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SAMELSON PRODUCTS IN FUNCTION SPACES

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Abstract. We study Samelson products on models of function spaces. Given a map $f : X \longrightarrow Y$ between 1-connected spaces and its Quillen model $\mathbb{L}(f) : \mathbb{L}(V) \longrightarrow \mathbb{L}(W)$, there is an isomorphism of graded vector spaces $\Theta : H_*(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))) \longrightarrow H_*(\mathbb{L}(W) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W)))$. We define a Samelson product on $H_*(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)))$.

1. Introduction

Throughout this paper, spaces are assumed to be 1-connected finite CW-complexes. Given a map $f : X \longrightarrow Y$ between 1-connected spaces and its Quillen model $\mathbb{L}(f) : \mathbb{L}(V) \longrightarrow \mathbb{L}(W)$, let $\mathrm{map}(X, Y; f)$ and $\mathrm{map}_*(X, Y; f)$ denote the path component containing f in the space of base point-free and base point-preserving functions respectively. Lupton–Smith [3] extended the notion of derivation of a differential graded Lie algebra to a derivation with respect to a map of differential graded Lie algebras. They proved the following vector space isomorphisms:

$$\pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(s\mathbb{L}(W) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W); f)),$$

and

$$\pi_n(\mathrm{map}_*(X, Y; f)) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(\mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W)); f).$$

The authors in their paper [2] established an isomorphism

$$\pi_{n-1}(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))).$$

Lupton-Smith [4] have expressed the Whitehead product on

$$H_*(s\mathbb{L}(W) \oplus \mathrm{Der}\,(\mathbb{L}(V), \mathbb{L}(W)).$$

We study the Samelson product in the homology of $\mathrm{Hom}(P, \mathbb{L}(W))$ where $P = (TV \otimes (\mathbb{Q} \oplus sV), D)$ is the acyclic TV-differential module with a differential defined as follows:

$$Dv = dv \otimes 1, \quad Dsv = v \otimes 1 - S(dv \otimes 1)$$

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and S is the $\mathbb{Q}$-graded vector spaces map of (degree 1) defined by

$$\begin{cases} S(v \otimes 1) = 1 \otimes sv, \quad S(1 \otimes (\mathbb{Q} \oplus sV)) = 0, \\ S(ax) = (-1)^{|a|} aS(x), \;\; \forall a \in T(V), x \in \mathbb{Q} \oplus sV, \;\; |x| > 0, \\ S(1 \otimes x) = 0. \end{cases}$$

2. Preliminaries

Let (L, d) be a differential graded Lie algebra. Given a map $\phi : (L, d_L) \longrightarrow (K, d_K)$ of differential graded Lie algebras, define a ϕ-derivation of degree p to be a linear map $\theta : L \longrightarrow K$ that increases degree by p and satisfies $\theta([x, y]) = [\theta(x), \phi(y)] + (-1)^{p|x|}[\phi(x), \theta(y)]$ for $x,\, y \in L$. Consider the vector space of ϕ-derivations $\mathrm{Der}_n(L, K; \phi)$ which are derivations that increase degree by some positive number n. When $n = 1$, we restrict to derivations θ such that $d_K \circ \theta + \theta \circ d_L = 0$. The differential D is defined by $D(\theta) = d_K \circ \theta - (-1)^{|\theta|} \theta \circ d_L$. Following [5, p. 46], if $\phi : A \longrightarrow B$ is a morphism of chain complexes, the mapping cone of ϕ is defined by $\mathrm{Cone}(\phi) = sA \oplus B$ with the differential $d(sa, b) = (-sd_A(a), \phi(a) + d_B(b))$.

Let (L, d) be a differential graded Lie algebra. Consider the adjoint mapping $\mathrm{ad} : L \longrightarrow \mathrm{Der}\, L$, defined by $(\mathrm{ad}\, x)(y) = [x, y]$. We have $\mathrm{Cone}(\mathrm{ad}) = sL \oplus \mathrm{Der}\, L$ where $(sL \oplus \mathrm{Der}(L), D)$ is a differential graded Lie algebra with the bracket defined as follows [6]:

$$\begin{cases} [sx, sy] = 0, \\ [sx, \theta] = (-1)^{|x||\theta|} s\theta(x), \\ [sx_1 + \theta_1,\, sx_2 + \theta_2] = (-1)^{|\theta_1|} s\theta_1(x_2) - (-1)^{|\theta_2||x_1|} s\theta_2(x_1) + [\theta_1, \theta_2]. \end{cases}$$

The differential is defined by $D(sx) = -s\delta x + \mathrm{ad}\, x$. Note that $sx + \theta$ is a cycle if and only if $d(x) = 0$ and $[d, \theta] + \mathrm{ad}\, x = 0$.

If $L = (\mathbb{L}(V), d)$, there is a differential isomorphism of degree 1

$$\Phi : \mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV, L)) \longrightarrow sL \oplus \mathrm{Der}\, L$$

defined by $\Phi(f) = sf(1) + (-1)^{|f|}\theta$ where $\theta(v) = f(sv)$ [1]. This gives a bracket (of degree 1) on $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV, \mathbb{L}(V)))$ defined on generators as follows: For $f,\, g \in \mathrm{Hom}(P, \mathbb{L}(V))$, $f(1)$, $g(1) \in \mathbb{L}(V)$, and $f(sv) \in \mathbb{L}(V) \subset T(V)$, then

$$\{f, g\}\,(sv) = -(-1)^{|f|}[f(S(g(sv))) - (-1)^{(|f|+1)(|g|+1)} f(S(g(sv)))],$$
$$\{f, g\}\,(1) = f(S(g(1))) - (-1)^{|f||g|} g(S(f(1))).$$

3. Whitehead product in differential graded Lie algebras

Let L be a differential graded Lie algebra and sL the suspension on L. A bilinear pairing on sL is defined by $[sx, sy] = (-1)^{|x|} s[x, y]$ where $x,\, y \in L$. A Whitehead product denoted by $[,]_W$ is a bilinear pairing satisfying the following identities:

(1) $|[\alpha, \beta]_W| = |\alpha| + |\beta| - 1$,
(2) $[\alpha, \beta]_W = (-1)^{|\alpha||\beta|}[\beta, \alpha]_W$ and

(3) $[\alpha,[\beta,\gamma]_W]_W = (-1)^{|\alpha|+1}[[\alpha,\beta]_W,\gamma]_W + (-1)^{(|\alpha|+1)(|\beta|+1)}[\beta,[\alpha,\gamma]_W]_W$ for $\alpha,\ \beta,\ \gamma \in sL$.

Lupton-Smith [4] defined a pairing on a differential graded Lie algebra L by $\{x,y\} = (-1)^{|x|+1}[x, d_L(y)]$, which induces a Whitehead product on the homotopy of L. For $\Phi : (L, d_L) \longrightarrow (K, d_K)$ they defined a pairing on its mapping cone $(\mathrm{Cone}(\Phi), \delta_\Phi)$ which induces a Whitehead product $[,]_W$ on $H_*(\mathrm{Cone}(\Phi), \delta_\Phi)$. The product is defined explicitly as follows:
Let $a,\ b \in L;\ |a| = p-1,\ |b| = q-1$,

$$[[(a,\alpha),(b,\beta)]] = ((-1)^p[a,b], \{\alpha,\beta\}) = ((-1)^p[a,b], (-1)^{p+1}[\alpha, d_K\beta])$$

for $(a,\alpha) \in \mathrm{Cone}_p(\Phi),\ (b,\beta) \in \mathrm{Cone}_q(\Phi)$ where $\alpha \in K_p,\ \beta \in K_q$.

4. Samelson product on $H_*(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)))$

Let $f : X \longrightarrow Y$ be a map, $\Phi : \mathbb{L}(V) \longrightarrow \mathbb{L}(W)$ its Quillen model. Consider the map $\mathrm{ad}_\Phi : \mathbb{L}(W) \longrightarrow \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W), \Phi)$. The mapping cone of ad_Φ is $(s\mathbb{L}(W) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W)), D)$ where $D(\theta) = \delta_W\theta - (-1)^{|\theta|}\theta\delta_V$ and $D(sx) = -s\delta_V x + \mathrm{ad}_\Phi(x)$. If $(sa,\theta), (sb,\theta')$ are cycles of degree p and q respectively, Lupton-Smith [4] defined a Whitehead product

$$H_p \otimes H_q \longrightarrow H_{p+q-1}, \text{ where } H_p = H_p(s\mathbb{L}(W) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(W); \Phi))$$

and showed that it corresponds to the Whitehead product on

$$\pi_*(\mathrm{map}(X,Y;f)) \otimes \mathbb{Q}.$$

We define on $H_*(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)))$ a Samelson product $H_{p-1} \otimes H_{q-1} \longrightarrow H_{p+q-2}$, that models the Samelson product on $\pi_*(\Omega\mathrm{map}(X,Y;f)) \otimes \mathbb{Q}$.

Our approach follows Lupton-Smith [4]. Let $(\mathbb{L}(V), d)$ be a free differential graded Lie algebra.

Define $\mathbb{L}(V)(a,b) = (\mathbb{L}(V, a, b, V^a, V^b), \delta)$ where $|a| = p-1$, $|b| = q-1$, $V^a = s^pV$, $V^b = s^qV$ with the differential

$$\begin{aligned}
&\delta v = dv,\ \delta a = \delta b = 0,\\
&\delta(v^a) = (-1)^{p-1}[a,v] + (-1)^p S_a(dv),\\
&\delta(v^b) = (-1)^{q-1}[b,v] + (-1)^q S_b(dv),
\end{aligned}$$

where S_a is the unique derivation of degree p of $\mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,b); \lambda)$ such that $S_a(v) = v^a$ and zero on other generators and λ is the canonical inclusion $\lambda : \mathbb{L}(V) \longrightarrow \mathbb{L}(V)(a,b)$. Also, $S_b \in \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,b); \lambda)$ is similarly defined.

Consider the chain complex $(\mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,b); \lambda), D_\lambda)$, with the differential defined in §2. Moreover

$$D_\lambda(S_a) = (-1)^{p-1}\mathrm{ad}_\lambda a, \quad D_\lambda(S_b) = (-1)^{q-1}\mathrm{ad}_\lambda b.$$

Consider $\mathrm{ad}_\lambda : \mathbb{L}(V)(a,\ b) \longrightarrow \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,\ b); \lambda)$ and the mapping cone of ad_λ,

$$\mathrm{Cone}(\mathrm{ad}_\lambda) = (s\mathbb{L}(V)(a,b)) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,b)), D),$$

where $D(\theta) = D_\lambda(\theta)$ for $\theta \in \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,b))$ and $D(sx) = -s\delta\, x + \mathrm{ad}_\lambda x$ for $x \in \mathbb{L}(V)(a,\, b)$. We have $D((-1)^{|x|}sx - S_x) = (-1)^{|x|}(-s\delta x + \mathrm{ad}_\lambda x) - (-1)^{|x|}\mathrm{ad}_\lambda x = 0$ for $x = a, b$. In the complex $(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)(a,b), D)$, we consider ϕ_a and ϕ_b defined by

$$\begin{cases} \phi_a(1) = -(-1)^{|a|}a; & \phi_a(sv) = (-1)^{|a|}S_a(v) \\ \phi_b(1) = -(-1)^{|b|}b; & \phi_b(sv) = (-1)^{|b|}S_b(v) \end{cases}$$

or in a condensed form $\phi_x(1) = -(-1)^{|x|}x$; $\phi_x(sv) = (-1)^x S_x(v)$, $x = a,\, b$.

Proposition 1. *ϕ_a and ϕ_b are cycles of degrees $p-1$ and $q-1$ respectively.*

Proof. It can be easily verified that $(D\phi_x)(1) = 0$ for $x = a,\, b$. For $(D\phi_x)(sv)$ we have

$$\begin{aligned}(D\phi_x)(sv) &= \delta\phi_x(sv) - (-1)^{|x|}\phi_x(d(sv)) \\ &= \delta((-1)^{|x|}S_x(v)) - (-1)^{|x|}\phi_x(v \otimes 1 - s(dv \otimes 1)) \\ &= (-1)^{|x|}((-1)^{|x|}[x,v] - (-1)^{|x|}S_x(dv)) - (-1)^{|x|}(-1)^{|x||v|}[v, \phi_x(1)] \\ &\quad + (-1)^{|x|}\phi_x(s(dv \otimes 1)) \\ &= [x,v] - (-1)^{|x|}S_x(dv) + (-1)^{|x||v|}[v,x] + (-1)^{|x|}\phi_x(s(dv \otimes 1)) \\ &= 0. \end{aligned}$$

□

Define $\theta_a,\ \theta_b \in \mathrm{Der}(\mathbb{L}(V)(a,b))$ such that

$$\begin{cases} \theta_a(v) \ = \ v^a; & \text{zero otherwise,} \\ \theta_b(v) \ = \ v^b; & \text{zero otherwise.} \end{cases}$$

We have $[\delta, \theta_x] = (-1)^{|x|}\mathrm{ad}\, x$; $x = a, b$.

Observe that $S_a = \theta_a$ and $S_b = \theta_b$ if restricted to $\mathbb{L}(V)$. We define a universal Samelson product $[\phi_a, \phi_b]$ in $(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)(a,b), D)$ as follows:

$$\begin{aligned}[\phi_a, \phi_b](1) &= (-1)^{|b|+1}[\phi_a(1), \phi_b(1)] \\ &= (-1)^q[a,b], \\ [\phi_a, \phi_b](sv) &= -(-1)^{|a|+1}[\theta_a, [\delta, \theta_b]](v) \\ &= -(-1)^p[\theta_a, [\delta, \theta_b]](v). \end{aligned}$$

Proposition 2. *$\Upsilon = [\phi_a, \phi_b]$ is a cycle of degree $p+q-2$.*

Proof. There is a differential isomorphism of chain complexes

$F : \mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)(a,b)) \longrightarrow s\mathbb{L}(V)(a,\, b) \oplus \mathrm{Der}(\mathbb{L}(V), \mathbb{L}(V)(a,\, b))$

defined by $F(f) = sf(1) + (-1)^{|f|}\theta$ with $\theta(v) = f(sv)$ see [1, Theorem 1]. This is an isomorphism of degree $1\, (DF = -FD)$. In particular

$$\begin{aligned} F([\phi_a, \phi_b]) &= (-1)^{p+q}s[\phi_a, \phi_b](1) - (-1)^{p+q}(-1)^p[\theta_a, [\delta, \theta_b]] \circ \lambda \\ &= (-1)^p s[a,b] - (-1)^q[\theta_a, [\delta, \theta_b]] \circ \lambda. \end{aligned}$$

In order to check that Υ is a cycle, we need only to show that $(-1)^p s[a,b] - (-1)^q[\theta_a,[\delta,\theta_b]]\circ\lambda$ is a cycle in $s\mathbb{L}(V)(a,b)\oplus \mathrm{Der}(\mathbb{L}(V),\mathbb{L}(V)(a,b))$. Indeed

$$\begin{aligned} &D((-1)^p s[a,b] - (-1)^q[\theta_a,[\delta,\theta_b]]\circ\lambda)\\ =&\ -(-1)^p s\delta[a,b] + (-1)^p \mathrm{ad}_\lambda[a,b] - (-1)^q[\delta,[\theta_a,[\delta,\theta_b]]]\circ\lambda\\ =&\ (-1)^p\mathrm{ad}_\lambda[a,b] - (-1)^q[[\delta,\theta_a],[\delta,\theta_b]]\circ\lambda\\ =&\ (-1)^p\mathrm{ad}_\lambda[a,b] - (-1)^p[\mathrm{ad}\, a,\mathrm{ad}\, b]\circ\lambda\\ =&\ (-1)^p\mathrm{ad}_\lambda[a,b] - (-1)^p\mathrm{ad}_\lambda[a,b]\\ =&\ 0. \end{aligned}$$

□

We are now ready to define Samelson's products in $H_*(\mathrm{Hom}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W)))$. Recall that $f: X\longrightarrow Y$ has a Quillen model $\Phi:\mathbb{L}(V)\longrightarrow\mathbb{L}(W)$. Let α, β be cycles of respective degrees $p-1$, $q-1$ in $\mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W))$. Define $\Psi_{\alpha,\beta}:\mathbb{L}(V)(a,b)\longrightarrow\mathbb{L}(W)$ by:

$$\Psi_{\alpha,\beta}(a)=\alpha(1),\qquad \Psi_{\alpha,\beta}(b)=\beta(1),$$
$$\Psi_{\alpha,\beta}(v)=\Phi(v),\qquad \Psi_{\alpha,\beta}(v^a)=\alpha(sv),\ \Psi_{\alpha,\beta}(v^b)=\beta(sv).$$

Theorem 3. (1) *The composition*

$$TV\otimes(\mathbb{Q}\oplus sV)\overset{[\phi_a,\phi_b]}{\longrightarrow}\mathbb{L}(V)(a,b)\overset{\Psi_{\alpha,\beta}}{\longrightarrow}\mathbb{L}(W)$$

defines a Samelson product on $H_*(\mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W)))$.

(2) *It is explicitly defined on the generators by*

$$\begin{aligned} [\alpha,\beta](1) &= (-1)^{|\beta|+1}[\alpha(1),\beta(1)],\\ [\alpha,\beta](sv) &= (-1)^{|\alpha|+1}[\alpha(1),\beta(sv)] + (-1)^{(|\beta|+1)|\alpha|}[\beta(1),\alpha(sv)]\\ &\quad + (-1)^{|\beta||\alpha|}\Psi_{\alpha,\beta}(\theta_b\theta_a(dv)), \end{aligned}$$

where $\theta_a,\theta_b\in\mathrm{Der}(\mathbb{L}(V)(a,b))$.

Proof. The first assertion is a direct consequence of the isomorphism

$$\mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W))\cong s\mathbb{L}(W)\oplus\mathrm{Der}(\mathbb{L}(V),\mathbb{L}(W))$$

and [4, Theorem 3.6.]. For the second assertion consider α, β cycles of respective degrees $p-1$ and $q-1$ in $\mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W))$ and ϕ_a, ϕ_b corresponding cycles in $\mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(V)(a,b))$ (see Proposition 1), we have

$$\begin{aligned} &[\phi_a,\phi_b](sv)\\ =&\ -(-1)^p[\theta_a,[\delta,\theta_b]](v)\\ =&\ -(-1)^p(\theta_a[\delta,\theta_b](v) - (-1)^{p(q-1)}[\delta,\theta_b](\theta_a(v)))\\ =&\ -(-1)^p[\theta_a((-1)^{q-1}[b,v]) - (-1)^{p(q-1)}(\delta\theta_b\theta_a(v) - (-1)^q\theta_b\delta\theta_a(v))]\\ =&\ -(-1)^p(-1)^{q-1}(-1)^{p(q-1)}[b,\theta_a(v)] \end{aligned}$$

$$
\begin{aligned}
&- (-1)^p(-1)^{p(q-1)}(-1)^q[\theta_b((-1)^{p-1}[a,v] + (-1)^p S_a(dv))]\\
= &- (-1)^p(-1)^{q-1}(-1)^{p(q-1)}[b,\theta_a(v)]\\
&- (-1)^p(-1)^{p(q-1)}(-1)^q(-1)^{p-1}(-1)^{q(p-1)}[a,\theta_b(v)]\\
&- (-1)^p(-1)^{p(q-1)}(-1)^q(-1)^p\theta_b(S_a(dv))\\
= &(-1)^{q(p+1)}[b,\theta_a(v)] + (-1)^p[a,\theta_b(v)] + (-1)^{pq+p+q+1}\theta_b(\theta_a(dv))\\
= &(-1)^{q(p+1)}[b,\theta_a(v)] + (-1)^p[a,\theta_b(v)] + (-1)^{(p+1)(q+1)}\theta_b(\theta_a(dv)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Psi_{\alpha,\beta}([\phi_a,\phi_b](sv)) = &(-1)^{q(p+1)}[\beta(1),\alpha(sv)] + (-1)^p[\alpha(1),\beta(sv)]\\
&+ (-1)^{(p+1)(q+1)}\Psi_{\alpha,\beta}\theta_b(\theta_a(dv)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
[\alpha,\beta](sv) = &(-1)^{|\alpha|+1}[\alpha(1),\beta(sv)] + (-1)^{(|\beta|+1)|\alpha|}[\beta(1),\alpha(sv)]\\
&+ (-1)^{|\beta||\alpha|}\Psi_{\alpha,\beta}(\theta_b\theta_a(dv)).
\end{aligned}
$$ □

Corollary 4. *If $f : X \longrightarrow Y$ is null homotopic, then*

$$[\alpha,\beta](1) = (-1)^{|\beta|+1}[\alpha(1),\beta(1)] \text{ and}$$
$$[\alpha,\beta](sv) = (-1)^{|\alpha|+1}[\alpha(1),\beta(sv)] + (-1)^{(|\beta|+1)|\alpha|}[\beta(1),\alpha(sv)].$$

In particular $H_*(\mathbb{L}(W))$ is a sub Lie algebra of $H_*(\mathrm{Hom}(TV\otimes(\mathbb{Q}\oplus sV)),\mathbb{L}(W))$.

Proof. Let x be a cycle in $(\mathbb{L}(W),\delta)$. Define $\alpha_x \in \mathrm{Hom}_{TV}(TV\otimes(\mathbb{Q}\oplus sV),\mathbb{L}(W))$ by $\alpha_x(1) = x$ and $\alpha_x(sv) = 0$. Clearly α_x is a cycle in $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV),\mathbb{L}(W))$. Moreover, if x is a boundary in $\mathbb{L}(W)$, so is α_x. Therefore there is a map:

$$\mathrm{I} : H_*(\mathbb{L}(W),\delta) \longrightarrow H_*(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV),\mathbb{L}(W))).$$

If α_x is a boundary, then there is $\gamma \in \mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV),\mathbb{L}(W))$ such that $\alpha = D\gamma$. In particular $x = \alpha(1) = (D\gamma)(1) = \delta\gamma(1)$. Therefore x is a boundary. □

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