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HYPERBOLIC NOTIONS ON A PLANAR GRAPH OF BOUNDED FACE DEGREE

Byung-Geun Oh

Abstract. We study the relations between strong isoperimetric inequalities and Gromov hyperbolicity on planar graphs, and give an alternative proof for the following statement: if a planar graph of bounded face degree satisfies a strong isoperimetric inequality, then it is Gromov hyperbolic. This theorem was formerly proved in the author's paper from 2014 [12] using combinatorial methods, while geometric approach is used in the present paper.

1. Introduction

For a given connected infinite graph $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G, there are several notions for the *type* of G. For example, G is usually called parabolic (hyperbolic) if the simple random walk on G is recurrent (transient, respectively) [14]. Another examples are Vertex Extremal Length(VEL)-hyperbolicity versus VEL-parabolicity, and Edge Extremal Length(EEL)-hyperbolicity versus EEL-parabolicity [8], etc.

In this paper we are especially interested in a concept that is related to various hyperbolic notions: *strong isoperimetric inequalities*. To explain this concept, let G be given as above, and suppose $S = (V(S), E(S))$ is a finite subgraph of G. The *vertex boundary* of S, which we denote by dS, is defined as the set of vertices in $V(S)$ that have neighbors in $V(G) \setminus V(S)$. Then *Cheeger's constant* of G is

$$h(G) := \inf_S \frac{|dS|}{|V(S)|},$$

where S runs over all nonempty finite subgraphs of G and $|\cdot|$ denotes the cardinality of the set, and we say that G satisfies a strong isoperimetric inequality if $h(G) > 0$. It is known that if G is of bounded valence, that is, if degrees of the vertices of G are uniformly bounded, then the inequality $h(G) > 0$ implies

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all of the hyperbolic notions explained above (transient random walk, VEL-hyperbolicity, and EEL-hyperbolicity) in a rather strong way (see [14, p. 82] and [8]). For this reason a graph is sometimes called uniformly hyperbolic or strongly hyperbolic if it satisfies a strong isoperimetric inequality.

We remark here that the constant $h(G)$ is one of the several discrete analogues of the continuous version of Cheeger's constant [3]. For other types of discrete Cheeger's constant, see for example [12].

Another hyperbolic notion we want to study is Gromov hyperbolicity, which is defined as follows. Let (X, d) be a metric space. We say that X is hyperbolic in the sense of Gromov, or *Gromov hyperbolic*, if there exists a constant $\delta \geq 0$ such that every four points $x, y, z, w \in X$ satisfy the inequality

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta, \tag{1.1}$$

where

$$(a, b)_c = \frac{1}{2}\big(d(a, c) + d(b, c) - d(a, b)\big).$$

The quantity $(a, b)_c$ is called the *Gromov product* of a and b with respect to the *base point* c. If $X = \mathbb{R}^2$, then $(a, b)_c$ represents the distance from c to the closer intersection points of the triangle abc and its inscribed circle. See Figure 1.

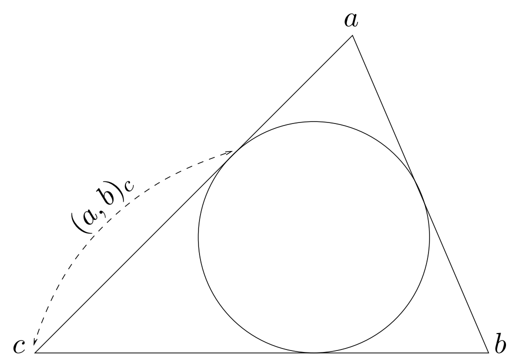

FIGURE 1. Gromov product in a Euclidean triangle

Every graph can be considered a one dimensional simplicial complex, hence we can equip it with the *simplicial metric*. That is, for a given graph G, we identify each edge of G with a line segment of length one, and define the distance between any two points $x, y \in G$ as the infimum of the lengths of the curves in G that connects x and y. Note that this definition is valid not only when x and y are vertices of G but also when they are just points on G. With this definition we can treat each graph as a (geodesic) metric space, hence it is possible to ask whether it is Gromov hyperbolic or not.

The concept of Gromov hyperbolicity was introduced in the theory of groups [7], where Gromov studied finitely generated free groups. See [4, 5, 6, 7] for more about Gromov hyperbolic spaces.

From now on, we will assume that G is a connected simple infinite *planar* graph embedded into $\mathbb{R}^2$ locally finitely such that $\deg v < \infty$ and $\deg f < \infty$ for every vertex $v \in V(G)$ and face $f \in F(G)$, where $F(G)$ is the face set of G and $\deg v$ (or $\deg f$) denotes the *degree* of v (or f, respectively). See Section 2 for the precise definitions of these terminologies.

Our main result is the following.

Theorem 1.1. *Suppose G is a planar graph whose face degrees are uniformly bounded. If G satisfies a strong isoperimetric inequality, then it is Gromov hyperbolic.*

In fact, Theorem 1.1 was proved in [12, Theorem 6(a)] using some combinatorial methods. The main tool was so called the *detour map* introduced in [2]. The purpose of the present paper is to give an alternative proof for Theorem 1.1 using a geometric method. We believe that this geometric method is more elementary and natural than the combinatorial one.

2. Preliminaries

A graph $G = (V(G), E(G))$ is a pair of the vertex set $V(G)$ and the edge set $E(G) \subset V(G) \times V(G)$. We always assume that G is *undirected*, which means that the edge $[v, w] \in E(G)$ is considered the same as $[w, v]$ for $v, w \in V(G)$. Therefore, $E(G)$ consists of some unordered pairs in $V(G) \times V(G)$. If $e = [v, w] \in E(G)$, v is called a *neighbor* of w, and vice versa. In this case we also say that e connects v and w, and v and w are incident to (or endpoints of) e. A *path* is a finite sequence of vertices $[v_0, v_1, \dots, v_n]$ such that $[v_{k-1}, v_k] \in E(G)$ for all $k = 1, 2, \dots, n$, and a path is called a *cycle* if $v_0 = v_n$. Also in this case we say that the path is of length n. A path or a cycle is called *simple* if it has no self intersections; i.e., $v_i \neq v_j$ for $i < j$, unless $i = 0$ and $j = n$. We say that G is infinite if $V(G)$ is an infinite set, and connected if it is connected as a one dimensional simplicial complex. The graph G is called *simple* if it has no multiple edges and no self loops. That is, for $v, w \in V(G)$ there are at most one edge $e \in E(G)$ connecting v and w, and for every $e = [v, w] \in E(G)$ we have $v \neq w$.

If there is a continuous injective map $\imath : G \hookrightarrow \mathbb{R}^2$, then the graph G is called *planar* and its image $\imath(G)$ is called an embedded graph. Strictly speaking, G and $\imath(G)$ are distinct objects, but we do not distinguish them and use the letter G for $\imath(G)$. Moreover, we also assume that G is embedded into $\mathbb{R}^2$ locally finitely; i.e., we assume that for every compact set $K \subset \mathbb{R}^2$ there are only finitely many vertices and edges of G that intersect with K.

When G is a connected planar graph embedded into $\mathbb{R}^2$ locally finitely, a component of $\mathbb{R}^2 \setminus G$ is called an *open face* of G, and its closure is called a closed face, or just a *face*, of G. We also denote by $F(G)$ the set of (closed) faces of G. Note that by the definition a face $f \in F(G)$ is the closure of the corresponding open face g, but in general g does not have to be the interior of f. See Figure 2 for this. However, g will become the interior of f when G has no *cut-vertices*; i.e., when there is no vertex $v \in V(G)$ such that $G \setminus \{v\}$ is disconnected. In fact if G has no cut-vertices, then the boundary ∂g of g must be a simple cycle. This is because, by the Jordan curve theorem, every repeated vertex of ∂g becomes a cut-vertex of G. Thus in this case we must have $\partial g = \partial f$, f is homeomorphic to the closed unit disk, and g is the interior of f.

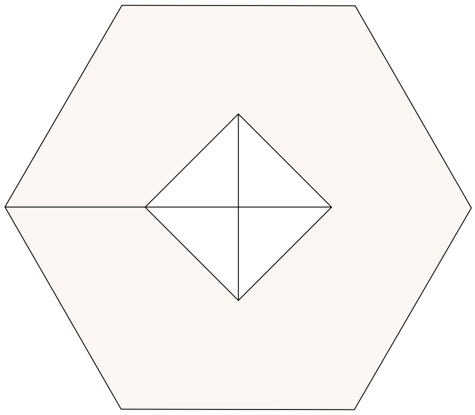

FIGURE 2. The shaded part represents a (closed) face that is not simply connected, and the corresponding open face is not the interior of the closed face

In the last section (Section 5) of this paper we will reduce Theorem 1.1 to the case when G has no cut-vertices, so until then we will always assume that G does not have any cut-vertices.

For every $v \in V(G)$, the *degree* of v is the number of neighbors of v, and it is denoted by $\deg v$. Similarly for every $f \in F(G)$, $\deg f$ denotes the *degree*, or the *girth*, of f and indicates the number of edges surrounding f. That is, we define $\deg f$ as the length of ∂f, which is in general a union of cycles. Perhaps, however, it might be correct to define $\deg f$ as the length of the cycle ∂g, where g is the open face corresponding to f. Recall that Theorem 1.1 requires face degrees to be uniformly bounded, so one might think that the definition for $\deg f$ should be fixed, since otherwise some confusions could arise. But as we will discuss in Section 5, for Theorem 1.1 it does not matter whether we interpret $\deg f$ as the length of ∂f or as the length of ∂g. Moreover, if G has no cut-vertices, we have $\partial f = \partial g$ and f becomes a topological k-gon for some $k \in \mathbb{N}$ with $3 \le k \le N$, where the inequality $3 \le k$ comes from the simplicity of G and N is an upper bound of face degrees.

A graph $S = (V(S), E(S))$ is called a subgraph of G if $V(S) \subset V(G)$ and $E(S) \subset E(G)$. In this case we use the notation $S \subset G$. The face set of S, denoted by $F(S)$, is defined as the *subset* of $F(G)$ such that $f \in F(S)$ if and only if $f \in F(G)$ and f is the closure of a component of $\mathbb{R}^2 \setminus S$. This notation is a little bit confusing, since $F(S)$ in fact means the intersection of $F(G)$ and the face set of S (closures of the components of $\mathbb{R}^2 \setminus S$). Thus if $f \in F(S)$, we must have $V(G) \cap f = V(S) \cap f$. A subgraph $S \subset G$ is called *simply connected* if it is connected and its Euler characteristic is one; i.e., $|V(S)| + |F(S)| - |E(S)| = 1$ if S is finite.

We finish this section by deriving a formula to be used later. Let S be a finite simply connected subgraph of G, and let $\mathrm{bd}(S)$ be the edges in $E(S)$ such that $e \subset \partial f$ for some $f \in F(G) \setminus F(S)$. Then it is easy to see that $e \in \mathrm{bd}(S)$ only if there is at most one $f \in F(S)$ such that $e \subset \partial f$. If $e \in E(S) \setminus \mathrm{bd}(S)$, there are exactly two such f's. Therefore we have $\sum_{f \in F(S)} \deg f \le 2|E(S)| - |\mathrm{bd}(S)|$. Now because $\deg f \ge 3$ for all $f \in F(G)$, which comes from the simplicity of G, we have

$$3|F(S)| \le \sum_{f \in F(S)} \deg f \le 2|E(S)| - |\mathrm{bd}(S)|.$$

Note that S is simply connected. Thus

$$|E(S)| + 1 = |V(S)| + |F(S)| \le |V(S)| + \frac{2}{3} \cdot |E(S)| - \frac{1}{3} \cdot |\mathrm{bd}(S)|,$$

or $|E(S)| \le 3|V(S)| - |\mathrm{bd}(S)| - 3$. Therefore

$$|F(S)| = |E(S)| - |V(S)| + 1 \le 2|V(S)| - |\mathrm{bd}(S)| - 2, \tag{2.1}$$

which is what we wanted to derive.

3. Rough isometries and Aleksandrov surfaces

Let X and Y be metric spaces with metrics d_X and d_Y, respectively.

Definition 3.1. A map $m : X \to Y$ is called a *rough isometry*, or a *quasi-isometry*, if there exist constants $A \ge 1$, $B \ge 0$, and $\epsilon > 0$ such that

(1) $A^{-1}\, d_X(a, b) - B \le d_Y(m(a), m(b)) \le A\, d_X(a, b) + B$ for all $a, b \in X$;
(2) the ϵ-neighborhood of $m(X)$ covers Y.

If there exists a rough isometry between X and Y, we say that X and Y are roughly isometric.

It is not difficult to see that rough isometries define an equivalence relation on metric spaces. The notion of rough isometries was first introduced by M. Gromov [6] and M. Kanai [9].

A metric space is called *intrinsic* if the distance between every two points is equal to the infimum of the lengths of the curves connecting these points, and an intrinsic metric space is called *geodesic* if every two points can be joined by a curve whose length is the same as the distance between the points. One may check using the Arzelá-Ascoli Theorem that a complete locally compact intrinsic space is actually geodesic. Therefore every connected graph G with $\deg v < \infty$ for every $v \in V(G)$ is a geodesic space if it is equipped with the simplicial metric.

Theorem 3.2 ([4], Theorem 2.2)**.** *Suppose X and Y are geodesic metric spaces that are roughly isometric to each other. Then X is Gromov hyperbolic if and only if Y is Gromov hyperbolic.*

Second subject of this section is Aleksandrov surfaces. However, because the precise definition of Aleksandrov surfaces is a little bit complicated, those who do not have any background knowledge about this concept may skip the definition, and accept the following well known statement: if a surface is obtained by pasting Euclidean polygons along sides of equal length, then it becomes a *polyhedral* surface, a typical example of Aleksandrov surfaces. In fact, this is all we need in this paper related to the definition of Aleksandrov surfaces.

Among many equivalent definitions for Aleksandrov surfaces we will use the analytic definition given in [13, §7], because it is shorter than the others. With the terminology *Aleksandrov surface*, we mean a two dimensional orientable topological manifold with an intrinsic metric whose length element is locally expressed in the form

$$e^u(z)|dz|,$$

where z is a local complex coordinate and u is a difference between two subharmonic functions such that $\exp u(z)$ is locally integrable on rectifiable curves in the z-plane.

There are two typical subclasses of Aleksandrov surfaces. Two dimensional Riemannian manifolds are thoses in the first class, and as indicated above, surfaces with polyhedral metrics are in the second class. A surface with a polyhedral metric, or a polyhedral surface, is a surface such that each point of it is locally isometric to a *cone* with the length element

$$|z|^{\alpha-1}|dz|$$

for some $\alpha > 0$. To study more about Aleksandrov surfaces or polyhedral surfaces, see for example [1, 10, 11, 13].

In the next section we will construct a geodesic polyhedral surface $\mathcal{S}$ that is roughly isometric to the given graph G. Then in order to prove Theorem 1.1, it will suffice to show that this polyhedral surface $\mathcal{S}$ is Gromov hyperbolic. Note that this comes from Theorem 3.2. On the other hand, we will show that the condition $h(G) > 0$ implies the inequality $h_1(\mathcal{S}) > 0$, where $h_1(\mathcal{S})$ denotes *Cheeger's constant* of $\mathcal{S}$ defined by

$$h_1(\mathcal{S}) = \inf_D \left(\frac{\text{Length}(\partial D)}{\text{Area}(D)} \right).$$

Here the infimum is taken over all Jordan domains $D \subset \mathcal{S}$ satisfying $\text{Area}(D) < \infty$. Then the following theorem will finish the proof.

Theorem 3.3 (Gromov)**.** *Every open simply connected Aleksandrov surface which satisfies a linear isoperimetric inequality is Gromov hyperbolic.*

A surface $\mathcal{S}$ is said to satisfy a linear isoperimetric inequality if $h_1(\mathcal{S}) > 0$. Also note that a surface without boundary is called *open* if it is not compact. Hence a simply connected surface is open if and only if it is topologically equivalent to the Euclidean plane. Theorem 3.3 was proved in [4, Chapter 6] when the surface is a two dimensional *complete* simply connected *Riemannian manifolds*, and then it was slightly extended in [11] to general simply connected Aleksandrov surfaces, even without the assumption about completeness.

The following two elementary lemmas will be used in the subsequent sections.

Lemma 3.4. *For $k \in \{3, 4, \ldots, N\}$, suppose f is a regular k-gon in $\mathbb{R}^2$ of side length one, and γ is a simple piecewise smooth arc in f with the endpoints on ∂f. Among the two subarcs of ∂f divided by the endpoints of γ, let λ be the subarc that contains less vertices of f. If both subarcs have the same number of vertices, we can choose either one. Then the following inequality holds:*

$$\text{Length}(\lambda) \leq N \cdot \text{Length}(\gamma). \tag{3.1}$$

Proof. It is easy to see that the inequality (3.1) holds with 1 in place of N if the endpoints of γ, which we denote by x and y, are on the same side of f. If λ contains more than one vertex, then the distance between x and y is definitely greater than one. But since $\text{Length}(\lambda) \leq N$, we have

$$\text{Length}(\lambda) \leq N = N \cdot 1 \leq N \cdot \|x - y\| \leq N \cdot \text{Length}(\gamma).$$

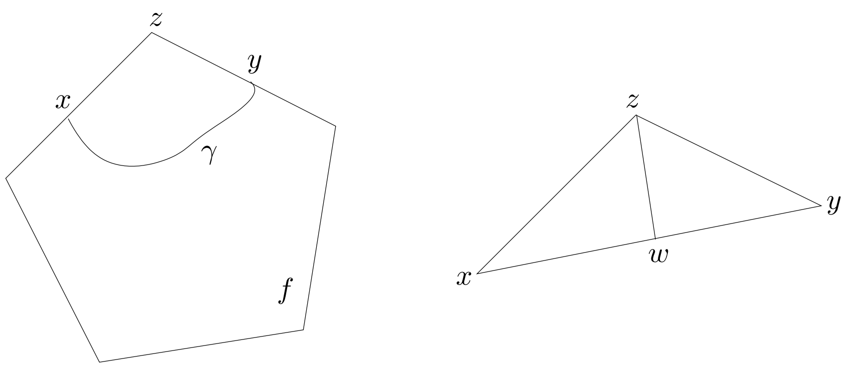

FIGURE 3. A curve γ with endpoints on ∂f, and the associated triangle

Now it remains the case when λ contains only one vertex, say z. Consider the triangle with vertices x, y, and z, and let w be the intersection point of the side xy and the bisector of $\angle z$. Then we have

$$\frac{\pi}{6} \leq \angle xzw \leq \frac{\pi}{2},$$

hence $\sin(\angle xzw) \geq 1/2$. Therefore,

$$\|x - z\| = \frac{\sin(\angle zwx)}{\sin(\angle xzw)} \cdot \|x - w\| \leq 2\|x - w\|,$$

and similarly $\|y - z\| \leq 2\|y - w\|$. Now we see that

$$\begin{aligned} \text{Length}(\lambda) &= \|x - z\| + \|y - z\| \leq 2\|x - w\| + 2\|y - w\| \\ &= 2\|x - y\| \leq 2 \cdot \text{Length}(\gamma), \end{aligned}$$

as desired. □

Lemma 3.5. *For $k \in \{3, 4, \ldots, N\}$, suppose f is a regular k-gon in $\mathbb{R}^2$ of side length one, and let D be a subset of f with piecewise smooth boundary. Then there exists $C = C(N)$ such that*

$$\text{Area}(D) \leq C \cdot \text{Length}(\partial D). \tag{3.2}$$

Proof. If $\text{Length}(\partial D) \leq 1$, then the isoperimetric inequality in $\mathbb{R}^2$ implies that

$$\text{Area}(D) \leq \frac{1}{4\pi} \cdot \{\text{Length}(\partial D)\}^2 \leq \frac{1}{4\pi} \cdot \text{Length}(\partial D).$$

If $\text{Length}(\partial D) \geq 1$, then we have (3.2) with C equal to

$$C_0 := \frac{N}{4} \cdot \cot\left(\frac{\pi}{N}\right), \tag{3.3}$$

since $D \subset f$ and $\text{Area}(f)$ is bounded by (3.3), the area of a regular N-gon of side length one. □

4. The Aleksandrov surface constructed from G

Suppose G is a planar graph such that every face of G is at most N-gon for some $N \in \mathbb{N}$. We also assume that G does not have any cut-vertices, so that every face of G is homeomorphic to the closed unit disk. Then for each face of G which is a k-gon, $k \leq N$, we associate a copy of a Euclidean regular k-gon of side length one, and paste those regular polygons along sides exactly in the same way as the faces of G are pasted. This is possible since all polygons considered here have the same side lengths (= 1). Now let us denote this obtained surface by $\mathcal{S}$.

The surface $\mathcal{S}$ is locally Euclidean possibly except at the vertices of pasted polygons. Therefore the natural metric on $\mathcal{S}$ is the intrinsic metric inherited from this locally Euclidean metric; i.e., the distance between two points $x, y \in \mathcal{S}$ is defined by the infimum of the lengths of the curves connecting x and y, and the area of a Borel set D is defined by $\sum_f \text{Area}(f \cap D)$, where the sum is over all the polygons $f \in F(G)$. Then because G was originally embedded into $\mathbb{R}^2$ locally finitely, $\mathcal{S}$ becomes a complete *locally compact* polyhedral surface. Therefore as we explained in the previous section, S becomes a geodesic metric space. Moreover, G is naturally embedded into $\mathcal{S}$, thus we can treat G as a subset of $\mathcal{S}$.

From now on, we will regard a face of G as a subset of $\mathcal{S}$. Similarly, the notations Length$(\cdot)$ and Area$(\cdot)$ will denote the $\mathcal{S}$-length and the $\mathcal{S}$-area of the considered objects, respectively, lying in $\mathcal{S}$.

Lemma 4.1. *The inclusion map $\iota : G \hookrightarrow \mathcal{S}$, defined by $\iota(x) = x$ for $x \in G$, is a rough isometry.*

Proof. It is easy to see that ι is a local isometry except at the vertices of G. Now in order to prove the lemma, let d_G and $d_{\mathcal{S}}$ be the intrinsic metrics on G and $\mathcal{S}$, respectively. Then $d_{\mathcal{S}}(x, y) \le d_G(x, y)$ for every $x, y \in G$. Furthermore, $\mathcal{S}$ is definitely contained in the N-neighborhood of G, since every point in a regular k-gon, $k \le N$, of side length one has a distance less than N from the boundary of the polygon. Therefore, to prove that ι is a rough isometry, it suffices to show that $d_G(x, y) \le C \cdot d_{\mathcal{S}}(x, y)$ for every $x, y \in G$, where C is a constant independent of the choices of x and y.

Suppose $x, y \in G \subset \mathcal{S}$ are given. Since $\mathcal{S}$ is a geodesic space, there exists a curve $\gamma \subset \mathcal{S}$ connecting x and y such that Length$(\gamma) = d_{\mathcal{S}}(x, y)$. Then there exists a sequence of points $x = x_0, x_1, \dots, x_{m-1}, x_m = y$ on $\gamma \cap G$ such that the subarc of γ connecting x_j and x_{j+1} lies on a face of G for each $j = 0, 1, \dots, m-1$. But Lemma 3.4 implies that $d_G(x_j, x_{j+1}) \le N \cdot d_{\mathcal{S}}(x_j, x_{j+1})$, hence we have

$$d_G(x, y) \le \sum_{j=0}^{m-1} d_G(x_j, x_{j+1}) \le N \sum_{j=0}^{m-1} d_{\mathcal{S}}(x_j, x_{j+1}) = N \cdot d_{\mathcal{S}}(x, y),$$

showing that G and $\mathcal{S}$ are roughly isometric. □

For $D \subset \mathcal{S}$, recall that the interior, closure, and boundary of D are denoted by D°, $\overline{D}$, and ∂D, respectively.

Lemma 4.2. *Suppose $h(G) > 0$ and the degrees of the faces of G are bounded above by N. Let D be an open set in $\mathcal{S}$ consisting of a finite number of faces; i.e., D is of the form*

$$D = (f_1 \cup f_2 \cup \cdots \cup f_\ell)^\circ \tag{4.1}$$

for some $f_1, f_2, \dots, f_\ell \in F(G)$. Then there exists a constant C such that

$$\text{Area}(D) \le C \cdot \text{Length}(\partial D). \tag{4.2}$$

The constant C depends only on N and $h(G)$.

Proof. By considering each component of D separately, and then by adding to D all the bounded components of $\mathcal{S} \setminus D$ if necessary, we may assume that D is simply connected. Let T be the maximal subgraph of G contained in $\overline{D}$. That is, we assume that T is the graph consisting of edges and vertices in $\overline{D}$. Also let bd(T) be the set of edges of T lying on ∂D. Then because D consists of faces, this definition coincides with the one given in Section 2; i.e., $e \in$ bd(T) if and only if $e \in \partial f$ for some $f \in F(G) \setminus F(S)$. Therefore, because D is simply connected, the number of vertices in dT is the same as the number of edges in $\mathrm{bd}(T)$. But because every edge has length one, we must have

$$|dT| = |\mathrm{bd}(T)| = \mathrm{Length}(\partial D).$$

On the other hand, we know that every face is at most N-gon, hence the area of each face is at most C_0, where the constant C_0 was defined in (3.3). Since $f \subset \overline{D}$ if and only if $f \in F(T)$, we have from (2.1) that

$$\begin{aligned}\mathrm{Area}(D) &= \sum_{f\in F(T)} \mathrm{Area}(f) \le C_0|F(T)| \le C_0(2\,|V(T)| - |\mathrm{bd}(T)| - 2)\\ &\le C_0\left(\frac{2}{h(G)}\,|dT| - |\mathrm{bd}(T)|\right) = \frac{C_0\big(2-h(G)\big)}{h(G)}\,|\mathrm{bd}(T)|\\ &= \frac{C_0\big(2-h(G)\big)}{h(G)}\cdot \mathrm{Length}(\partial D),\end{aligned}$$

as desired. □

5. Proof of Theorem 1.1

In this section we will first reduce Theorem 1.1 to the case when graphs in consideration have no cut-vertices. To do this, suppose G_0 is a planar graph satisfying the assumptions in Theorem 1.1. Then there exist a natural number N such that $\deg f \le N$ for every $f \in F(G_0)$, and we have $|V(S)| \le c|dS|$ for every finite subgraph $S \subset G_0$, where $c = h(G_0)^{-1}$.

Let $\{f_1, f_2, \ldots\}$ be an enumeration of $F(G_0)$, and suppose g_1 is the open face of G_0 corresponding to f_1. If ∂g_1 is a simple cycle, we set $G_1 := G_0$. If not, we choose a repeated vertex $v \in \partial g_1$ lying on the boundary of the unbounded component of $\mathbb{R}^2 \setminus g_1$. Note that such v must exist, and there could be more than one such vertex. Then because v is a cut-vertex of G_0, $G_0 \setminus \{v\}$ has two or more components. However, all such components except one should be finite, because G_0 is embedded into $\mathbb{R}^2$ locally finitely. Now we claim that every finite component of $G_0 \setminus \{v\}$ contains at most $c = h(G_0)^{-1}$ vertices. In fact, if T' is a finite component of $G_0 \setminus \{v\}$, then $T := \overline{T'} = T' \cup \{v\}$ should be a finite subgraph of G_0 with $dT = \{v\}$. Therefore, because $h(G_0) > 0$, we have $|V(T)| \le c|dT| = c$, proving the claim. This implies that if we remove from G_0 all the finite components of $G_0 \setminus \{v\}$, and if we do the same thing to every repeated vertices on ∂g_1 that is also lying on the boundary of the unbounded component of $\mathbb{R}^2 \setminus g_1$, then the obtained graph G_1 is an infinite subgraph of G_0 such that its c-neighborhood contains G_0. We also remark that face degrees of G_1 are uniformly bounded, because they are at most the maximum of face degrees of G_0. This is true no matter whether we define the face degree of $f \in F(G_0)$ as the length of ∂f, or as the length of the boundary cycle of the corresponding open face.

Suppose we have constructed G_{j-1} for some $j \in \mathbb{N}$. We then choose the smallest natural number n_j such that the face f_{n_j} is in $F(G_{j-1})$ as well as in

$F(G_0)$, and has not been used in the construction of $G_1, G_2, \ldots, G_{j-1}$. Note that some faces of G_0 might not be in the face set of G_{j-1} if it was *enclosed* by f_{n_i} for some $i \in \{1, 2, \ldots, j-1\}$, so we require f_{n_j} to be a face in $F(G_{j-1})$. Now we repeat the same procedure as above: if ∂g_{n_j} is a simple cycle, where g_{n_j} is the open face corresponding to f_{n_j}, then we set $G_j := G_{j-1}$; if not, we remove all the finite components of $G_{j-1} \setminus \{v\}$ for every repeated vertex v of ∂g_{n_j} that also lies on the boundary of the unbounded component of $\mathbb{R}^2 \setminus g_{n_j}$, and we obtain a new graph G_j. Then because we use different faces in every step, G_j must be an infinite subgraph of G_0 such that its c-neighborhood contains G_0. Moreover, face degrees of G_j are at most the maximum of face degrees of G_0.

We repeat this process and obtain a sequence of graphs $\{G_j\}$ such that

$$G_0 \supset G_1 \supset G_2 \supset \cdots .$$

Then by the construction one can see that the graph $G := \bigcap_{i=0}^{\infty} G_j$ is an infinite subgraph of G_0 such that its c-neighborhood contains G_0. Moreover for every two points $x, y \in G$, the G-distance between x and y is the same as G_0-distance between x and y, because each part we have removed from G_0 was connected to G_0 only through one cut-vertex, hence a geodesic curve between x and y cannot go inside the removed parts. Thus the inclusion map $\iota : G \hookrightarrow G_0$ is a rough isometry, hence Theorem 3.2 implies that G is Gromov hyperbolic if and only if G_0 is Gromov hyperbolic. Moreover, face degrees of G are at most the maximum of face degrees of G_0, so they are uniformly bounded.

We next claim that $h(G) > 0$. To see this, suppose S is a finite subgraph of $G \subset G_0$, and let dS and d_0S be the vertex boundaries of S as a subgraph of G and G_0, respectively. Then definitely $dS \subset d_0S$. However, if we define S_0 as the union of S and all the finite components of $G_0 \setminus S$, then S_0 becomes a subgraph of G_0 satisfying $dS \supset d_0S_0$. Therefore, because $h(G_0) > 0$,

$$|V(S)| \le |V(S_0)| \le h(G_0)^{-1}|d_0S_0| \le h(G_0)^{-1}|dS|.$$

Since S is an arbitrary finite subgraph of G, we have $h(G) \ge h(G_0) > 0$, proving the claim. Finally, by the construction every closed face of G is homeomorphic to the closed unit disk, so one may check without difficulties that G has no cut-vertices.

Now by assuming that G has no cut vertices, we prove Theorem 1.1. For this, let us sketch our strategy discussed before. Let $\mathcal{S}$ be the polyhedral surface constructed from G as in the previous section. Then since both G and $\mathcal{S}$ are geodesic metric spaces and they are roughly isometric by Lemma 4.1, we see from Theorem 3.2 that Gromov hyperbolicity of $\mathcal{S}$ implies that of G. On the other hand, Theorem 3.3 says that Gromov hyperbolicity of $\mathcal{S}$ comes from a linear isoperimetric inequality on $\mathcal{S}$. Therefore in order to prove Theorem 1.1, we only need to show $h_1(\mathcal{S}) > 0$ as explained in Section 3. Equivalently, all we have to show is an inequality of the form

$$\text{Area}(D) \le C \cdot \text{Length}(\partial D) \tag{5.1}$$

for some constant C independent of D, where D is an arbitrary Jordan domain in $\mathcal{S}$.

For (5.1) we will apply a method similar to the one used in [10]. Suppose V is an open set in $\mathcal{S}$ such that ∂V is piecewise smooth. Also suppose that for some $f \in F(G)$ we have $V \cap f^\circ \neq \emptyset$, and $\overline{(f^\circ \cap \partial V)} = \bigcup_{k=1}^n \gamma_k$, where γ_k's are disjoint piecewise smooth simple curves meeting ∂f at their endpoints but not at any other points of the curves. Then each γ_k divides ∂f into two subarcs. Among these two subarcs of ∂f, let λ_k be the one containing less or the same number of vertices of f as in Lemma 3.4. Also let f_k be the closure of the component of $f \setminus \gamma_k$ that is enclosed by γ_k and λ_k; i.e., f_k must satisfy $\partial f_k = \gamma_k \cup \lambda_k$. In these notations, we have from Lemma 3.4 that

$$\sum_{k=1}^n \operatorname{Length}(\lambda_k) \le N \cdot \sum_{k=1}^n \operatorname{Length}(\gamma_k) = N \cdot \operatorname{Length}(f^\circ \cap \partial V), \tag{5.2}$$

because f is a k-gon for some $k \in \{3, 4, \ldots, N\}$. Similarly Lemmas 3.4 and 3.5 imply that

$$\begin{aligned} \sum_{k=1}^n \operatorname{Area}(f_k) &\le C_0 \cdot \sum_{k=1}^n \operatorname{Length}(\partial f_k) \\ &= C_0 \cdot \sum_{k=1}^n \operatorname{Length}(\gamma_k \cup \lambda_k) \le C_0(1+N) \cdot \operatorname{Length}(f^\circ \cap \partial V), \end{aligned} \tag{5.3}$$

where C_0 is the constant in (3.3). Note that C_0 depends only on N, and $C_0 \ge 1/(4\pi)$ since $N \ge 3$.

We claim that one of the following holds: either $(\partial f \cap V) \subset \bigcup_{k=1}^n \lambda_k$, or $(\partial f \setminus \overline{V}) \subset \bigcup_{k=1}^n \lambda_k$. If this were not true, there would exist $x \in \partial f \cap V$ and $y \in \partial f \setminus \overline{V}$ with $x, y \notin \bigcup_{k=1}^n \lambda_k$. Then since $x \in V$ and $y \notin \overline{V}$, x and y must be separated by ∂V. This means that there exists $j \in \{1, 2, \ldots, n\}$ such that γ_j separates x and y inside f, thus we must have $x \in \lambda_j$ or $y \in \lambda_j$ by the definition of λ_j. This contradicts our assumption, hence the claim has been proved.

We next claim that if $(\partial f \setminus \overline{V}) \not\subset \bigcup_{k=1}^n \lambda_k$, then $(V \cap f) \subset \bigcup_{k=1}^n f_k$. To prove this second claim, suppose there exists $y \in \partial f \setminus \overline{V}$ such that $y \notin \bigcup_{k=1}^n \lambda_k$. Then for each $x \in V \cap f$, a similar argument as above shows that there exists $j \in \{1, 2, \ldots, n\}$ with γ_j separating x and y inside f. Thus we have either $x \in f_j$ or $y \in f_j$. But since $y \notin \lambda_j = f_j \cap \partial f$ and $y \in \partial f$, it must be true that $y \notin f_j$. We conclude that $x \in f_j \subset \bigcup_{k=1}^n f_k$, and this proves the claim since $x \in V \cap f$ is arbitrary.

Define $U = V \cup f^\circ$ if $(\partial f \setminus \overline{V}) \subset \bigcup_{k=1}^n \lambda_k$, and $U = V \setminus f$ otherwise. Then we must have

$$\operatorname{Length}(\partial U) \le \operatorname{Length}(\partial V) + C_1 \cdot \operatorname{Length}(f^\circ \cap \partial V) \quad \text{and} \tag{5.4}$$

$$\operatorname{Area}(U) \ge \operatorname{Area}(V) - C_1 \cdot \operatorname{Length}(f^\circ \cap \partial V) \tag{5.5}$$

for some constant $C_1 = C_1(N)$. To verify these inequalities, we first consider the case $U = V \setminus f$; i.e., the case when $(\partial f \setminus \overline{V}) \not\subset \bigcup_{k=1}^n \lambda_k$. Then we have $(\partial f \cap V) \subset \bigcup_{k=1}^n \lambda_k$ by the first claim above. But since $\partial U \subset \big(\partial V \cup (\partial f \cap V)\big)$, we obtain from (5.2) that

$$\begin{aligned}\text{Length}(\partial U) &\leq \text{Length}(\partial V) + \text{Length}(\partial f \cap V) \\ &\leq \text{Length}(\partial V) + \sum_{k=1}^n \text{Length}(\lambda_k) \leq \text{Length}(\partial V) + N \cdot \text{Length}(f^\circ \cap \partial V),\end{aligned}$$

so the inequality (5.4) is satisfied. On the other hand, because $(V \cap f) \subset \bigcup_{k=1}^n f_k$ by the second claim above, we have from (5.3) that

$$\text{Area}(V \cap f) \leq \sum_{k=1}^n \text{Area}(f_k) \leq C_0(1+N) \cdot \text{Length}(f^\circ \cap \partial V).$$

Now (5.5) follows because $\text{Area}(U) = \text{Area}(V) - \text{Area}(V \cap f)$. We next consider the case $U = V \cup f^\circ$. But in this case we have $(\partial f \setminus \overline{V}) \subset \bigcup_{k=1}^n \lambda_k$ by the definition of U. Thus (5.4) comes from (5.2) and a computation similar to the above, because $\partial U \subset \partial V \cup (\partial f \setminus \overline{V})$. Finally one can easily see that (5.5) holds in this case, since $\text{Area}(U) \geq \text{Area}(V)$.

By summarizing all of these, we obtain the following lemma.

Lemma 5.1. *Let $V \subset \mathcal{S}$ be an open set such that ∂V is piecewise smooth, and suppose $f \in F(G)$ is a face such that $f^\circ \cap \partial V$ is nonempty and consists of finitely many disjoint simple arcs with endpoints on ∂f. Then by either adding f° to V or subtracting f from V, we obtain another open set U satisfying the inequalities* (5.4) *and* (5.5).

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose G is a planar graph such that $h(G) > 0$ and $\deg f \leq N$ for all $f \in F(G)$, where $N \in \mathbb{N}$ is as before. Then as we discussed at the beginning of this section, we may assume that G is a graph without cut-vertices, and then verifying a linear isoperimetric inequality (5.1) on $\mathcal{S}$ will prove Theorem 1.1, where the constant C in (5.1) should be independent of the Jordan domain $D \subset \mathcal{S}$. Thus suppose that a Jordan domain $D \subset \mathcal{S}$ is given. Then without loss of generality we may assume that ∂D is piecewise smooth, since otherwise we can approximate D by domains with piecewise smooth boundaries. Moreover in this approximation we can make the inequality (5.1) remain valid without changing the constant C. Similarly, we can assume that ∂D meets G only at finitely many points. Under these assumptions, one can easily check that there are only finitely many faces $f_1, f_2, \ldots, f_n \in F(G)$ such that $f_j^\circ \cap \partial D \neq \emptyset$, $j = 1, 2, \ldots, n$. Moreover for each $j \in \{1, 2, \ldots, n\}$, we can assume that $f_j^\circ \cap \partial D$ is a disjoint union of piecewise smooth simple arcs with endpoints on ∂f_j. Therefore we can apply Lemma 5.1 to $D_0 := D$ and

f_1, hence either by adding f_1° to D or by subtracting f_1 from D, we obtain D_1 satisfying

$$\text{Length}(\partial D_{j+1}) \le \text{Length}(\partial D_j) + C_1 \cdot \text{Length}(f_{j+1}^\circ \cap \partial D_j) \quad \text{and} \tag{5.6}$$

$$\text{Area}(D_{j+1}) \ge \text{Area}(D_j) - C_1 \cdot \text{Length}(f_{j+1}^\circ \cap \partial D_j) \tag{5.7}$$

for $j = 0$.

Suppose we have obtained $D_0, D_1, \ldots, D_k$ that satisfy the inequalities (5.6) and (5.7) for all $j = 0, 1, \ldots, k-1$. If $k < n$, then because the interiors of f_j's are all disjoint, one can easily check that D_k and f_{k+1} also satisfy the assumptions in Lemma 5.1. Therefore we can obtain D_{k+1}, either by adding f_{k+1}° to D_k or by subtracting f_{k+1} from D_k, so that they satisfy the inequalities (5.6) and (5.7) for $j = k$. Now we repeat this process until we obtain a sequence of open sets $D_0, D_1, \ldots, D_n$, which satisfy (5.6) and (5.7) for all $j = 0, 1, \ldots, n-1$.

Note that D_n must consist of faces. That is, D_n must satisfy the assumption (about D) of Lemma 4.2, since from the construction there is no $f \in F(G)$ such that $f^\circ \cap \partial D_n \neq \emptyset$. Thus there exists a constant C_2 depending only on N and $h(G)$ such that $\text{Area}(D_n) \le C_2 \cdot \text{Length}(\partial D_n)$. Also recall that the interiors of f_j's are mutually disjoint, thus we must have

$$\sum_{j=0}^{n-1} \text{Length}(f_{j+1}^\circ \cap \partial D_j) \le \text{Length}(\partial D). \tag{5.8}$$

Therefore from (5.6), (5.7), and (5.8) we have

$$\begin{aligned}
&\text{Area}(D) = \text{Area}(D_0) \le \text{Area}(D_1) + C_1 \cdot \text{Length}(f_1^\circ \cap \partial D_0) \\
&\le \text{Area}(D_2) + C_1 \cdot \text{Length}(f_1^\circ \cap \partial D_0) + C_1 \cdot \text{Length}(f_2^\circ \cap \partial D_1) \\
&\qquad \vdots \\
&\le \text{Area}(D_n) + C_1 \cdot \sum_{j=0}^{n-1} \text{Length}(f_{j+1}^\circ \cap \partial D_j) \\
&\le C_2 \cdot \text{Length}(\partial D_n) + C_1 \cdot \text{Length}(\partial D) \\
&\le C_2 \cdot \text{Length}(\partial D_{n-1}) + C_2 C_1 \cdot \text{Length}(f_n^\circ \cap \partial D_{n-1}) + C_1 \cdot \text{Length}(\partial D) \\
&\qquad \vdots \\
&\le C_2 \cdot \text{Length}(\partial D_0) + C_2 C_1 \sum_{j=0}^{n-1} \text{Length}(f_{j+1}^\circ \cap \partial D_j) + C_1 \cdot \text{Length}(\partial D) \\
&\le (C_2 + C_2 C_1 + C_1) \cdot \text{Length}(\partial D),
\end{aligned}$$

showing the inequality (5.1). Since both C_1 and C_2 are independent of D, this completes the proof of Theorem 1.1. □

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Department of Mathematics Education
Hanyang University
Seoul 133-791, Korea
E-mail address: bgoh@hanyang.ac.kr