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ON SOME SUBGROUPS OF D^* WHICH SATISFY A GENERALIZED GROUP IDENTITY

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Abstract. Let D be a division ring and $w(x_1, x_2, \ldots, x_m)$ be a generalized group monomial over D^*. In this paper, we investigate subnormal subgroups and maximal subgroups of D^* which satisfy the identity $w(x_1, x_2, \ldots, x_m) = 1$.

1. Introduction

A *group monomial* is a non-trivial word

$$u(x_1, x_2, \ldots, x_m) = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_t}^{\alpha_t}$$

in the free (multiplicative) group F_m generated by indeterminates $x_1, x_2, \ldots, x_m$ for some positive integer m. Let G be a group with center $Z(G) = \{a \in G \mid ab = ba \text{ for any } b \in G\}$. An element

$$w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$$

of the free product $G * F_m$ of G and F_m for some positive integer m is called a *generalized group monomial* over G if, for any $j = 1, 2, \ldots, t-1$, whenever $i_j = i_{j+1}$ and $\alpha_j \alpha_{j+1} < 0$, one has $a_{j+1} \notin Z(G)$ (see [17]). The integer $\alpha(w) = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_t|$ is called the *length* of w. Let H be a subgroup of G. If $w(c_1, c_2, \ldots, c_m) = 1$ (resp. $u(c_1, c_2, \ldots, c_m) = 1$) for any $c_1, c_2, \ldots, c_m \in H$, then we say that H *satisfies* $w(x_1, x_2, \ldots, x_m) = 1$ (resp. $u(x_1, x_2, \ldots, x_m) = 1$) or that $w(x_1, x_2, \ldots, x_m) = 1$ (resp. $u(x_1, x_2, \ldots, x_m) = 1$) is a *generalized group identity* (resp. *group identity*) of H.

Generalized group identities are a good technique to "link" a group and its subgroups. We use this nice technique to investigate some classes of subgroups of the multiplicative group of a division ring.

We recall briefly some known results on division rings whose multiplicative groups satisfy a group identity and, more generally, a generalized group identity. Let D be a division ring with center F. The first result comes from Amitsur.

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In 1966, he showed in [2] that if F is infinite and $D^* = D\backslash\{0\}$ satisfies a group identity, then $D = F$. In 1982, Golubchik and Mikhalev proved that if F is infinite and D^* satisfies a generalized group identity over D^*, then $D = F$ (see [9]). In the proofs of these results, the condition "F is infinite" is really essential. In 2004, in their paper [5], Chebotar and Lee considered the case when F is finite and showed that if F contains sufficiently many elements, then Amitsur's result still holds. In fact, they showed that if the cardinality of F is greater than $\frac{3\alpha(w)}{2}$, then $D = F$. Recently, there are some articles on some subgroups of D^* which satisfy a group identity or some special group identity (see [7, 10, 12, 14]): Ramezan-Nassab and Kiani proved in [14] that subnormal subgroups of D^* satisfying the n-Engel condition are contained in F. It is proved in [12] that every maximal subgroup of D^* satisfying a group identity is the multiplicative group of a maximal subfield of D if $F[M] \neq D$ and $F[M]$ is algebraic over F. Here, $F[M]$ is the subring of D generated by M and F.

The goal of this paper is to investigate subnormal or maximal subgroups of D^* satisfying a generalized group identity in case when F contains sufficiently many elements. In Section 2, we generalize two classical results [9, Theorem 2] and [5, Theorem 4]. In Section 3, we study subnormal subgroups of D^* satisfying a generalized group identity, and then two interesting corollaries are presented as well. In Section 4, we focus our attention on maximal subgroups of D^* satisfying a generalized group identity.

2. Division rings satisfying a generalized group identity

In this section, we prove that for a division ring D whose multiplicative group D^* satisfies a generalized group identity over D^*, if the center of D contains sufficiently many elements, then D is commutative. This is an extension of both [9, Theorem 2] and [5, Theorem 4].

We first recall notation used in this paper. For a set S, the notation $|S|$ denotes the cardinality of S. For any division ring D with center F and an indeterminate x, we denote by $D((x))$ the division ring of Laurent series. Hence, $F((x))$ is the center of $D((x))$ [13, Proposition 14.2]. Denote by $D(x)$ the division subring of $D((x))$ generated by D and x. Assume that $y_1, y_2, \ldots, y_n$ are $n \geq 1$ indeterminates, by $F\langle y_1, y_2, \ldots, y_n\rangle$ the free F-algebra on $y_1, y_2, \ldots, y_n$ and by $D[[y_1, y_2, \ldots, y_n]]$ the universal division ring of fractions of the free product of D and $F\langle y_1, y_2, \ldots, y_n\rangle$ over F. An element of a group G is called *central* if it is in the center $Z(G)$ of G. Otherwise, it is called *non-central*.

Lemma 2.1. *Let D be a division ring, x an indeterminate and $D(x)$ as above. For any $a, a_1, a_2, b \in D$, there exist $c, d \in D$ such that*

$$(1 + ax)a_1(1 + bx)^{-1}a_2 = a_1a_2(1 + cx)^{-1}(1 + dx).$$

Proof. We leave the proof for readers with

$$c = (a_2^{-1}ba_2 - a_2^{-1}a_1^{-1}aa_1a_2)a_2^{-1}ba_2(a_2^{-1}ba_2 - a_2^{-1}a_1^{-1}aa_1a_2)^{-1}$$

and $d = (a_2^{-1}ba_2 - a_2^{-1}a_1^{-1}aa_1a_2)a_2^{-1}a_1^{-1}aa_1a_2(a_2^{-1}ba_2 - a_2^{-1}a_1^{-1}aa_1a_2)^{-1}$. □

Lemma 2.2. *Let F be a field, R a ring with center $Z(R) = F$ and $p(x) = v_n x^n + v_{n-1}x^{n-1} + \cdots + v_0$ a polynomial over R with $n \geq 0$ and $v_n \neq 0$. Then $p(x)$ has at most n roots in F.*

Proof. This lemma is a corollary of [15, Proposition 2.3.27]. Assume that $p(x) = 0$ has $n+1$ roots in F, namely $c_0, c_1, \ldots, c_n$, which are distinct in F. By [15, Proposition 2.3.27], $|c_0, c_1, \ldots, c_n| v_n = 0$ where $|c_0, c_1, \ldots, c_n|$ is the Vandermonde determinant of $c_0, c_1, \ldots, c_n$. By [15, Proposition 2.3.26], $|c_0, c_1, \ldots, c_n| = \Pi_{0 \leq i < j \leq n}(c_i - c_j) \neq 0$. Thus, $\Pi_{0 \leq i < j \leq n}(c_i - c_j)v_n = 0$. Therefore,

$$v_n = (\Pi_{0 \leq i < j \leq n}(c_i - c_j))^{-1} \Pi_{0 \leq i < j \leq n}(c_i - c_j) v_n = 0.$$

Contradiction! □

Lemma 2.3. *Let D be a division ring with center F and $w(x_1, x_2, \ldots, x_m)$ a generalized group monomial over D^*. Denote by $D(x)$ the division ring as in Lemma 2.2 for some indeterminate x. For any m elements $u_1, u_2, \ldots, u_m \in D^*$, put $f(x) = w(1 + u_1 x, 1 + u_2 x, \ldots, 1 + u_m x)$, an element of $D(x)$. Then, if $|F| > \alpha(w) + m$ and D^* satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$, then $f(x) \equiv 1$.*

Proof. Assume that $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ and $f(x) = w(1 + u_1 x, 1 + u_2 x, \ldots, 1 + u_m x) \not\equiv 1$. Then by Lemma 2.1, $f(x)$ has a form $bg_1(x)^{-1} g_2(x)$ where $g_1(x)$ and $g_2(x)$ are polynomial of degree $\leq \alpha(w)$. Put $S = \{ c \in F \,|\, 1 + u_i c \neq 0, i = 1, 2, \ldots, m \}$. Now the cardinality of S is greater than $\alpha(w)$ and $f(c) = bg_1^{-1}(c) g_2(c) = w(1 + u_1 c, 1 + u_2 c, \ldots, u_m c) = 1$. Therefore, the polynomial $g_2(x) = g_1(x) b^{-1}$ has at least $\alpha(w) + 1$ roots in F. This contrasts with Lemma 2.2 since $g_2(x) \not\equiv g_1(x) b^{-1}$. Thus, $f(x) \equiv 1$. □

Lemma 2.4. *Let D be a division ring with center F and $w(x_1, x_2, \ldots, x_m)$ a generalized group monomial over D^*. If F is infinite and D^* satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$, then $D = F$.*

Proof. This lemma is from [9, Theorem 2]. □

Proposition 2.5. *Let D be a division ring with center F and $w(x_1, x_2, \ldots, x_m)$ a generalized group monomial over D^*. Then if $|F| > \alpha(w) + m$ and D^* satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$, then $D = F$.*

Proof. We assume that $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$. If the center F is infinite, then D is commutative by Lemma 2.4. Suppose that F is finite. Let $x, y_1, y_2 \ldots, y_m$ be $m+1$ indeterminates. Consider the division ring $K = D[[y_1, y_2, \ldots, y_m]]((x))$ and the division subring $H = D((x))$ of K. It is easy to see that $F((x))$ is infinite, so that, by Lemma 2.4, H does not satisfy $w(x_1, x_2, \ldots, x_m) = 1$. That means, there exist $u_1, u_2, \ldots, u_m \in H$ such that $w(u_1, u_2, \ldots, u_m) \neq 1$. Hence, $w_1(y_1, y_2, \ldots, y_m) = w(1 + y_1 x, 1 + y_2 x, \ldots, 1 +$

$y_m x)$ is an expression which does not coincide with 1 identically. Observe that (or see [15, Remark 8.2.10]) in the division ring K, for any $1 \leq i \leq m$,

$$(1 + y_i x)^{-1} = 1 + \sum_{j=0}^{\infty} (-y_i)^j x^j.$$

One has, $w_1(y_1, y_2, \ldots, y_m) = w(1 + y_1 x, 1 + y_2 x, \ldots, 1 + y_m x)$ has a form

$$= 1 + \sum_{j=1}^{\infty} f_j(y_1, y_2, \ldots, y_m) x^j.$$

Where $f_j(y_1, y_2, \ldots, y_m)$, $j \geq 1$, are generalized polynomials over D in the indeterminates $y_1, y_2, \ldots, y_m$. Notice that there is some j_0 such that $f_{j_0}(y_1, y_2, \ldots, y_m) \not\equiv 0$ since $w_1(y_1, y_2, \ldots, y_m) \not\equiv 1$. We claim that $f_{j_0}(y_1, y_2, \ldots, y_m)$ is a generalized polynomial identity of D. Since the cardinality of F is greater than $\alpha(w) + m$ and by Lemma 2.3, one has $w_1(u_1, u_2, \ldots, u_m) \equiv 1$ for any $u_1, u_2, \ldots, u_m \in D$. It implies $f_{j_0}(u_1, u_2, \ldots, u_m) = 0$ for any $u_1, u_2, \ldots, u_m \in D$. Therefore, $f_{j_0}(y_1, y_2, \ldots, y_m)$ is a generalized polynomial identity of D. By [3, Theorem 6.1.9], D is finite-dimensional over F and hence D is a (finite) commutative field. □

The following result is an extension of both [9, Theorem 2] and [5, Theorem 4].

Theorem 2.6. *Let D be a division ring with center F and $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ be a generalized group monomial over D^*. If $|F| > \min\{\, 2t + m, \alpha(w) + m \,\}$ and D^* satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$, then $D = F$.*

Proof. By Proposition 2.5 , it suffices to prove that if the cardinality of F is greater than $2t + m$, then $D = F$. Assume that $D \neq F$. We substitute $y_i b y_i^{-1}$ for x_i, $i = 1, 2, \ldots, m$, in $w(x_1, x_2, \ldots, x_m)$, and $b \notin F$. Then

$$\begin{aligned} w_1(y_1, y_2, \ldots, y_m) &= w(y_1 b y_1^{-1}, y_2 b y_2^{-1}, \ldots, y_m b y_m^{-1}) \\ &= a_1 y_{i_1} b^{\alpha_1} y_{i_1}^{-1} a_2 y_{i_2} b^{\alpha_2} y_{i_2}^{-1} a_3 \cdots y_{i_t} b^{\alpha_t} y_{i_t}^{-1} a_{t+1}. \end{aligned}$$

One has $w_1(y_1, y_2, \ldots, y_m) = 1$ is a generalized group identity of D^* and $\alpha(w_1) = 2t$. Applying again Proposition 2.5, we have $D = F$. Contradiction! □

3. Subnormal subgroups of D^* satisfying a generalized group identity

In this section, we show that every subnormal subgroup of D^* with center F which satisfies a generalized group identity is contained in F if F contains sufficiently many elements. Two interesting corollaries will be proved then. We first prove some basic lemmas on generalized group monomials over an arbitrary group G.

Lemma 3.1. *Let $w(x_1, x_2, \ldots, x_m)$ be a generalized group monimial over a group G. Then there exists a generalized group monomial*

$$w_1(x_1, x_2, \ldots, x_m) = b_1 x_{i_1}^{\beta_1} b_2 x_{i_2}^{\beta_2} \cdots a_t x_{i_t}^{\beta_t} b_{t+1}$$

over G such that two following conditions hold.

(1) *For any $1 \le j \le t-1$, whenever $i_j = i_{j+1}$, one has $b_{j+1} \notin Z(G)$.*
(2) *$w(c_1, c_2, \ldots, c_m) = w_1(c_1, c_2, \ldots, c_m)$ for any $c_1, c_2, \ldots, c_m \in G$.*
(3) *$\alpha(w_1) = \alpha(w)$.*

Proof. Assume that $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$, where $\alpha_j \in \mathbb{Z}\backslash\{0\}$, $i_j \in \{1, 2, \ldots, m\}$ and $a_i \in G$ such that for any $j = 1, 2, \ldots, t-1$, whenever $i_j = i_{j+1}$ and $\alpha_j \alpha_{j+1} < 0$, one has $a_{j+1} \notin Z(G)$. Without loss of generality, we suppose that only $a_2 \in Z(G)$. Then $i_1 \neq i_2$ or $\alpha_1 \alpha_2 > 0$. If $i_1 \neq i_2$, then put

$$w_1(x_1, x_2, \ldots, x_m) = a_1 a_2 x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} a_3 \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}.$$

If $i_1 = i_2$ and $\alpha_1 \alpha_2 > 0$, then put

$$w_1(x_1, x_2, \ldots, x_m) = a_1 a_2 x_{i_1}^{\alpha_1 + \alpha_2} a_3 \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}.$$

In both cases, we have $w(c_1, c_2, \ldots, c_m) = w_1(c_1, c_2, \ldots, c_m)$ for any $c_1, c_2, \ldots, c_m \in G$ and $w_1(x_1, x_2, \ldots, x_m)$ satisfies (1) and (3). $\square$

Assume a is a non-central element of a group G and y is an indeterminate. Put

$$u_0(y) = y, u_\ell(y) = u_{\ell-1}(y) a u_{\ell-1}(y)^{-1}$$

for any $\ell \ge 1$. It is clear that $u_\ell(y)$ is a generalized group monomial over G and the length $\alpha(u_\ell(y)) = 2^\ell$ for $\ell > 0$.

Lemma 3.2. *Let $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ be a generalized group monomial over G such that for $1 \le j \le t-1$, whenever $i_j = i_{j+1}$, one has $a_{j+1} \notin Z(G)$. Let $y_1, y_2, \ldots, y_m$ be m indeterminates. Then $w'(y_1, y_2, \ldots, y_m) = w(u_\ell(y_1), u_\ell(y_2), \ldots, u_\ell(y_m))$ is also a generalized group monomial over G for any $\ell \ge 1$.*

Proof. There is nothing to do if $\ell = 0$. Assume that $\ell > 0$. We have $u_\ell^\alpha = u_{\ell-1}(y) a^\alpha u_{\ell-1}(y)^{-1}$ and, therefore,

$$w'(y_1, y_2, \ldots, y_m) = w(u_\ell(y_1), u_\ell(y_2), \ldots, u_\ell(y_m))$$
$$= a_1 \underbrace{y_{i_1} a \cdots a^{\alpha_1} \cdots y_{i_1}^{-1}}_{u_\ell(y_{i_1})} a_2 \underbrace{y_{i_2} a \cdots a^{\alpha_2} \cdots y_{i_2}^{-1}}_{u_\ell(y_{i_2})} a_3 \cdots a_t \underbrace{y_{i_t} a \cdots a^{\alpha_t} \cdots y_{i_t}^{-1}}_{u_\ell(y_{i_t})} a_{t+1}.$$

Since a is non-central, it is easy to check that the inside of all $\underbrace{*}_{u_\ell(y_{i_j})}$ satisfies conditions of the definition of generalized group monomials. Now for any $0 < j < t$, if $i_j = i_{j+1}$, then by the hypothesis of $w(x_1, x_2, \ldots, x_m)$, $a_i \notin Z(G)$. Thus, $w'(y_1, y_2, \ldots, y_m)$ is a generalized group monomial over G. $\square$

A subgroup H of a group G is called an *r-subnormal subgroup* of G for some positive integer r if there exist r subgroups $N_1, N_2, \ldots, N_r$ of G such that

$$N = N_r \lhd N_{r-1} \lhd \cdots \lhd N_0 = G.$$

Theorem 3.3. *Let D be a division ring with center F, N be an r-subnormal subgroup of D^* and $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ be a generalized group monomial over D^*. Then if N satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$ and $|F| > 2^r t + m$, then N is contained in F.*

Proof. If $r = 0$, i.e., $N = D^*$, then by Theorem 2.6, $N = D = F$. Assume that $r \geq 1$ and N is not contained in F. By Lemma 3.1, we assume that

$$w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$$

which satisfies the condition whenever $i_j = i_{j+1}$, one has $b_{j+1} \notin F$ for any $1 \leq j \leq m-1$. Since N is r-subnormal in D^*, there exist a positive integer r and subgroups $N_1, N_2, \ldots, N_r$ of D^* such that

$$N = N_r \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd N_0 = D^*.$$

Fix a non-central element a in N. Let $u_0(y) = y, u_\ell(y) = u_{\ell-1}(y) a u_{\ell-1}(y)^{-1}$ for any $\ell = 1, 2, \ldots, r$ as in Lemma 2.4. Observe that $u_\ell(b) \in N_\ell$ for any b in D^* and $\ell = 1, 2, \ldots, r$, so that, by Lemma 2.4,

$$w'(y_1, y_2, \ldots, y_m) = w(u_r(y_1), u_r(y_2), \ldots, u_r(y_m)) = 1$$

is a generalized group identity of D^*. Since $|F| > 2^r t + m = \alpha(w') + m$ and by Lemma 2.6, $D = F$. Thus, $N \subseteq F$. Contradiction! □

Recall that for a division ring D with center F, an element x of D is called *algebraic* over F if x is a root of a non-zero polynomial over F. A subset S of D is called *algebraic* over F if every element of S is algebraic over F. Notice that for an algebraic element x of D^* over F, we have that x^{-1} belongs to $\in F[x]$, the subring of D generated by x and F. It implies that every subring H of D which contains F and is algebraic over F is a division subring of D. This fact will be used several times in this paper.

Proposition 3.4. *Let D be a division ring with center F and $w(x_1, x_2, \ldots, x_m)$ be a generalized group monomial over D^*. Assume that N is a subnormal subgroup of D^* such that N is algebraic over F. If N satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$, then it is contained in F.*

Proof. If F is infinite, then, by Theorem 3.3, N is contained in F. Assume that F is finite. Then for any $x \in N$, the subring $F[x]$ of D generated by x and F is a finite division ring. Hence, x is torsion. Now by [11, Theorem 8], one has $N \subseteq F$. □

To prove next interesting result, we need a basic lemma.

Lemma 3.5. *Let G be a group and N be a subnormal subgroup of G. For any subgroup H of G, the subgroup $H \cap N$ is subnormal subgroup of H.*

Proof. Assume that $N = N_r \lhd N_{r-1} \lhd \cdots \lhd N_0 = G$. Put $M_i = H \cap N_i$ for any $i = 0, 1, \ldots, r$. It is easy to check $N \cap H = M_r \lhd M_{r-1} \lhd \cdots \lhd M_0 = H$. □

Let D be a division ring with center F and N be a subnormal subgroup of D^*. Goncalves and Mandel showed that if there exists a positive integer n such that $x^n \in F$, then N is contained in F [10, Theorem 5.2], which answer partially the Herstein Conjecture [11, Conjecture 3]. Here, we prove the following result.

Proposition 3.6. *Let D be a division ring with center F and N be a subnormal subgroup of D^*. Assume that K is a subfield of D. If there exists a positive integer n such that $x^n \in K$ for any $x \in N$, then N is contained in F.*

Proof. Since K is a field, N satisfies the identity $x^n y^n x^{-n} y^{-n} = 1$. If F is infinite, then N is contained in F by Proposition 3.3. Now we consider the case when F is finite. Without loss of generality, assume that K is a maximal subfield of D. We claim that $N \subseteq K$. Suppose that $N \not\subseteq K$. Then there exists $a \in N$ such that $a \notin K$ and $a^n \in K$. Put $L = C_D(a^n) = \{x \in D \mid a^n x = x a^n\}$, the centralizer of a^n in D. Observe that L is a division subring of D whose center $Z(L)$ contains $F(a^n)$ and that K is contained in L. There are two cases:

Case 1. a^n is not algebraic over F. Then, the field $F(a^n)$ is infinite. Put $N_1 = N \cap L$. One has N_1 is a subnormal subgroup of L^* (Lemma 3.5) and satisfies the identity $x^n y^n x^{-n} y^{-n} = 1$. By Proposition 3.4, $N_1 \subseteq Z(L)$. Notice that K is a maximal subfield of L since K is a maximal subfield of D. Hence, $a \in N_1 \subseteq Z(L) \subseteq K$. Contradiction!

Case 2. a^n is algebraic over F. Since F is finite, so is $F(a^n)$. It implies that a is torsion. By [4, Proposition 2.2], there exists a division subring D_1 of D such that D_1 is algebraic over $Z(D_1)$ and $a \in D_1 \backslash Z(D_1)$. Therefore, $N_1 = D_1^* \cap N$ is a subnormal subgroup of D_1 (Lemma 3.5) and N_1 also satisfies the identity $x^n y^n x^{-n} y^{-n} = 1$. By Lemma 3.4, $N_1 \subseteq Z(D_1)$. In particular, $a \in Z(D_1)$. Contradiction!

Thus the claim is proved. It means that $N \subseteq K$, which implies $N \subseteq F$ by [16, 14.4.4]. □

4. Maximal subgroups of D^* satisfying a generalized group identity

In this section, we focus on maximal subgroups of D^* which satisfying a generalized group identity. The following lemmas are basic.

Lemma 4.1. *Let D be a division ring and M a maximal subgroup of D^*. Then either $F(M) = D$ or $M \cup \{0\}$ is a division subring of D, where $F(M)$ is the division subring of D generated by F and M.*

Proof. We have $M \subseteq F(M)^*$. Hence either $F(M) = D$ or $M = F(M)^*$. If $F(M) \neq D$, then $M \cup \{0\} = F(M)$, a division subring of D. □

Let G be a (multiplicative) group. The subgroup $[G, G]$ of G generated by all commutators $xyx^{-1}y^{-1}$, where x, y range over G, is called the *derived subgroup* of G.

Lemma 4.2. *Let G be a group with center $Z(G)$. If M is a maximal subgroup of G, then either M contains $Z(G)$ or M contains the derived subgroup $[G,G]$ of G.*

Proof. Assume that M does not contain $Z(G)$. Then by the maximality of M in G, $G = Z(G)M$. One has $[G,G] = [Z(G)M, Z(G)M] \subseteq M$. □

Theorem 4.3. *Let D be a division ring with center F. Assume that M is a maximal subgroup of D^* which satisfies $w(x_1, x_2, \ldots, x_m) = 1$ with $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ a generalized group monomial over M. Then, if $|F| > 2t + m$ and $F(M) \neq D$, then M is abelian.*

Proof. By Lemma 4.2, either M contains the derived subgroup $[D^*, D^*]$ of D^* or M contains F^*. If the derived subgroup $[D^*, D^*] \subseteq M$, then $[D^*, D^*]$ satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$. By Theorem 3.3, $D = F$, which implies M is abelian. Now we assume that M contains F^*. Then, by Lemma 4.1, $L = M \cup \{0\}$ is a division subring of D. Hence $L^* = M$ satisfies the identity $w(x_1, x_2, \ldots, x_m) = 1$. Since the center K of L contains F, the cardinality of K is greater than $2t + m$. By Theorem 2.6, M is abelian. □

Remark 4.4. The condition "$F(M) \neq D$" is essential in Theorem 4.3 because there are examples of division rings D whose multiplicative groups D^* contain maximal subgroups M such that M satisfies a (generalized) group identity and is not abelian. For example, consider $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}j$, the Hamilton real quaternion. By [1, Theorem 1], $\mathbb{H}^*$ contains a maximal subgroup $M = \mathbb{C}^* \cup \mathbb{C}^* j$ which is not abelian but M satisfies a group identity since M is solvable ($|M/\mathbb{C}^*| = 2$ and $\mathbb{C}^*$ is abelian).

A division ring D with center F is called a *division F-algebra* or a *centrally finite division ring* if D is a finite-dimensional vector space over F [13, Definition 14.1]. In this case, $\dim_F D = (\dim_F K)^2$ where K is a maximal subfield of D, and $\dim_F K$ is called the *degree* of D. Two following results extend partially of [12, Theorem 4.1] and [1, Theorem 4] for a division ring.

Proposition 4.5. *Let D be a division ring with center F. Assume that M is a maximal subgroup of D^* such that $F[M]$, the subring of D generated by M and F, is algebraic over F and $F[M] \neq D$. Then, if M satisfies a generalized group identity $w(x_1, x_2, \ldots, x_m) = 1$ over M, then D is centrally finite, and M is the multiplicative group of a maximal subfield K of D and there is no field $F \subsetneq L \subsetneq K$.*

Proof. We have $F[M]$ is a division subring of D which contains F. If F is infinite, then by Theorem 4.3, M is abelian. If F is finite, then $F[M]$ is a division ring which algebraic over the finite field F. Hence, by Jacobson's Theorem $F[M]$ is commutative [13, Theorem 13.11]. In both cases, M is abelian. In particular, $F[M]$ is a PI-ring, so that D is centrally finite by [12, Theorem 3.5], and M is the multiplicative group of the maximal subfield $K = F[M]$ of D. For the last statement, assume that there exists a subfield L of K such that $F \subsetneqq L \subsetneqq K$. By [13, Theorem 15.4], $1 < \dim_{C_D(L)} D = \dim_F L < n = \dim_K D$, where n is the degree of D. Therefore, $M \subsetneqq C_L(D)^* \subsetneqq D^*$, which contradicts with the hypothesis. □

Proposition 4.6. *Let D be a division ring with center F. Assume that M is a maximal subgroup of D^* such that $M \backslash F$ contains an algebraic element over F and $F(M) \neq D$. Let $w(x_1, x_2, \ldots, x_m) = a_1 x_{i_1}^{\alpha_1} a_2 x_{i_2}^{\alpha_2} \cdots a_t x_{i_t}^{\alpha_t} a_{t+1}$ be a generalized group monomial over M. Then, if M satisfies $w(x_1, x_2, \ldots, x_m) = 1$ and $|F| > 2t+m$, then D is centrally finite and M is the multiplicative group of a maximal subfield K of D and there is no field $F \subsetneqq L \subsetneqq K$.*

Proof. By Theorem 4.3, M is abelian, which implies that $M = F[M]$ is a subfield of D. Let $x \in M \backslash F$ be an algebraic element of F. Then $1 < \dim_F F(x) = \dim_{C_D(F(x))} D < \infty$ by [13, Theorem 15.4]. Since $M \subseteq C_D(F(x))^*$ and the maximality of M, $C_D(F(x)) = M \cup \{0\}$. By [1, Lemma 6], D is centrally finite. Now using the same argument in the proof of Proposition 4.5, the last statement is proved. □

Remarks 4.7. The description of the maximal subgroups in Propositions 4.5 and 4.6 is the best one we know because non-commutative division rings whose multiplicative groups contain abelian maximal subgroups are unknown [1, Conjecture C]. More generally, non-commutative division rings D with center F which satisfy following properties:

(1) $\mathrm{char}(F) = p$, where p is a prime number,
(2) D is centrally finite of degree p,
(3) and, D contains a maximal subfield K such that K^* is a maximal subgroup of D^* and $x^p \in F$ for each $x \in K$

are unknown.

In [6, Theorem 2.10] (or see [8]), it is proved that if D^* contains a maximal subgroup M such that $M/(M \cap F)$ is locally finite, then D satisfies three properties in Remark 4.7. Here, a group G is called *locally finite* if every finitely generated subgroup of G is finite. We end this paper with an analogue on maximal subgroups whose elements are periodic module F of bounded order.

Proposition 4.8. *Let D be a division ring with center F. Assume that M is a maximal subgroup of D^* such that $F(M) \neq D$. Then, if there exists a positive integer n such that $x^n \in F$ for any $x \in M$, then*

(1) *$\mathrm{char}(F) = p$, where p is a prime number.*
(2) *D is centrally finite of degree p.*
(3) *$K = M \cup \{0\}$ is a maximal subfield of D and $x^p \in F$ for each $x \in K$.*

Proof. We claim that $K = M \cup \{0\}$ is a maximal subfield of D and D is centrally finite. Indeed, by Lemma 4.2, either M contains the derived subgroup $[D^*, D^*]$ of D^* or M contains F^*. If the derived subgroup $[D^*, D^*] \subseteq M$, then $a^n \in F$

for any $a \in [D^*, D^*]$. By Proposition 3.6 and [13, Corollary 13.16], $D = F$, which contradicts with the hypothesis. Therefore, M contains F^*. Then, by Lemma 4.1, K is a division subring of D. Notice that $K^* = M$ satisfies the identity $x^n y^n x^{-n} y^{-n} = 1$, so that by Proposition 4.5, K is a maximal subfield of D and D is centrally finite. The claim is proved.

By [13, Proposition 15.13], we have $\mathrm{char}(F) = p$ and either $x^p \in F$ for any $x \in K$ or K is algebraic over F_p, the finite field of p elements. If F is algebraic over F_p, then so is D. By Jacobson's Theorem [13, Thereom 13.11], D is commutative, which contradicts with the hypothesis. Thus, $a^p \in F$ for any $a \in K$ and hence $\dim_F K = p$. The proposition is proved completely. □

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