].)

Lemma 5.3. *There is $K \in \mathbb{N}$ such that for all $\pi \in \mathcal{B}_K(X)$ and $u \in \mathcal{B}_{K+1}(Y)$ with $\phi(\pi)$ being a prefix of u, there exist $\pi' \in \mathcal{B}_{K+1}(X)$ with $\pi_1 = \pi'_1$ and $\phi(\pi') = u$.*

Proof. Let $d := (|\mathcal{B}_1(X)|)^2 + 1$ and $K := d + R + 1$ and consider two words $\pi = x_{[-R-d,0]} \in \mathcal{B}_K(X)$ and $u = y_{[-R-d,1]} \in \mathcal{B}_{K+1}(Y)$ with $\phi(x_{[-R-d,0]}) = y_{[-R-d,0]}$.

We can choose a ϕ-preimage $\bar{x}_{[-R-d,1]} \in \mathcal{B}(X)$ of $y_{[-R-d,1]}$. By the pigeon hole principle, there exist $-R - d \le I < J \le -R - 1$ such that $x_I = x_J$ and $\bar{x}_I = \bar{x}_J$. Build a sequence $x_1 x_2 x_3 \cdots$ that extends $x_{[-R-d,0]}$ and $\bar{x}_2 \bar{x}_3 \cdots$ that extends $\bar{x}_{[-R-d,1]}$. Now $x_{[-R-d,\infty)}$ and $\bar{x}_{[-R-d,\infty)}$ are right-infinite sequences allowed in X. The point $\hat{x} := (x_{[I,J-1]})^\infty x_{[J,-1]} \cdot x_{[0,\infty)}$ is in X. The point $(\bar{x}_{[I,J-1]})^\infty \bar{x}_{[J,-1]}.\bar{x}_{[0,\infty)}$ is also in X and its image $\hat{y} := (y_{[I,J-1]})^\infty y_{[J,-1]} \cdot y_0 y_1 \phi(\bar{x}_2) \phi(\bar{x}_3) \cdots$ is in Y and we have $\phi(\hat{x}_i) = \hat{y}_i$, $i \le 0$. By applying the previous lemma to $\hat{x}$ and $\hat{y}$, we can see that there exist $x'_{[-R,1]} \in \mathcal{B}(X)$ satisfying $x'_{-R} = \hat{x}_{-R} = x_{-R}$ and $\phi(x'_{[-R,1]}) = \hat{y}_{[-R,1]} = y_{[-R,1]}$.

Define $x'_{[-R-d,-R-1]} := x_{[-R-d,-R-1]}$, then we have $x'_{[-R-d,1]} \in \mathcal{B}(X)$ and $\phi(x'_{[-R-d,1]}) = y_{[-R-d,1]}$. $\square$

Remark 5.1. In order to prove the above lemma, one might be temped to just set $K := R$ and start by extending the word π to a point $x \in X$ such that $x_{[-R,0]} = \pi$ and then somehow extend y to apply Lemma 5.2. But choosing

an appropriate $y \in Y$ requires showing that $x_{[-\infty, -R-1]}u \in \mathcal{B}(Y)$, which you cannot show at this point because Y is not yet known to be an SFT, let alone an R-step SFT.

Lemma 5.4. *For words* $u, v \in \mathcal{B}(Y)$ *and a symbol* $a \in \mathcal{B}_1(Y)$ *such that* $uv \in \mathcal{B}(Y)$, $va \in \mathcal{B}(Y)$ *and* $|v| = K$, *we have* $uva \in \mathcal{B}(Y)$.

Proof. There exist $\pi, \tau \in \mathcal{B}(X)$ such that $\pi\tau \in \mathcal{B}(X)$, $\phi(\pi\tau) = uv$ and $|\tau| = K$. Applying the previous lemma to τ and va, we see that there is $\tau' \in \mathcal{B}(X)$ with $\tau_1 = \tau_1'$ and $\phi(\tau') = va$. Clearly $\pi\tau' \in \mathcal{B}(X)$ and $\phi(\pi\tau') = uva$. □

Proposition 5.5. Y *is a* K-*step SFT.*

Proof. It follows easily from the previous lemma. □

The proof of Theorem 5.1 is now complete.

Remark 5.2. Since Y is now shown to be an SFT, Proposition 2.1 applies and therefore ϕ is topologically equivalent to a right-eresolving code from a 1-step SFT onto another 1-step SFT.

Remark 5.3. This result implies in particular that there is no right continuing code from an SFT onto a strictly sofic shift. This contrasts with the fact that every sofic shift is a right closing factor of an SFT.

Question 5.6. We showed that Y is a K-step SFT where $K = R + 2 + (|\mathcal{B}_1(X)|)^2$. Is there a better bound on K?

Remark 5.4. An example of Y not being an R-step SFT is as follows. Let R, R' be arbitrary positive integers. Let Y be any $(R + R')$-step SFT that is not an $(R + R' - 1)$-step SFT. Let X be the $(R + R' + 1)$-th higher block shift of Y. Then X is a 1-step SFT conjugate to Y. Let $\phi : X \to Y$ be a 1-block factor map defined by $\phi(y_{[-R', R]}) = y_0$. Since ϕ is a conjugacy, it is right continuing. One can easily check that R is a retract of ϕ.

6. Decreasing the retract of a right continuing map

Theorem 6.1. *Given a right continuing code* ϕ *with a retract from a shift space* X *onto a shift space* Y, *there exists a topological conjugacy* ψ *from* X *to a shift space* $\bar{X}$ *such that* $\phi \circ \psi^{-1} : \bar{X} \to Y$ *is a right continuing* 1-*block code with retract* 0.

Proof. We may assume that ϕ is a 1-block code and its retract is R.

Define a block code ψ on X by

$$(\psi x)_i = (x_{i-R}, (\phi x)_{[i-R, i]})$$

for all $x \in X$ and $i \in \mathbb{Z}$. This code is injective, therefore a conjugacy onto its image $\bar{X}$.

Let $\bar{\phi} := \phi \circ \psi^{-1}$. This is a 1-block code because for $\bar{x} \in \bar{X}$ and $x := \psi^{-1}(\bar{x})$ we have

$$(\bar{\phi}\bar{x})_0 = (\phi x)_0$$
$$= F((\psi x)_0)$$
$$= F(\bar{x}_0),$$

where F is a function that maps $(a, b_0 b_1 \cdots b_R)$ to b_R.

It remains to show that $\bar{\phi}$ is right continuing with retract 0.

Suppose that we are given $\bar{x} \in \bar{X}$, $y \in Y$ with $(\bar{\phi}\bar{x})_{(-\infty,0]} = y_{(-\infty,0]}$. Then the point $x := \psi^{-1}\bar{x}$ satisfies $(\phi x)_{(-\infty,0]} = y_{(-\infty,0]}$. But since ϕ has retract R, there is $z \in X$ satisfying $x_{(-\infty,-R]} = z_{(-\infty,-R]}$ and $\phi z = y$. The point $\bar{z} := \psi z$ satisfies the desired properties: for $i \leq 0$

$$\bar{z}_i = (z_{i-R}, (\phi z)_{[i-R,i]})$$
$$= (z_{i-R}, y_{[i-R,i]})$$
$$= (x_{i-R}, (\phi x)_{[i-R,i]})$$
$$= \bar{x}_i$$

and $\bar{\phi}\bar{z} = y$. $\qquad\square$

For the next theorem, we need some definitions.

Definition 6.1. A block code from a shift space onto another shift space is *bi-continuing* if it is left continuing and right continuing.

Definition 6.2. A bi-continuing code ϕ from a shift space X onto another shift space Y is said to have *retracts* $n_1, n_2 \in \mathbb{N}$ if n_1 is a retract of ϕ as a right continuing code and if n_2 is a retract of ϕ as a left continuing code.

Theorem 6.2. *Given a bi-continuing factor map ϕ with retracts from a shift space X to a shift space Y, there are conjugacies $\psi : X \to \bar{X}$ and $\theta : Y \to \bar{Y}$ such that $\theta \circ \phi \circ \psi^{-1} : \bar{X} \to \bar{Y}$ is a bi-continuing 1-block factor map with retracts $0, 0$.*

Proof. We may assume ϕ is a 1-block code with retracts R, R. Let $\bar{X}$ and $\bar{Y}$ be images of block codes ψ and θ defined by

$$(\psi x)_i = (x_i, (\phi x)_{[i-R,i+R]})$$
$$(\theta y)_i = y_{[i-R,i+R]}$$

for $i \in \mathbb{Z}$, $x \in X$ and $y \in Y$. They are easily checked to be injective, hence they are conjugacies.

Let $\bar{\phi} := \theta \circ \phi \circ \psi^{-1} : \bar{X} \to \bar{Y}$, then it is a 1-block code since for all $\bar{x} \in \bar{X}$ and $x := \psi^{-1}\bar{x}$ we have

$$(\bar{\phi}\bar{x})_0 = (\theta \phi x)_0$$
$$= (\phi x)_{[i-R,i+R]}$$

$$= F((\psi x)_0)$$
$$= F(\bar{x}_0),$$

where the function F maps (a, b) to b.

To show that $\bar{\phi}$ is right continuing with retract 0, suppose we are given $\bar{x} \in \bar{X}$ and $\bar{y} \in \bar{Y}$ with $(\bar{\phi}\bar{x})_{(-\infty,0]} = \bar{y}_{(-\infty,0]}$ and let $x = \psi^{-1}\bar{x}$ and $y = \theta^{-1}\bar{y}$. Then we have $y_{(-\infty,R]} = (\phi x)_{(-\infty,R]}$ and since ϕ is right continuing with retract R, there is $z \in X$ such that $x_{(-\infty,0]} = z_{(-\infty,0]}$ and $\phi z = y$. Let $\bar{z} = \psi z$, then we have for $i \le 0$,

$$\bar{x}_i = (x_i, (\phi x)_{[i-R,i+R]})$$
$$= (x_i, y_{[i-R,i+R]})$$
$$= (z_i, (\phi z)_{[i-R,i+R]})$$
$$= \bar{z}_i$$

and that $\bar{\phi}\bar{z} = \bar{y}$

Similarly, $\bar{\phi}$ is also left continuing with retract 0. $\square$

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RESEARCH INSTITUTE OF BASIC SCIENCES
AJOU UNIVERSITY
SUWON 443-749, KOREA
E-mail address: jisangy@kaist.ac.kr