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THE LIMITING CASE OF SEMICONTINUITY OF AUTOMORPHISM GROUPS

Steven G. Krantz

Abstract. In this paper we study the semicontinuity of the automorphism groups of domains in multi-dimensional complex space. We give examples to show that known results are sharp (in terms of the required boundary smoothness).

1. Introduction

The paper [4] was the first work to study the semicontinuity of automorphism groups of domains in complex space. The main result there is as follows:

Theorem 1.1. *Let $\Omega^* \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. Then there is a neighborhood $\mathcal{U}$ of Ω^* in the C^∞ topology on domains (that is to say, $\mathcal{U}$ is a collection of domains) so that, if $\Omega \in \mathcal{U}$, then* $\mathrm{Aut}\,(\Omega)$ *is a subgroup of* $\mathrm{Aut}\,(\Omega^*)$. *Moreover, there is a C^∞ mapping Ψ from $\overline{\Omega}$ to $\overline{\Omega}_0$ so that*

$$\mathrm{Aut}\,(\Omega) \ni \varphi \longmapsto \Psi \circ \varphi \circ \Psi^{-1}$$

is an injective group homomorphism from $\mathrm{Aut}\,(\Omega)$ *to* $\mathrm{Aut}\,(\Omega^*)$.

Over the years, the hypothesis of smooth or C^∞ boundary in this theorem has been weakened. In the paper [5], the hypothesis was weakened (using an entirely different argument) to C^2 boundary smoothness. In the paper [3], yet another approach to the C^2 boundary smoothness situation was described. The paper [2] treats the case of C^1 boundary smoothness. Also the paper [1] treats other points of view, such as the dependence on the dimension of the automorphism group.

In the present paper we show that C^2 boundary smoothness is sharp for this type of result.

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2. Notation, terminology, and enunciation of results

For us a *domain* in $\mathbb{C}^n$ is a connected, open set. We generally denote a domain by Ω. We let the *automorphism group* of Ω, denoted by $\mathrm{Aut}\,(\Omega)$, be the collection of biholomorphic selfmaps of Ω. These form a group with the binary relation of composition. The topology on $\mathrm{Aut}\,(\Omega)$ is the compact-open topology (equivalently, the topology of uniform convergence on compact sets).

If Ω is a domain with at least C^1 boundary, then we equip it with a defining function ρ. This is a C^1 function defined on a neighborhood U of $\partial\Omega$ so that

$$\Omega \cap U = \{z \in U : \rho(z) < 0\}.$$

We generally require that $\nabla\rho \neq 0$ on $\partial\Omega$, so that the outward normal vector is well defined at each boundary point. We say that Ω has C^k boundary if it has a defining function that is C^k (that is to say, k-times continuously differentiable).

Now fix a domain $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ with C^k boundary. Let $\epsilon > 0$. We say that a collection of domains $\mathcal{U}$ is a subbasic C^k *neighborhood of* Ω if

$$\mathcal{U} = \left\{\Omega' : \Omega' = \{w \in \mathbb{C}^n : \rho'(w) < 0\} \text{ and } \|\rho - \rho'\|_{C^k} < \epsilon\right\}.$$

These subbasic sets of course generate a C^k topology on the set of all bounded domains with C^k boundary. In this paper we focus our attention on bounded domains. We also will have use for the $C^{1,1}$ topology on domains, and it is defined similarly.

Now the main result of the present paper is as follows.

Theorem 2.1. *There exists a domain $\Omega^* \subseteq \mathbb{C}^2$ having $C^{1,1}$ boundary and a sequence of smoothly bounded domains Ω_K so that*

(a) *The domains Ω_K converge, as $j \to \infty$, to Ω^* in the $C^{1,1}$ topology.*
(b) *Each domain Ω_K has automorphism group containing $\mathbb{Z}$.*
(c) *The domain Ω^* is rigid. That is to say, Ω^* has no automorphism except for the identity mapping.*

So we see from this result that there is no semicontinuity theorem in the $C^{1,1}$ topology.

3. Proof of the theorem

Let B denote the unit ball in $\mathbb{C}^2$. We restrict attention to $\mathbb{C}^2$ just to simplify the notation a bit. The proof of an analogous result in n dimensions is quite similar.

Let φ be a C_c^∞ function supported in the Euclidean ball $B(0,\delta)$ of center the origin and radius $\delta > 0$. We assume that φ is identically equal to 1 on $B(0,\delta/2)$. Now, for $K \geq 1$ we specify the defining function

$$\widetilde{\rho}_K(w_1,w_2) = |w_2|^2 - \mathrm{Im}\, w_1 - \sum_{k=0}^{K}\sum_{j=-\infty}^{\infty} 2^{2jk}\varphi\left(k + 2^{-2jk}w_1, 2^{-jk}w_2\right).$$

Let

$$\widetilde{\Omega}_K = \{(w_1, w_2) \in \mathbb{C}^2 : \widetilde{\rho}_K(w_1, w_2) < 0\}.$$

This new domain should be compared to the defining function for the Siegel upper half space which is given by

$$\widetilde{\rho}_{\mathbf{U}} = |w_2|^2 - \operatorname{Im} w_1.$$

Of course it is well known that the Siegel upper half space $\mathbf{U}$ is biholomorphic to the unit ball B. Indeed, the relevant mappings are

$$\begin{array}{rcl} \Phi : B & \to & \mathbf{U} \\ (z_1, z_2) & \mapsto & \left(i \cdot \dfrac{1 - z_1}{1 + z_1}, \dfrac{z_2}{1 + z_1} \right) \end{array}$$

and

$$\begin{array}{rcl} \Phi^{-1} : \mathbf{U} & \to & B \\ (w_1, w_2) & \mapsto & \left(\dfrac{i - w_1}{i + w_1}, \dfrac{2iw_w}{i + w_1} \right). \end{array}$$

So we think of $\widetilde{\rho}_K$ as defining a perturbation of the Siegel upper half space $\mathbf{U}$. Now we use Φ^{-1} to pull this perturbed domain back to a perturbation of the unit ball B. Now let

$$\Omega_K = \Phi^{-1}(\widetilde{\Omega}_K).$$

Of course, by inspection, Ω_K has C^∞ smooth boundary except at the points $(\pm 1, 0)$ where the "bumps" coming from the translates of φ accumulate. We need to say something about the boundary smoothness at those two exceptional points, and we need to say something about the automorphism group of Ω_K for each K. Finally, we need to specify what the limit of the domains Ω_K is as $K \to +\infty$.

Examining our list of desiderata, we see that, with

$$\widetilde{\Omega}^* = \{(w_1, w_2) \in \mathbb{C}^2 : \widetilde{\rho}^*(w_1, w_2) < 0\}$$

and

$$\widetilde{\rho}^* = |w_2|^2 - \operatorname{Im} w_1 - \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} 2^{2jk} \varphi\Big(k + 2^{-2jk} w_1, 2^{-jk} w_2\Big),$$

and

$$\Omega^* = \Phi^{-1}(\widetilde{\Omega}^*),$$

then $\Omega_K \to \Omega^*$ in some sense as $K \to +\infty$. In point of fact, the Ω_K certainly converge to Ω^* in the Hausdorff metric on sets. And it is also clear from inspection that the defining functions ρ_K for the Ω_K (obtained by pulling back the defining functions for the $\widetilde{\Omega}_K$) are bounded in the C^2 topology. In point of fact, a simple calculation with Φ^{-1} shows that the j^{th} bump in the k^{th} group has height $\approx 2^{-2|j|k}$ and diameter $\approx 2^{-|j|k}$. The main point being that the decay is quadratic as the two extreme points are approached. That is why we get boundedness in the C^2 norm.

It follows then, by a version of the Landau inequalities (for which see [6]), that Ω_K converges to Ω^* in the $C^{1,1}$ topology.

Now what about the automorphism group of Ω_K? It is easiest to instead examine the automorphism group of $\widetilde{\Omega}_K$ (which is of course the same group). Thanks to work in [9], we know that any automorpism of $\widetilde{\Omega}_K$ must be an automorphism of the Siegel upper half space $\mathbf{U}$ which preserves the "bumps" that are created by the translates of φ. We conclude that the only possible automorphisms are dilations of the Siegel upper half space (see [8, Ch. 10]). Examining the definition of $\widetilde{\rho}_K$, we see that a dilation

$$\alpha_\delta(w_1, w_2) = (\delta^2 w_1, \delta w_2)$$

can leave this defining function invariant if and only if $\delta = 2^m$ and m is divisible by $1, 2, \ldots, K$. In detail,

$$\begin{aligned}\widetilde{\rho}_K(2^{2m}w_1, 2^m w_2) &= |2^m w_2|^2 - \operatorname{Im}\left(2^{2m}w_1\right)\\ &\quad - \sum_{k=0}^{K}\sum_{j=-\infty}^{\infty} 2^{2jk}\varphi\Big(k + 2^{-2jk}2^{2m}w_1, 2^{-jk}2^m w_2\Big)\\ &= |2^m w_2|^2 - \operatorname{Im}\left(2^{2m}w_1\right)\\ &\quad - \sum_{k=0}^{K}\sum_{j=-\infty}^{\infty} 2^{2jk}\varphi\Big(k + 2^{2m-2jk}w_1, 2^{m-jk}w_2\Big).\end{aligned}$$

Multiplying by 2^{-2m}, we see that we must examine

$$|w_2|^2 - \operatorname{Im} w_1 - \sum_{k=0}^{K}\sum_{j=-\infty}^{\infty} 2^{2jk-2m}\varphi\Big(k + 2^{-2jk}2^{2m}w_1, 2^{-jk}2^m w_2\Big).$$

We want to shift the index of summation by replacing j with $j + m/k$, but we can only do so if m is divisible by k for every $k = 1, 2, \ldots, K$. The result of this shift is

$$|w_2|^2 - \operatorname{Im} w_1 - \sum_{k=0}^{K}\sum_{j=-\infty}^{\infty} 2^{2jk}\varphi\Big(k + 2^{-2jk}w_1, 2^{-jk}w_2\Big).$$

So we see that the defining function has been preserved under the dilation. Hence $w \mapsto \alpha_{2^m}(w)$ is an automorphism of $\widetilde{\Omega}_K$ provided that $k \mid m$ for $k = 1, 2, \ldots, K$. In particular, we must demand that $m \geq K$. Since iterates of this dilation are also automorphisms, we conclude that the automorphism group of $\widetilde{\Omega}_K$ contains a copy of $\mathbb{Z}$.

The analysis in the last paragraph also shows that the automorphism group of $\widetilde{\Omega}^*$ does not contain any nontrivial dilations. For, if it did, then it would have to be a dilation of magnitude 2^m with $m \geq K$ for every positive K. And that is impossible. So the automorphism group of $\widetilde{\Omega}^*$ is trivial—it contains only the identity map.

In conclusion, we have verified all the required properties of the $\widetilde{\Omega}_K$ (and hence also of the Ω_K) and of Ω^*. Thus the theorem is proved.

References

[1] B. Fridman, D. Ma, and E. Poletski, *Upper semicontinuity of the dimensions of automorphism groups of domains in* $\mathbb{C}^N$, Amer. J. Math. **125** (2003), no. 2, 289–299.

[2] R. E. Greene and K.-T. Kim, *Stably-interior points and the semicontinuity of the automorphism group*, Math. Z. **277** (2014), no. 3-4, 909–916.

[3] R. E. Greene, K.-T. Kim, S. G. Krantz, and A.-R. Seo, *Semicontinuity of automorphism groups of strongly pseudoconvex domains: the low differentiability case*, Pacific J. Math. **262** (2013), no. 2, 365–395.

[4] R. E. Greene and S. G. Krantz, *The automorphism groups of strongly pseudoconvex domains*, Math. Ann. **261** (1982), no. 4, 425–446.

[5] ———, *Normal families and the semicontinuity of isometry and automorphism groups*, Math. Z. **190** (1985), no. 4, 455–467.

[6] S. G. Krantz, *Lipschitz spaces, smoothness of functions, and approximation theory*, Exposition. Math. **3** (1983), no. 3, 193–260.

[7] ———, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[8] ———, *Explorations in Harmonic Analysis*, Birkhäuser Publishing, Boston, 2009.

[9] L. Lempert and L. Rubel, *An independence result in several complex variables*, Proc. Amer. Math. Soc. **113** (1991), no. 4, 1055–1065.

Department of Mathematics
Washington University in St. Louis
St. Louis, Missouri 63130, USA
E-mail address: sk@math.wustl.edu