THE STRONG MORI PROPERTY
IN RINGS WITH ZERO DIVISORS

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Abstract. An SM domain is an integral domain which satisfies the ascending chain condition on w-ideals. Then an SM domain also satisfies the descending chain condition on those chains of v-ideals whose intersection is not zero. In this paper, a study is begun to extend these properties to commutative rings with zero divisors. A Q_0-SM ring is defined to be a ring which satisfies the ascending chain condition on semiregular w-ideals and satisfies the descending chain condition on those chains of semiregular v-ideals whose intersection is semiregular. In this paper, some properties of Q_0-SM rings are discussed and examples are provided to show the difference between Q_0-SM rings and SM rings and the difference between Q_0-SM rings and Q_0-Mori rings.

1. Introduction

In this paper, we assume that R is a commutative ring with an identity and S is a multiplicatively closed set of R. Let $Z(R)$ denote the set of zero divisors of R and let $T(R)$ be the total quotient ring of R.

Let I be an ideal of R. Then I is called a *regular ideal* if I contains a regular element and I is called a *semiregular ideal* of R if I contains a finitely generated ideal I_0 which has no nonzero annihilator. Let S^0 denote the set of finitely generated semiregular ideals. T. Lucas used semiregular ideals to define the ring of finite fractions $Q_0(R)$ and tried to use $Q_0(R)$ instead of the quotient field of a domain to research multiplicative ideal theory of commutative rings with zero divisors, where

$$Q_0(R) = \{u \in T(R[X]) \mid Iu \subseteq R \text{ for some } I \in S^0\}.$$

If u is an element of $Q_0(R)$, then u can be written in the form $u = \frac{\sum_{i=0}^{n} a_i X^i}{\sum_{i=0}^{n} b_i X^i} \in T(R[X])$ with $a_i b_j = a_j b_i$ for each i and j. If each semiregular ideal of R is regular, then $Q_0(R) = T(R)$.

For a domain D with the quotient field K, an ideal I of D is called a *v-ideal* if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in K \,|\, xI \subseteq R\}$. A Mori domain, i.e., a domain which satisfies the ascending chain condition on v-ideals, has many good properties and various researchers have focused on it. The definition of Mori domains is analogous to that of Noetherian domains, but many classical theorems on Noetherian domains cannot be generalized to Mori domains. In order to give a description in the star operation theory corresponding to the classical theory of Noetherian domains, in 1997, Wang and McCasland gave the definitions of the w-operation and SM domains (strong Mori domains) [8, Definition 2 and Definition 4]. Let J be a finitely generated ideal of a domain D. If $J^{-1} = D$, then J is called a *GV-ideal*, denoted by $J \in$GV(D). A torsion-free module M is called a *w-module* of D if whenever $Jx \subseteq M$ for some $J \in$ GV(D) and $x \in K \bigotimes_D M$, then $x \in M$. An SM domain is an integral domain which satisfies the ascending chain condition on w-ideals. Wang and McCasland also have shown that SM domains satisfy the Principal Ideal Theorem [9, Corollary 1.11], the Hilbert Basis Theorem [9, Theorem 1.13] and the Krull Intersection Theorem [9, Theorem 1.8] (Note that these three theorems do not hold for Mori domains in general).

In 2004, T. Lucas introduced the Q_0-Mori ring and discussed the Mori property in commutative rings with zero divisors. The Q_0-Mori ring is defined to be a ring which satisfies ACC (i.e., the ascending chain condition) on semiregular v-ideals and satisfies the restricted DCC on semiregular v-ideals (i.e., the descending chain condition on those chains of semiregular v-ideals whose intersection is semiregular) (see [5, page 1]), where a semiregular v-ideal is not only a semiregular ideal but also a v-ideal. Following his idea, in this paper, the definition of Q_0-SM rings is provided and we use $Q_0(R)$ to discuss the SM property in commutative rings with zero divisors.

2. The reducible property of commutative rings

We say that an R-module $B \subseteq Q_0(R)$ is a *fractional ideal* of R if there is a semiregular ideal $I \subseteq R$ such that $IB \subseteq R$. If, in addition, B contains an element that is not a zero divisor and there is a regular element $r \in R$ such that $rB \subseteq R$, then it is a *regular fractional ideal* of R. For B to be considered a *semiregular fractional ideal*, B must contain a finitely generated fractional ideal that has no nonzero annihilator and there must be a semiregular ideal I of R such that $IB \subseteq R$.

We let

$$R_{\langle S \rangle} = \{q \in Q_0(R) \,|\, rq \in R \text{ for some } r \in S\}.$$

It is clear that $R_{\langle S \rangle}$ is a ring and we call $R_{\langle S \rangle}$ the Q_0-*quotient ring with respect to S*. For an ideal I of R, we set

$$I_{\langle S \rangle} = \{q \in R_{\langle S \rangle} \,|\, rq \in I \text{ for some } r \in S\}.$$

For a prime ideal P of R, $S = R \setminus P$ is a multiplicatively closed set, then we consider $R_{\langle S \rangle}$ as $R_{\langle P \rangle}$ and consider $I_{\langle S \rangle}$ as $I_{\langle P \rangle}$. Here we call $R_{\langle P \rangle}$ the Q_0-quotient ring with respect to P. It is easy to see that if I is not contained in P, then $I_{\langle P \rangle} = R_{\langle P \rangle}$.

Definition 2.1. Let $A \subseteq Q_0(R)$ be an R-module. Then A is called an S-reducible module if whenever $sx \in A$ where $s \in S$ and $x \in Q_0(R)$, we have $x \in A$. For a prime ideal P of R and $S = R \setminus P$, we say that A is P-reducible if A is S-reducible.

For R and S, R may not be its own S-reducible ideal. For example, let R be the ring of integers $\mathbb{Z}$ and $S = R \setminus \{0\}$. Then $Q_0(R) = \mathbb{Q}$ where $\mathbb{Q}$ is the rational number field. We choose $s \in S$ and $1/s \in Q_0(R)$, then $s \cdot \frac{1}{s} = 1 \in R$, but $1/s \notin R$. We also have that not all $R_{\langle S \rangle}$-modules are S-reducible. For example, let $s \in R, s \neq 0$ and $s^2 = 0$. Then $S = \{0, 1, s\}$ is a multiplicatively closed set of R. Clearly, $C = 0$ is an $R_{\langle S \rangle}$-module. Since $ss = 0 \in C$ and $s \notin C$, we have C is not S-reducible.

Lemma 2.2. *Let S be a multiplicatively closed set in R.*
  (1) Let A be an ideal of $R_{\langle S \rangle}$ and let $B = A \cap R$. Then $A \subseteq B_{\langle S \rangle}$.
  (2) I is a proper S-reducible ideal of $R_{\langle S \rangle}$ if and only if $I = I_{\langle S \rangle} = (I \cap R)_{\langle S \rangle}$ and $I \cap S = \emptyset$.
  (3) If A is a proper ideal of R, then $A_{\langle S \rangle}$ is an S-reducible ideal of $R_{\langle S \rangle}$ and $A_{\langle S \rangle} = (A_{\langle S \rangle})_{\langle S \rangle}$.

In order to discuss the relationship between reducible ideals and w-ideals, we need to know the concept of w-modules of commutative rings in [10]. Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism, then J is called a *GV-ideal*, denoted by $J \in \mathrm{GV}(R)$. An R-module M is called a *GV-torsion-free module* if whenever $Jx = 0$ for some $J \in \mathrm{GV}(R)$ and $x \in M$, we have $x = 0$. A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}_R^1(R/J, M) = 0$ for each $J \in \mathrm{GV}(R)$ and the w-envelope of M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. Therefore M is a w-module if and only if $M_w = M$. In this paper, let $w\text{-}\mathrm{Max}(R)$ denote the set of all maximal w-ideals of R. For R, there is at least one maximal w-ideal and each maximal w-ideal is prime.

Lemma 2.3. *Let P be a prime w-ideal of R. If I is a P-reducible ideal of $R_{\langle P \rangle}$, then I is a w-module of R.*

Proof. Let $x \in I_w$. Then there is some $J \in \mathrm{GV}(R)$ such that $Jx \subseteq I$. Since $J \nsubseteq P$, there is some $s \in J \setminus P$ such that $sx \in I$. Because $I_w \subseteq (Q_0(R))_w = Q_0(R)$ and I is P-reducible, we have $x \in I$. Thus I is a w-module of R. $\square$

Theorem 2.4. *Let P be a prime w-ideal of R. If R satisfies ACC on semiregular w-ideals, then $R_{\langle P \rangle}$ satisfies ACC on semiregular P-reducible ideals.*

Proof. Let I be a semiregular P-reducible ideal of $R_{\langle P \rangle}$ and let $A = I \cap R$. We first show that A is a semiregular w-ideal of R. Let $B = (b_1, b_2, \ldots, b_n) \subseteq I$ be a finitely generated semiregular ideal of $R_{\langle P \rangle}$. Then for each i there is a finitely generated semiregular ideal I_{α_i} of R such that $b_i I_{\alpha_i} \subseteq R$. Let $I_0 = I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_n}$. Then I_0 is also a semiregular ideal of R and $B I_0 \subseteq I \cap R = A$. Since $B I_0$ is a finitely generated semiregular ideal of R, we have A is semiregular. By Lemma 2.3, I is a w-module of R. We have A is a w-ideal of R.

Let $\{I_n\}$ be an ascending chain of semiregular P-reducible ideals of $R_{\langle P \rangle}$. Then $\{I_n \cap R\}$ is an ascending chain of semiregular w-ideals of R. Let $A_i = I_i \cap R$. There is an integer n such that $I_k \cap R = I_n \cap R$ for each $k \geq n$. So $(A_k)_{\langle P \rangle} = (A_n)_{\langle P \rangle}$. Because I_i is a P-reducible ideal, $I_i = (I_i \cap R)_{\langle P \rangle} = (A_i)_{\langle P \rangle}$ by Lemma 2.2(2). Then $I_k = I_n$. It follows that $R_{\langle P \rangle}$ satisfies ACC on semiregular P-reducible ideals. $\square$

For a semiregular fractional ideal I, let $I^{-1} = (R : I) = \{t \in Q_0(R) \,|\, tI \subseteq R\}$ and let $I_v = (I^{-1})^{-1}$. Then it is easy to show that $(R : I)$ is a semiregular fractional ideal as well and $(R : I) = (R : I)_v$. Set $I_t = \bigcup(I_0)_v$ with the union taken over the finitely generated fractional ideals that are contained in I (see [5, Introduction]). If $I = I_v$ (resp., $I = I_t$), then I is said to be a *v-fractional ideal* (resp., *t-fractional ideal*). For each semiregular fractional ideal I, we have $I_w \subseteq I_t \subseteq I_v$ and if I is finitely generated, we have $I_v = I_t$.

For a finitely generated semiregular ideal J of R, J is a GV-ideal if and only if $J^{-1} = R$. Let M be a semiregular prime ideal of R. Then M is a maximal w-ideal if and only if M is a maximal t-ideal. We use $t\text{-Max}(R)$ to denote the set of all maximal t-ideals of R.

Lemma 2.5 ([5, Lemma 2.15]). *Let I be a semiregular ideal of R.*
  (1) *If I is a v-ideal, then $I = \bigcap\{(I_{\langle M \rangle})_v \,|\, M \in t\text{-Max}(R)\}$.*
  (2) *If I is finitely generated, then $I_v = \bigcap\{(I_{\langle M \rangle})_v \,|\, M \in t\text{-Max}(R)\}$.*

Lemma 2.6. *Let A and B be semiregular w-ideals of R. Then $A = B$ if and only if $A_{\langle M \rangle} = B_{\langle M \rangle}$ for each semiregular maximal w-ideal M.*

Proof. Let $x \in A$ and let $I = \{r \in R \,|\, rx \in B\}$. It is easy to show that I is a semiregular w-ideal of R and $I \nsubseteq M$ for each semiregular maximal w-ideal M, so $I = R$. Hence we can get $x \in B$, which implies $A \subseteq B$. We can prove $B \subseteq A$ in the same way. $\square$

Theorem 2.7. *Let R be a ring in which each semiregular ideal is contained in at most finitely many maximal w-ideals.*

  (1) *If for each semiregular maximal w-ideal M, $R_{\langle M \rangle}$ satisfies ACC on semiregular M-reducible ideals, then R satisfies ACC on semiregular w-ideals.*

(2) *If for each semiregular maximal w-ideal M, $R_{\langle M \rangle}$ satisfies the restricted DCC on semiregular v-ideals, then R satisfies the restricted DCC on semiregular v-ideals.*

Proof. (1) Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a chain of semiregular w-ideals of R and let $M_1, M_2, \ldots, M_t$ be the semiregular maximal w-ideals that contain I_1. For each M_i, $(I_1)_{\langle M_i \rangle} \subseteq (I_2)_{\langle M_i \rangle} \subseteq \cdots \subseteq (I_n)_{\langle M_i \rangle} \subseteq \cdots$ is a chain of semiregular M_i-reducible ideals of $R_{\langle M_i \rangle}$. Thus there is an integer n_i such that $(I_{n_i})_{\langle M_i \rangle} = (I_k)_{\langle M_i \rangle}$ for each $k \geq n_i$. Let $n = \text{Max}\{n_1, n_2, \ldots, n_t\}$. Then for each M_i and $k \geq n$, $(I_n)_{\langle M_i \rangle} = (I_k)_{\langle M_i \rangle}$. For all other semiregular maximal w-ideals M, we have $(I_n)_{\langle M \rangle} = R_{\langle M \rangle} = (I_k)_{\langle M \rangle}$, so $(I_n)_{\langle M \rangle} = (I_k)_{\langle M \rangle}$ for each semiregular maximal w-ideal M. Hence $I_n = I_k$ by Lemma 2.6.

(2) Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a chain of semiregular v-ideals of R with $\bigcap I_n$ semiregular and let $I = \bigcap I_n$. Then I is contained in only finitely many semiregular maximal w-ideals. As above let $M_1, M_2, \ldots, M_t$ denote these ideals. For each M_i, $((I_1)_{\langle M_i \rangle})_v \supseteq ((I_2)_{\langle M_i \rangle})_v \supseteq \cdots \supseteq ((I_n)_{\langle M_i \rangle})_v \supseteq \cdots$ is a chain of semiregular v-ideals of $R_{\langle M_i \rangle}$ with $\bigcap((I_n)_{\langle M_i \rangle})_v$ semiregular. In each ring $R_{\langle M_i \rangle}$ there is an integer m_i such that $((I_{m_i})_{\langle M_i \rangle})_v = ((I_k)_{\langle M_i \rangle})_v$ for each $k \geq m_i$. Let $m = \text{Max}\{m_1, m_2, \ldots, m_t\}$. Then $((I_m)_{\langle M_i \rangle})_v = ((I_k)_{\langle M_i \rangle})_v$ for each $k \geq m$. For all other semiregular maximal w-ideals N we have $((I_m)_{\langle N \rangle})_v = R_{\langle N \rangle} = ((I_k)_{\langle N \rangle})_v$. Since semiregular maximal w-ideals are equal to semiregular maximal t-ideals, $I_k = I_m$ by Lemma 2.5(1). $\square$

3. Q_0-SM rings

In this section, we need the concept of finite-type R-modules in [7]. Now, we introduce this concept. Let B and N be R-modules and let $f \in \text{Hom}_R(B, N)$. If for each maximal w-ideal M of R, $f_M : B_M \to N_M$ is a monomorphism (resp., an epimorphism, an isomorphism), we call f a *w-monomorphism* (resp., a *w-epimorphism*, a *w-isomorphism*). A module B is said to be of *finite type* if there exist a finitely generated free module F and a w-epimorphism $f : F \to B$ [7, Definition 1.3]. If B is a GV-torsion-free module, then B is of finite type if and only if there is a finitely generated submodule A of B such that $B_w = A_w$ [7, Proposition 1.2] and a fractional ideal I is said to be a *t-finite-type fractional ideal* (resp., *v-finite-type fractional deal*) if there is a finitely generated fractional ideal J contained in I such that $I_t = J_t$ (resp., $I_v = J_v$).

The following is an SM domain analogue whose proof is similar to that of the domain case.

Lemma 3.1. *The following are equivalent for R.*

(1) *R satisfies ACC on semiregular w-ideals.*

(2) *Each semiregular w-ideal of R is of finite type.*

(3) *Each nonempty collection of semiregular w-ideals of R has maximal elements.*

(4) *Each semiregular prime w-ideal of R is of finite type.*

(5) *Each semiregular ideal of R is of finite type.*

Let I be a w-ideal of R. If I is semiregular, then I is said to be a semiregular w-ideal. In the following we define Q_0-SM rings similarly to Q_0-Mori rings.

Definition 3.2. A Q_0-SM ring is defined to be a ring which satisfies ACC on semiregular w-ideals and satisfies the restricted DCC on semiregular v-ideals.

T. Lucas has defined Q_0-Mori rings in [5]: A Q_0-Mori ring is defined to be a ring which satisfies ACC on semiregular v-ideals and satisfies the restricted DCC on semiregular v-ideals. Since v-ideals are w-ideals, a Q_0-SM ring is a Q_0-Mori ring. Each semiregular fractional ideal of a Q_0-Mori ring is a v-finite-type fractional ideal ([5, Theorem 2.5 and Theorem 2.7]), so is each semiregular fractional ideal of a Q_0-SM ring. Here we also have that each semiregular ideal of a Q_0-SM ring is contained in at most finitely many semiregular maximal w-ideals. In order to get this result, we need to define and discuss Q_0-H-rings and Q_0-TV rings. A ring R is said to be a Q_0-*H-ring* if I is a semiregular ideal of R and $I^{-1} = R$, then there is some $J \in \mathrm{GV}(R)$ such that $J \subseteq I$ and a ring R is said to be a Q_0-*TV ring* if each semiregular t-fractional ideal is a semiregular v-fractional ideal.

The following is an H-domain analogue whose proof is similar to that of the domain case.

Lemma 3.3. *The following assertions are equivalent for R.*
  (1) *R is a Q_0-H-ring.*
  (2) *Each semiregular maximal w-ideal of R is a v-ideal.*
  (3) *Let I be a semiregular ideal of R. If $I^{-1} = R$, then $I_w = R$.*

Since semiregular maximal t-ideals are maximal w-ideals, we have that a Q_0-TV ring is a Q_0-H-ring by Lemma 3.3.

Lemma 3.4. *Let I be a semiregular v-ideal of R and let $\{B_i\}$ be a nonempty collection of v-ideals which contain I. Then $(\sum_i B_i^{-1})_v \subseteq I^{-1}$.*

Proof. Since $I \subseteq \bigcap_i B_i = \bigcap_i (B_i)_v = (\sum_i B_i^{-1})^{-1}$, we have $(\sum_i B_i^{-1})_v \subseteq I^{-1}$. □

Lemma 3.5. *Let R be a Q_0-TV ring, I be a semiregular t-ideal of R, and M be a maximal w-ideal containing I. If $\{B_i\}$ is the nonempty collection of semiregular t-ideals of R which contain I but are not contained in M, then $\bigcap_i B_i \nsubseteq M$.*

Proof. For $B \in \{B_i\}$, we have $B \nsubseteq M$. Since M is a maximal v-ideal, we have $(M + B)_v = R$. Hence $M^{-1} \cap B^{-1} = R$. Let $x \in M^{-1} \setminus R$, where $x \notin B^{-1}$ for every $B \in \{B_i\}$. Then we have $(\sum_i B_i^{-1})_v \subseteq I^{-1}$ by Lemma 3.4, so there is a finitely generated semiregular ideal I_0 contained in I such that $I_0(\sum_i B_i^{-1})_v \subseteq R$. Hence $I_0 \sum_i B_i^{-1} \subseteq R$. Therefore $\sum_i B_i^{-1}$ is a fractional ideal of R. Note that $(\sum_i B_i^{-1})_v = (\sum_i B_i^{-1})_t$ since R is a Q_0-TV ring and that $x \notin (\sum_i B_i^{-1})_v = (\sum_i B_i^{-1})_t$. If not, there is a finitely generated submodule F contained in $\sum_i B_i^{-1}$ such that $x \in F_v$, so there are finitely many elements

$B_1, \ldots, B_n$ of $\{B_i\}$ such that $x \in (\sum_{j=1}^n B_j^{-1})_v$, that is, $x \in (\bigcap_{j=1}^n B_j)^{-1}$. Since $\bigcap_{j=1}^n B_j$ is also an element of $\{B_i\}$, we can get a contradiction. We have $x \notin (\sum_i B_i^{-1})_v = (\bigcap_i B_i)^{-1}$ and $M^{-1} \not\subseteq (\bigcap_i B_i)^{-1}$. Hence $\bigcap_i B_i \not\subseteq M$.   □

Lemma 3.6. *Let R be a Q_0-TV ring and let I be a semiregular t-ideal of R. Then I is contained in at most finitely many semiregular maximal w-ideals.*

Proof. Let $\{M_i\}$ denote the collection of maximal w-ideals containing I, $T_i = \bigcap_{j \neq i} M_j$, and $T = \sum_i T_i$ for each i. We can get $T_i \not\subseteq M_i$ by Lemma 3.5, so $T \not\subseteq M_i$ for each i. And because $I \subseteq T$, we have $T \not\subseteq M$ for each semiregular maximal w-ideal M. It follows that $T_t = R$, i.e., $1 \in T_t$. Hencce there are finitely many elements $T_1, \ldots, T_n$ of $\{T_i\}$ such that $1 \in (\sum_{i=1}^n T_i)_t = (\sum_{i=1}^n T_i)_v$. Let $M_1, \ldots, M_n$ be the maximal w-ideals corresponding to $T_1, \ldots, T_n$. Then $M_1, \ldots, M_n$ are the all maximal w-ideals that contain I. If not, there is a maximal w-ideal M_j which is not equal to each M_i ($i = 1, 2, \ldots, n$) such that $I \subseteq M_j$. Then $T_i \subseteq M_j, i = 1, \ldots, n$. Hence $\sum_{i=1}^n T_i \subseteq M_j$. Therefore $1 \in (\sum_{i=1}^n T_i)_t \subseteq M_j$, which contradicts that $M_j \neq R$.   □

Lemma 3.7. *Let R be a Q_0-SM ring. Then each semiregular maximal w-ideal of R is a maximal v-ideal and each semiregular ideal of R is contained in at most finitely many semiregular maximal w-ideals.*

Proof. Let I be a semiregular t-fractional ideal of R. Then there is a finitely generated semiregular fractional ideal J contained in I such that $I_v = J_v$. Thus $I = I_t \subseteq I_v = J_v = J_t \subseteq I_t = I$, so $I = I_v$ is a v-fractional ideal. It follows that Q_0-SM rings are Q_0-TV rings. Hence each semiregular maximal w-ideal of R is a maximal v-ideal. Because the semiregular maximal w-ideal containing I contains I_t and I_t is contained in at most finitely many semiregular maximal w-ideals by Lemma 3.6, I is contained in at most finitely many semiregular maximal w-ideals.   □

Definition 3.8. For R and S, R is said to be a Q_{0S}-Noetherian ring if R satisfies ACC on semiregular S-reducible ideals and satisfies the restricted DCC on semiregular v-ideals. For a prime ideal P of R and $S = R \setminus P$, we say that R is a Q_{0P}-Noetherian ring if R is a Q_{0S}-Noetherian ring.

Next we use the Q_0-quotient ring $R_{\langle S \rangle}$ with respect to S to describe Q_0-SM rings.

Theorem 3.9. *Let P be a prime w-ideal of R. If R is a Q_0-SM ring, then $R_{\langle P \rangle}$ is a Q_{0P}-Noetherian ring.*

Proof. It is clear that $R_{\langle P \rangle}$ satisfies ACC on semiregular P-reducible ideals by Theorem 2.4.

Next we need to prove that $R_{\langle P \rangle}$ satisfies the restricted DCC on semiregular v-ideals.

Let J be a semiregular v-ideal of $R_{\langle P \rangle}$ and let $A = J \cap R$. We first show that A is a semiregular v-ideal of R. We can prove that A is a semiregular

ideal of R in the same way as in Theorem 2.4. Because each semiregular ideal of a Q_0-SM ring is a v-finite-type ideal, there is a finitely generated semiregular ideal C contained in A such that $C_v = A_v$. We may assume $C = (c_1, c_2, \ldots, c_n)$. Let $u \in (R_{\langle P \rangle} : J)$. Then $uJ \subseteq R_{\langle P \rangle}$. Since $C \subseteq A \subseteq J$, there is some $s \in R \setminus P$ such that $uc_i s \in R$ for each i $(i = 1, 2, \ldots, n)$. Then $us \subseteq (R : C) = (R : C)_v = (R : C_v) = (R : A_v)$. Thus $usA_v \subseteq R$, which implies $uA_v \subseteq R_{\langle P \rangle}$. We have $(R_{\langle P \rangle} : J)A_v \subseteq R_{\langle P \rangle}$. Therefore $A_v \subseteq J \cap R = A$ and $A \subseteq A_v$ is obvious. Hence $A = A_v$, and so is a v-ideal of R.

To complete the proof, we need to show that $J = A_{\langle P \rangle}$. For $x \in A_{\langle P \rangle}$, we have $xs \in A \subseteq J$ for some $s \in R \setminus P$. Then $xs(R_{\langle P \rangle} : J) \subseteq R_{\langle P \rangle}$. Thus $x(R_{\langle P \rangle} : J) \subseteq R_{\langle P \rangle}$, which implies $x \in J_v = J$. So $A_{\langle P \rangle} \subseteq J$. The containment $J \subseteq A_{\langle P \rangle}$ is clear by Lemma 2.2(1). Thus we have $A_{\langle P \rangle} = J$.

Let $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$ be a descending chain of semiregular v-ideals of $R_{\langle P \rangle}$ with $\bigcap J_n$ semiregular and let $A_i = J_i \cap R$ for each i. By the proof above, $J_1 \cap R \supseteq J_2 \cap R \supseteq \cdots \supseteq J_n \cap R \supseteq \cdots$ is a descending chain of semiregular v-ideals of R with $\bigcap(J_n \cap R)$ semiregular. Thus there is an integer n such that $J_k \cap R = J_n \cap R$ for each $k \geq n$, i.e., $A_k = A_n$. We have $J_k = (A_k)_{\langle P \rangle} = (A_n)_{\langle P \rangle} = J_n$. Then we have that $\{J_n\}$ stabilizes. Hence $R_{\langle P \rangle}$ is a Q_{0P}-Neotherian ring. $\qquad \square$

Corollary 3.10. *R is a Q_0-SM ring if and only if each semiregular ideal of R is contained in at most finitely many semiregular maximal w-ideals and $R_{\langle M \rangle}$ is a Q_{0M}-Noetherian ring for each semiregular maximal w-ideal M.*

Proof. The sufficiency follows from Theorem 2.7, while the necessity follows from Lemma 3.7 and Theorem 3.9. $\qquad \square$

4. Examples

We know that an SM domain is a Mori domain, but the converse does not hold. It is also known that a Q_0-SM ring is a Q_0-Mori ring. The first example is provided to show that the converse of this fact does not hold yet. The last two examples are provided to show the difference between Q_0-SM rings and SM rings.

Let D be a domain with the quotient field K, $\mathcal{P}$ be a nonempty set of prime ideals of D where $P_\alpha^{-1} \neq D$ for each $P_\alpha \in \mathcal{P}$, and K_α be the quotient field of D/P_α. Then $B = \bigoplus K_\alpha$ is a D-module. Let $R = D(+)B$ be the idealization of B in R, where for each $r, s \in D$ and $a, b \in B$, $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, as + rb)$ (see [1, Section 25]). Let

$$\mathcal{F} = \{J \,|\, J \text{ is an ideal of } D \text{ and for some finite subset } A \text{ of } J,$$
$$A \nsubseteq P_\alpha \text{ for each } P_\alpha \in \mathcal{P}\},$$

$\mathcal{F}_0$ be the collection of finitely generated ideals in $\mathcal{F}$, and

$$E = \{x \in K \,|\, xA \subseteq D \text{ for some } A \in \mathcal{F}_0\}.$$

Lemma 4.1. *Let the notation be as above. Then the following hold.*

(1) $(r, b) \in Z(D(+)B)$ *if and only if* $r \in P_\alpha$ *for some* $P_\alpha \in \mathcal{P}$.

(2) I *is a semiregular ideal of* R *if and only if* $I = I_0(+)B = I_0R$ *where* $I_0 \in \mathcal{F}$.

(3) *If* $I = I_0(+)B$ *is a semiregular ideal of* R, *then* $\mathrm{Hom}_R(I, R) \cong I_0^{-1}(+)B$.

(4) $Q_0(R)=E(+)B$.

(5) $J \in GV(R)$ *if and only if* $J = J_0(+)B$ *where* $J_0 \in GV(D)$.

(6) *If* I *is a semiregular ideal of* R, *then* $I_W = (I_0)_w(+)B$ *where* $I_0 \in \mathcal{F}$.

(7) *If* D *is an* SM *domain, then* $R = D(+)B$ *is a* Q_0-SM *ring.*

Proof. (1) Let $r \in D$ and $P_\alpha \in \mathcal{P}$. If $r \notin P_\alpha$, then $rK_\alpha = K_\alpha$. If $r \in P_\alpha$, then $rc_\alpha = 0$ for each $c_\alpha \in K_\alpha$. Hence if $r \in P_\alpha$, then $(r, b) \in Z(D(+)B)$ for each $b \in B$. Conversely, since D is a domain, if $(r, b) \in Z(D(+)B)$, then r must be an element of some $P_\alpha \in \mathcal{P}$.

(2), (3) See [3, Theorem 11].

(4) See [3, Theorem 11] and [4, Theorem 3].

(5) Let $J \in GV(R)$. Then J is a finitely generated semiregular ideal of R and $J = J_0(+)B = J_0R$ where $J_0 \in \mathcal{F}$ by (2). Since $J \in GV(R)$, $\mathrm{Hom}_R(J, R) \cong R$. By (3), $\mathrm{Hom}_R(J, R) \cong J_0^{-1}(+)B$. So $J_0^{-1} \cong D$. Since J is finitely generated, J_0 is a finitely generated ideal of D. We have $J_0 \in GV(D)$.

Conversely, if $J_0 \in GV(D)$, then J_0 is a finitely generated ideal of D and $J_0 \in \mathcal{F}_0$. If not, $J_0 \subseteq P_\alpha$ for some $P_\alpha \in \mathcal{P}$. Then $P_\alpha^{-1} \subseteq J_0^{-1} = D$. Hence $P_\alpha^{-1} = D$, which is a contradiction to the choice of P_α, so we have $J = J_0R = J_0(+)B$ is a finitely generated ideal of R and $J = J_0R = J_0(+)B$ is a finitely generated semiregular ideal of R by (2). By (3), $\mathrm{Hom}_R(J, R) \cong J_0^{-1}(+)B = D(+)B = R$. So $J \in GV(R)$.

(6) Let I be a semiregular ideal of R. Then $I = I_0(+)B$ where $I_0 \in \mathcal{F}$. Let $x \in I_W$. Then there is some $J \in GV(R)$ such that $xJ \subseteq I$ where $J = J_0(+)B$ and $J_0 \in GV(D)$. Because $I_W \subseteq (Q_0(R))_W = Q_0(R) = E(+)B \subseteq K(+)B$, x can be written as $x = (e, b)$ where $e \in E, b \in B$. So $eJ_0 \subseteq I_0$. We have $x \in (I_0)_w(+)B$. Conversely, if $x \in (I_0)_w(+)B \subseteq D_w(+)B = D(+)B$, x can be written as $x = (d, b)$ where $d \in D$ and $b \in B$, so there is some $J_0 \in GV(D)$ such that $dJ_0 \subseteq I_0$. By (5), we have $J = J_0(+)B \in GV(R)$. For each $(a_0, b_0) \in J$, $(d, b)(a_0, b_0) = (da_0, ba_0 + db_0) \in I_0(+)B$. Then $(d, b)J \subseteq I_0(+)B$. Hence $x \in (I_0(+)B)_W = I_W$. Therefore $I_W = (I_0)_w(+)B$.

(7) Since D is an SM domain, then D is a Mori domain. We have $D(+)Q_0(B) = D(+)B = R$ is a Q_0-Mori ring by [5, Theorem 4.4], so R satisfies the restricted DCC on semiregular v-ideals. Let I be a semiregular W-ideal of R. Then $I = (I_0)_w(+)B$ by (6). D is an SM domain, then D satisfies ACC on w-ideals, so R satisfies ACC on semiregular W-ideals. Hence we have R is a Q_0-SM ring. $\square$

Example 4.2. Let D be a domain, not a field and $\mathcal{P} = \{0 \,|\, 0$ is the zero ideal of $D\}$. Then $\mathcal{F} = \{J \,|\, J$ is a nonzero ideal of $D\}$. If D is a Mori domain but not an SM domain, we claim that $R = D(+)B$ is a Q_0-Mori ring but not a

Q_0-SM ring. Since D is not an SM domain, there is an infinite ascending chain $\{I_{0n}\}$ of w-ideals of D. By Lemma 4.1(2) and (6), $\{I_{0n}(+)B\}$ is an infinite ascending chain of semiregular w-ideals of R. Thus R does not satisfy ACC on semiregular w-ideals. Now we give a concrete example of a Q_0-SM ring. Let D denote the ring of integers $\mathbb{Z}$ and $\mathcal{P}=\mathrm{Spec}(D) \setminus \{2D\}$. Then $R = D(+)B$ is a Q_0-SM ring.

The next example is provided to show that a Q_0-SM ring is an SM ring, but the converse does not hold.

Let K be a field and $\mathcal{X} = \{X_n\}_{n=1}^{\infty} = \{X_1, X_2, \ldots, X_n, \ldots\}$, where each X_i is an indeterminate over K. Let $\mathcal{F}$ denote the set of finite products of elements from $\mathcal{X}$. And for each positive integer m, let $\mathcal{X}^{[m]} = \{X_n^m\}_{n=1}^{\infty}$.

Example 4.3 ([5, Example 3.5])**.** Let $D = K[Y, Z, \mathcal{X}^{[2]}, \mathcal{X}^{[3]}, Y\mathcal{F}, Z\mathcal{F}]$ and $\mathcal{P}$ denote the set of primes of D which do not contain both Y and Z. Let R be the $A + B$ ring corresponding to D and $\mathcal{P}$(see[1, Section 26]). Then $R = T(R)$, so R is the only regular ideal of R. Hence each regular ideal of R is of finite type. Then R is an SM ring. Since R does not satisfy the restricted DCC on semiregular v-ideals, we have that R is not a Q_0-SM ring.

The last example is provided to show that SM rings which also satisfy the restricted DCC on semiregular v-ideals may not be Q_0-SM rings, i.e., ACC on regular w-ideals does not imply ACC on semiregular w-ideals.

Example 4.4 ([5, Example 3.6])**.** Let $\mathcal{Q} = \{q_i = \frac{p_i+1}{2p_i} \,|\, p_1 < p_2 < \cdots,$ each p_i is an odd prime$\}$ and $\mathcal{T} = \{t \in \mathbb{Q} \,|\, 0 \leq t = m + a_1 q_1 + \cdots + a_n q_n$ with m and each a_i nonnegative integers$\}$ and $\{W_i\}$ be a set of indeterminates indexed over the positive integers. For each integer $k \geq 0$, let $\mathcal{W}_k = \{\prod_{i>k}^{n} W_i^{r_i} \,|\, 0 \leq r_i \in \mathbb{Z}, n > k\}$. We assume that $1 \in \mathcal{W}_k$ for each k. And let $\mathcal{U} = \{UY^r Z^s \,|\, U \in \mathcal{W}_0, r, s \in \mathcal{T}$ with $r + s \geq 1\}$ and $\mathcal{V} = \{VY^{q_i}, VZ^{q_i} \,|\, V \in \mathcal{W}_i, i \geq 1\}$. Let $D = K[Y, Z, \mathcal{U}, \mathcal{V}]$ where K is a field and $\mathcal{P}$ denote the set of primes of D which do not contain both Y and Z. Let R be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then $R = T(R)$, so R is an SM ring and R satisfies the restricted DCC on semiregular v-ideals. Let $Q = (Y, Z, \mathcal{U})D, Q_1 = Q + (Y^{q_1}\mathcal{W}_1)D + (Z^{q_1}\mathcal{W}_1)D, \ldots, Q_i = Q_{i-1} + (Y^{q_i}\mathcal{W}_i)D + (Z^{q_i}\mathcal{W}_i)D$. Then the chain $QR \subseteq Q_1 R \subseteq Q_2 R \subseteq \cdots$ is a proper infinite ascending chain of semiregular v-ideals. Hence this chain is a proper infinite ascending chain of semiregular w-ideals. Therefore R is not a Q_0-SM ring.

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