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SKEW COMPLEX SYMMETRIC OPERATORS AND WEYL TYPE THEOREMS

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Abstract. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be skew complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = -CT^*C$. In this paper, we study properties of skew complex symmetric operators including spectral connections, Fredholmness, and subspace-hypercyclicity between skew complex symmetric operators and their adjoints. Moreover, we consider Weyl type theorems and Browder type theorems for skew complex symmetric operators.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\rho(T)$, $\sigma(T)$, $\sigma_{su}(T)$, $\sigma_{comp}(T)$, $\sigma_r(T)$, $\sigma_c(T)$, $\sigma_a(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ for the resolvent set, for the spectrum, the surjective spectrum, the compression spectrum, the residual spectrum, the continuous spectrum, the approximate point spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum of T, respectively.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An *antiunitary* operator is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$ and *skew complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$. Many standard operators such as normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and some Volterra integration operators are

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included in the class of complex symmetric operators. Several authors have studied the structure of complex symmetric operators (see [12]-[14], [20], and [18] for more details). On the other hand, less attention has been paid to skew complex symmetric operators. There are several motivations for such operators. In particular, skew symmetric matrices have many applications in pure mathematics, applied mathematics and even in engineering disciplines. Real skew symmetric matrices are very important in applications, including function theory, the solution of linear quadratic optimal control problems, robust control problems, model reduction, crack following in anisotropic materials, and others. In view of these applications, it is natural to study skew symmetric operators on the Hilbert space $\mathcal{H}$ (see [22], [25], and [27] for more details).

In this paper, we study properties of skew complex symmetric operators including spectral connections, Fredholmness, and subspace-hypercyclicity between skew complex symmetric operators and their adjoints. Moreover, we consider Weyl type theorems and Browder type theorems for skew complex symmetric operators.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if T has closed range and $\dim \ker(T) < \infty$, and $T \in \mathcal{L}(\mathcal{H})$ is called *lower semi-Fredholm* if T has closed range and $\dim(\mathcal{H}/\operatorname{ran}(T)) < \infty$. When T is upper semi-Fredholm or lower semi-Fredholm, T is said to be *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$, denoted $\operatorname{ind}(T)$, is given by

$$\operatorname{ind}(T) = \dim \ker(T) - \dim(\mathcal{H}/\operatorname{ran}(T))$$

and this value is an integer or $\pm\infty$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Weyl* if it is Fredholm of index zero. If there is a nonnegative integer m such that $\ker(T^m) = \ker(T^{m+1})$, then T is said to have *finite ascent*. If there is a nonnegative integer n satisfying $\operatorname{ran}(T^n) = \operatorname{ran}(T^{n+1})$, then T is said to have *finite descent*. We say that $T \in \mathcal{L}(\mathcal{H})$ is *Browder* if it has finite ascent and finite descent. We define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

It is evident that $\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T)$. We say that *Weyl's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \dim \ker(T - \lambda) < \infty\}$, $\pi_{0f}(T)$ is the set of the eigenvalue of finite multiplicity, and $iso\Delta$ denotes the set of all isolated points of Δ. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if for any

$\lambda \in iso\ \sigma(T)$, $\lambda \in \mathbb{C}$ is an eigenvalue of T. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$, or equivalently, $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$ where $p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$. It is known that

$$\text{Weyl's theorem} \Longrightarrow \text{Browder's theorem.}$$

We refer the reader to [1], [10], [17], and [20] for more details.

We recall the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi_{00}(T) := \{\lambda \in iso\ \sigma(T) : 0 < \dim\ker(T - \lambda) < \infty\}$$

and

$$\pi^a_{00}(T) := \{\lambda \in iso\ \sigma_a(T) : 0 < \dim\ker(T - \lambda) < \infty\}.$$

Let $T \in \mathcal{L}(\mathcal{H})$. We say that

(i) *a-Browder's theorem holds for* T if $\sigma_{ea}(T) = \sigma_{ab}(T)$ or equivalently, $\sigma_a(T) \setminus \sigma_{ea}(T) = p^a_{00}(T)$ where $p^a_{00}(T) = \sigma_a(T) \setminus \sigma_{ab}(T)$;
(ii) *a-Weyl's theorem holds for* T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi^a_{00}(T)$;
(iii) T has *the property* (w) if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$.

It is known that

$$\begin{array}{ccc} \text{Property } (w) & \Longrightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.} \end{array}$$

We refer the reader to [1], [10], [17] for more details.

Let $T_n = T|_{\mathrm{ran}(T^n)}$ for each nonnegative integer n; in particular, $T_0 = T$. If T_n is upper semi-Fredholm for some nonnegative integer n, then T is called a *upper semi-B-Fredholm* operator. In this case, by [6], T_m is a upper semi-Fredholm operator and $\mathrm{ind}(T_m) = \mathrm{ind}(T_n)$ for each $m \geq n$. Thus, one can consider the *index* of T, denoted by $\mathrm{ind}_B(T)$, as the index of the semi-Fredholm operator T_n. Similarly, we define lower semi-B-Fredholm operators. We say that $T \in \mathcal{L}(\mathcal{H})$ is *B-Fredholm* if it is both upper and lower semi-B-Fredholm. In [6], Berkani proved that $T \in \mathcal{L}(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. Let $\mathcal{A}$ be a unital algebra. Recall that $x \in \mathcal{A}$ is Drazin invertible of degree k if there exists an elements $a \in \mathcal{A}$ such that $x^k ax = x^k, aca = a$, and $xa = ax$.

Let $SBF^-_+(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators such that $\mathrm{ind}_B(T) \leq 0$, and let

$$\sigma_{SBF^-_+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF^-_+(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}.$$

We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if it has finite ascent, i.e., $a(T) < \infty$ and $\operatorname{ran}(T^{a(T)+1})$ is closed where $a(T) = \dim \ker(T)$. The notation $p_0(T)$ (respectively, $p_0^a(T)$) denotes the set of all poles (respectively, left poles) of T, while $\pi_0(T)$ (respectively, $\pi_0^a(T)$) is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$ (respectively, $\sigma_a(T)$).

Let $T \in \mathcal{L}(\mathcal{H})$. We say that

(i) T satisfies *generalized Browder's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus p_0(T)$;
(ii) T satisfies *generalized a-Browder's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus p_0^a(T)$;
(iii) T satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$;
(iv) T satisfies *generalized a-Weyl's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \pi_0^a(T)$.

It is known that

$$\begin{array}{ccc} \text{generalized } a\text{-Weyl's theorem} & \Longrightarrow & \text{generalized Weyl's theorem} \\ \Downarrow & & \Downarrow \\ \text{generalized } a\text{-Browder's theorem} & \Longrightarrow & \text{generalized Browder's theorem.} \end{array}$$

We refer the reader to [5]-[8], and [21] for more details.

3. Main results

In this section, we study some properties of skew complex symmetric operators. For a fixed conjugation C on $\mathcal{H}$, set $S_C(\mathcal{H}) = \{X \in \mathcal{L}(\mathcal{H}) \mid CXC = -X^*\}$ and we call $S_C(\mathcal{H})$ the set of skew complex symmetric operators. For a simple example, if $A \in \mathcal{L}(\mathcal{H})$ is complex symmetric, then it is easy to show that $A \oplus (-A)$ is skew complex symmetric on $\mathcal{H} \oplus \mathcal{H}$. We also know that if $A \in \mathcal{L}(\mathcal{H})$ is skew complex symmetric, then $-A \oplus (-A)$ is skew complex symmetric on $\mathcal{H} \oplus \mathcal{H}$. We next explain how to get some skew complex symmetric operators from given any complex symmetric operators.

Example 3.1. *If $R \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with a conjugation C, then an easy computation shows that*

$$T_1 = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}, \ T_2 = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix}, \ \textit{and } T_3 = \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix}$$

are skew complex symmetric operators on $\mathcal{H} \oplus \mathcal{H}$.

We first provide some relations among spectra of skew complex symmetric operators and their adjoint. Since the proof of the following lemma is similar to one in [20], we state it without its proof.

Lemma 3.2. *If $T \in S_C(\mathcal{H})$, then the following relations are valid:*

(i) $\sigma_p(T)^* = -\sigma_p(T^*)$, $\sigma_a(T)^* = -\sigma_a(T^*)$, $\sigma_{su}(T)^* = -\sigma_{su}(T^*)$, $\sigma_{comp}(T)^* = -\sigma_{comp}(T^*)$, $\sigma_r(T)^* = -\sigma_r(T^*)$, *and* $\sigma_c(T)^* = -\sigma_c(T^*)$.

(ii) $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$, $\sigma_{re}(T)^* = -\sigma_{re}(T^*)$, *and* $\sigma_e(T)^* = -\sigma_e(T^*)$.

(iii) $\sigma_p(T) = -\sigma_{comp}(T)$ *and* $\sigma_p(T^*) = -\sigma_{comp}(T^*)$.

Example 3.3. Assume that T and S are in $\mathcal{L}(\mathbb{C}^3)$ which admit the following representations:

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & -1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

Then T is a skew complex symmetric operator with respect to the conjugation $C(\alpha_1, \alpha_2, \alpha_3) = (-\overline{\alpha_3}, \overline{\alpha_2}, -\overline{\alpha_1})$ and a simple calculation shows that $\sigma(T)^* = \sigma_p(T)^* = \{-1, 0, 1\} = -\sigma_p(T^*) = -\sigma(T^*)$. However, we know that $\sigma_p(S)^* \neq -\sigma_p(S^*)$. Therefore, S is not a skew complex symmetric operator by Lemma 3.2, while S is a complex symmetric with the conjugation $C(\alpha_1, \alpha_2, \alpha_3) = (\overline{\alpha_3}, \overline{\alpha_2}, \overline{\alpha_1})$.

We next give more examples for skew complex symmetric operators on a 3-dimensional space.

Example 3.4. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{C}^3$. If C is a conjugation on $\mathbb{C}^3$ defined by the one of the followings;

$$\begin{cases} \text{(i) } Cx = \pm\overline{\alpha_1}e_1 \pm \overline{\alpha_2}e_2 \pm \overline{\alpha_3}e_3 \\ \text{(ii) } Cx = \pm\overline{\alpha_1}e_1 \mp \overline{\alpha_2}e_2 \pm \overline{\alpha_3}e_3 \\ \text{(iii) } Cx = \pm\overline{\alpha_1}e_1 \pm \overline{\alpha_3}e_2 \pm \overline{\alpha_2}e_3 \\ \text{(iv) } Cx = \pm\overline{\alpha_2}e_1 \pm \overline{\alpha_1}e_2 \pm \overline{\alpha_3}e_3 \\ \text{(v) } Cx = \pm\overline{\alpha_3}e_1 \pm \overline{\alpha_2}e_2 \pm \overline{\alpha_1}e_3 \end{cases}$$

where $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, it is easy to show that the following matrices are skew complex symmetric matrices;

(1) $\begin{pmatrix} 0 & a & b \\ \mp a & 0 & c \\ \mp b & -c & 0 \end{pmatrix}$ with respect to $Cx = \pm\overline{\alpha_1}e_1 \pm \overline{\alpha_2}e_2 \pm \overline{\alpha_3}e_3$, respectively.

(2) $\begin{pmatrix} 0 & a & b \\ \pm a & 0 & c \\ \mp b & c & 0 \end{pmatrix}$ with respect to $Cx = \pm\overline{\alpha_1}e_1 \mp \overline{\alpha_2}e_2 \pm \overline{\alpha_3}e_3$, respectively.

(3) $\begin{pmatrix} 0 & a & b \\ \mp b & c & 0 \\ \mp a & 0 & -c \end{pmatrix}$ with respect to $Cx = \pm\overline{\alpha_1}e_1 \pm \overline{\alpha_3}e_2 \pm \overline{\alpha_2}e_3$, respectively.

(4) $\begin{pmatrix} a & 0 & b \\ 0 & -a & c \\ \mp c & \mp b & 0 \end{pmatrix}$ with respect to $Cx = \pm\overline{\alpha_2}e_1 \pm \overline{\alpha_1}e_2 \pm \overline{\alpha_3}e_3$, respectively.

(5) $\begin{pmatrix} a & b & 0 \\ c & 0 & \mp b \\ 0 & \mp c & -a \end{pmatrix}$ with respect to $Cx = \pm\overline{\alpha_3}e_1 \pm \overline{\alpha_2}e_2 \pm \overline{\alpha_1}e_3$, respectively.

As some applications of Lemma 3.2, we get the following proposition.

Proposition 3.5. *If $T \in S_C(\mathcal{H})$, then the following relations hold*:

(i) $\sigma(T) = \sigma_a(T) \cup [-\sigma_a(T)]$ *and* $\sigma(T^*) = \sigma_{su}(T^*) \cup [-\sigma_{su}(T^*)]$.

(ii) $\sigma_e(T) = [-\sigma_{re}(T)] \cup \sigma_{re}(T) = \sigma_{le}(T) \cup [-\sigma_{le}(T)]$ *and* $\sigma_e(T^*) = [-\sigma_{re}(T^*)] \cup \sigma_{re}(T^*) = \sigma_{le}(T^*) \cup [-\sigma_{le}(T^*)]$.

Proof. (i) For any $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^*$ by [15, Corollary, page 222]. Since $\sigma_a(T) = -\sigma_a(T^*)^*$ by Lemma 3.2, we obtain

$$\sigma(T) = \sigma_a(T) \cup [-\sigma_a(T)].$$

On the other hand, since $\sigma_{su}(T^*) = \sigma_a(T)^*$ for any $T \in \mathcal{L}(\mathcal{H})$ and $T \in S_C(\mathcal{H})$, it follows from Lemma 3.2 that

$$-\sigma(T^*)^* = \sigma_a(T) \cup [-\sigma_a(T)] = \sigma_{su}(T^*)^* \cup [-\sigma_{su}(T^*)^*].$$

Hence we get $\sigma(T^*) = \sigma_{su}(T^*) \cup [-\sigma_{su}(T^*)]$.

(ii) Since $\sigma_{le}(T)^* = \sigma_{re}(T^*)$ for any $T \in \mathcal{L}(\mathcal{H})$, $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$, and $\sigma_{re}(T)^* = -\sigma_{re}(T^*)$ by Lemma 3.2, we obtain

$$\sigma_{le}(T) = -\sigma_{re}(T) \text{ and } \sigma_{le}(T^*) = -\sigma_{re}(T^*).$$

Hence we conclude that

$$\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T) = [-\sigma_{re}(T)] \cup \sigma_{re}(T) = \sigma_{le}(T) \cup [-\sigma_{le}(T)] \text{ and}$$
$$\sigma_e(T^*) = \sigma_{le}(T^*) \cup \sigma_{re}(T^*) = [-\sigma_{re}(T^*)] \cup \sigma_{re}(T^*) = \sigma_{le}(T^*) \cup [-\sigma_{le}(T^*)].$$

□

Corollary 3.6. *Let $T \in S_C(\mathcal{H})$. The following statements are equivalent*:

(i) $T - \lambda$ *is invertible.*

(ii) $T \pm \lambda$ *are bounded below.*

(iii) $T \pm \lambda$ *are one-to-one and have closed range.*

Proof. (i) $\Longrightarrow$ (ii) and (iii): If $T - \lambda$ is invertible, then $\lambda \notin \sigma_a(T) \cup [-\sigma_a(T)]$ from Proposition 3.5. Hence $\lambda \notin \sigma_a(T)$ and $-\lambda \notin \sigma_a(T)$. Therefore [9] implies that $T - \lambda$ and $T + \lambda$ are bounded below. Equivalently, $T \pm \lambda$ are one-to-one and have closed range.

(ii) $\Longleftrightarrow$ (iii): It is trivial from [9].

(ii) $\Longrightarrow$ (i): If $T \pm \lambda$ are bounded below, then $\pm\lambda \notin \sigma_a(T)$ and so $\lambda \notin \sigma_a(T) \cup [-\sigma_a(T)]$. Therefore, from Proposition 3.5, we have $\lambda \notin \sigma(T)$. □

The following corollary immediately follows from Proposition 3.5 and [9].

Corollary 3.7. *If $T \in S_C(\mathcal{H})$, the following arguments are equivalent*:

(i) $\lambda \notin \sigma_e(T)$.

(ii) $\dim \ker(T \pm \lambda) < \infty$ *and* $\operatorname{ran}(T \pm \lambda)$ *are closed.*

(iii) $\dim[\operatorname{ran}(T \pm \lambda)]^{\perp} < \infty$ *and* $\operatorname{ran}(T \pm \lambda)$ *are closed.*

Let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. Recall that T is *subspace-hypercyclic for* $\mathcal{M}$ if there exists $x \in \mathcal{H}$ such that $\mathcal{O}(T,x) \cap \mathcal{M}$ is dense in $\mathcal{M}$, where $\mathcal{O}(T,x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$. We call x a *subspace-hypercyclic* vector.

Proposition 3.8. *Let $T \in S_C(\mathcal{H})$. Then the following arguments hold.*

(i) *T is subspace-hypercyclic for $C\mathcal{M}$ if and only if $-T^*$ is subspace-hypercyclic for $\mathcal{M}$.*

(ii) *If T is subspace-hypercyclic for $\mathcal{M}$, then* $\ker(T + \overline{\lambda}) \subset C\mathcal{M}^{\perp}$ *for all* $\lambda \in \mathbb{C}$.

Proof. (i) Let C be a conjugation on $\mathcal{H}$. If T is subspace-hypercyclic for $C\mathcal{M}$, then there exists $x \in \mathcal{H}$ such that $\mathcal{O}(T,x) \cap C\mathcal{M}$ is dense in $C\mathcal{M}$. Hence we get that $C\overline{\mathcal{O}(T,x) \cap C\mathcal{M}} = C^2\mathcal{M} = \mathcal{M}$. Moreover, since $T \in S_C(\mathcal{H})$, it follows that $CT^n = (-T^*)^n C$ for all nonnegative integer n. Thus we obtain that

$$\mathcal{M} = \overline{\mathcal{O}(-T^*, Cx) \cap \mathcal{M}}.$$

Therefore, $-T^*$ is subspace-hypercyclic for $\mathcal{M}$. The converse implication follows from the same method as the above.

(ii) Suppose that T is subspace-hypercyclic for $\mathcal{M}$. Fix any $\lambda \in \mathbb{C}$. If $y \in \ker(T + \overline{\lambda})$, then $C(T^* - \lambda)Cy = 0$ and so $T^*Cy = \lambda Cy$. Let $f : \mathcal{M} \to \mathbb{C}$ be the functional defined by $f(x) = \overline{\langle x, y\rangle}$. If f is onto and $y \in \mathcal{M}^{\perp}$, then $f(x) = \overline{\langle x, y\rangle} = 0$ for all $x \in \mathcal{M}$. Thus $f \equiv 0$ on $\mathcal{M}$. Since $\mathcal{M}$ is a nonzero subspace of $\mathcal{H}$, this is a contradiction. Hence $y \notin \mathcal{M}^{\perp}$. Assume that $y \notin \mathcal{M}^{\perp}$. For all $z \in \mathbb{C}$, there exists $x = \overline{z}\frac{y}{\|y\|^2} \in \mathcal{M}$ such that $f(x) = \overline{\langle x, y\rangle} = z$. Thus f is onto. Therefore, we get that f is onto if and only if $y \notin \mathcal{M}^{\perp}$. Since $T \in S_C(\mathcal{H})$, it ensures that

$$\begin{aligned} \langle T^n Cx, Cy\rangle &= \langle Cx, T^{*n} Cy\rangle = \langle Cx, \lambda^n Cy\rangle \\ &= \overline{\lambda}^n \langle Cx, Cy\rangle = \overline{\lambda}^n \langle y, x\rangle = \overline{\lambda}^n \overline{\langle x, y\rangle}. \end{aligned}$$

Hence we obtain that

$$f(\mathcal{O}(T, Cx) \cap \mathcal{M}) = \{\overline{\lambda^n \langle x, y\rangle} \ : \text{there exists } n \text{ such that } T^n Cx \in \mathcal{M}\}. \tag{1}$$

On the other hand, if $\overline{\mathcal{O}(T, Cx) \cap \mathcal{M}} = \mathcal{M}$, then $f(\mathcal{O}(T, Cx) \cap \mathcal{M})$ should be dense in $\mathbb{C}$, otherwise f is not onto. Since the above set of (1) is obviously not dense in $\mathbb{C}$, it follows that f is not onto. So, $Cy \in \mathcal{M}^{\perp}$ and so $y \in C\mathcal{M}^{\perp}$. Hence we complete the proof. $\square$

Corollary 3.9. *If $\mathcal{H}$ is finite-dimensional and $T \in S_C(\mathcal{H})$, then T and T^* are not subspace-hypercyclic for any $\mathcal{M}$.*

Proof. The proof follows from [26, Theorem 4.9] and Proposition 3.8. □

We next obtain the following spectra relations from Lemma 3.2 and Proposition 3.5.

Lemma 3.10.

(i) $\pi_0(T)^* = -\pi_0(T^*)$, $\pi_{00}(T)^* = -\pi_{00}(T^*)$, *and* $\pi_{00}^a(T)^* = -\pi_{00}^a(T^*)$.

(ii) $p_{00}(T)^* = -p_{00}(T^*)$ *and* $p_{00}^a(T)^* = -p_{00}^a(T^*)$.

(iii) $\sigma_{ab}(T)^* = -\sigma_{ab}(T^*)$ *and* $\sigma_{ea}(T)^* = -\sigma_{ea}(T^*)$.

We say that $T \in \mathcal{L}(\mathcal{H})$ satisfies the property (b) if $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}(T)$, the property (ab) if $\sigma(T) \setminus \sigma_w(T) = p_{00}^a(T)$, and the property (aw) if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^a(T)$. In the following theorem, if $T \in \mathcal{L}(\mathcal{H})$ is a skew complex symmetric operator, then we provide an equivalence statement about a Fredholm, Drazin invertible, Weyl type theorems, and Browder type theorems, respectively.

Theorem 3.11. *If $T \in S_C(\mathcal{H})$, then the following statements hold*:

(i) *$T - \lambda$ is Fredholm if and only if $T^* + \overline{\lambda}$ is for all $\lambda \in \mathbb{C}$.*

(ii) *$T - \lambda$ is Drazin invertible if and only if $T^* + \overline{\lambda}$ is for all $\lambda \in \mathbb{C}$.*

(iii) *T satisfies Browder's theorem (respectively, a-Browder's theorem) if and only if T^* does.*

(iv) *If $T - \lambda$ has finite descent for all $\lambda \in \mathbb{C}$, then both T and T^* satisfy Browder's theorem.*

(v) *T satisfies Weyl's theorem (respectively, a-Weyl's theorem) if and only if T^* does.*

(vi) *T satisfies property (δ) if and only if T^* does where (δ) denotes (w), (aw), (b), and (ab).*

Proof. (i) Assume that $T - \lambda$ is Fredholm. Then $T - \lambda$ has closed range, $\ker(T - \lambda)$ and $\ker(T - \lambda)^*$ are finite dimensional. Since $T \in S_C(\mathcal{H})$, we know that $T^* + \overline{\lambda}$ has closed range. Since $\dim \ker(T - \lambda) = \dim C \ker(T^* + \overline{\lambda})$ and $\dim \ker(T - \lambda)^* = \dim C \ker(T + \lambda)$, it suffices to show that

$$\begin{cases} \dim C \ker(T^* + \overline{\lambda}) = \dim \ker(T^* + \overline{\lambda}) \\ \dim C \ker(T + \lambda) = \dim \ker(T + \lambda). \end{cases}$$

If $\dim \ker(T^* + \overline{\lambda}) = n$, there exists a basis $\{e_1, \ldots, e_n\} \in \ker(T^* + \overline{\lambda})$ and so $\{Ce_1, \ldots, Ce_n\} \in C \ker(T^* + \overline{\lambda})$. If $\sum_{i=1}^n \alpha_i Ce_i = 0$ for all complex numbers $\alpha_1, \ldots, \alpha_n$, then we have $C(\sum_{i=1}^n \alpha_i Ce_i) = \sum_{i=1}^n \overline{\alpha_i} e_i = 0$. Since the linear independence property of $\{e_1, \ldots, e_n\}$ in $\ker(T^* + \overline{\lambda})$, it follows that $\alpha_i = 0$ for all $i = 1, 2, \ldots, n$. On the other hand, for any $x \in C \ker(T^* + \overline{\lambda})$, we write $x = Cy$ where $y \in \ker(T^* + \overline{\lambda})$. By spanning property of $\{e_1, \ldots, e_n\}$ in $\ker(T^* + \overline{\lambda})$, $y = \sum_{i=1}^n \beta_i e_i$ where $\beta_1, \ldots, \beta_n$ are complex numbers. Thus we have

$$x = C(\sum_{i=1}^n \beta_i e_i) = \sum_{i=1}^n \overline{\beta_i} Ce_i$$

where $\overline{\beta_1}, \ldots, \overline{\beta_n}$ are complex numbers. This means that $\{Ce_1, \ldots, Ce_n\}$ is a basis of $C \ker(T^* + \overline{\lambda})$. Hence we conclude $\dim C \ker(T^* + \overline{\lambda}) = n$. The converse inclusion and $\dim C \ker(T + \lambda) = \dim \ker(T + \lambda)$ hold by using a similar method. Hence $T^* + \overline{\lambda}$ is Fredholm. Similarly, the converse implication holds.

(ii) If $T - \lambda$ is Drazin invertible, it is well-known that $T - \lambda$ has finite ascent and finite descent. Suppose that $\ker(T - \lambda)^n = \ker(T - \lambda)^{n+1}$ for some positive integer n. Since $\ker(T^* + \overline{\lambda})^n \subset \ker(T^* + \overline{\lambda})^{n+1}$ is clear, it suffices to show $\ker(T^* + \overline{\lambda})^{n+1} \subset \ker(T^* + \overline{\lambda})^n$. If $x \in \ker(T^* + \overline{\lambda})^{n+1}$, then $(T^* + \overline{\lambda})^{n+1} x = 0$. Since $T^* = -CTC$, it follows that $(CTC - \overline{\lambda})^{n+1} x = 0$ and hence

$$0 = (CTC - \overline{\lambda})^{n+1} x = [C(T - \lambda)C]^{n+1} x = [C(T - \lambda)^{n+1} C] x.$$

So, we have $(T - \lambda)^{n+1} Cx = 0$. Thus $Cx \in \ker(T - \lambda)^{n+1} = \ker(T - \lambda)^n$ which means $(T - \lambda)^n Cx = 0$. Hence we get that

$$0 = C(T - \lambda)^n Cx = (CTC - \overline{\lambda})^n x = [-(T^* + \overline{\lambda})]^n x.$$

Therefore, $x \in \ker(T^* + \overline{\lambda})^n$. Thus $\ker(T^* + \overline{\lambda})^{n+1} \subset \ker(T^* + \overline{\lambda})^n$. The converse implication holds by using a similar argument. Hence $T^* + \overline{\lambda}$ has finite ascent.

Assume that $\operatorname{ran}(T - \lambda)^n = \operatorname{ran}(T - \lambda)^{n+1}$ for some positive integer n. Since $\operatorname{ran}(T^* + \overline{\lambda})^{n+1} \subset \operatorname{ran}(T^* + \overline{\lambda})^n$ is trivial, we only need to show that $\operatorname{ran}(T^* + \overline{\lambda})^n \subset \operatorname{ran}(T^* + \overline{\lambda})^{n+1}$. If $y \in \operatorname{ran}(T^* + \overline{\lambda})^n$, then

$$\begin{aligned} y &= (T^* + \overline{\lambda})^n x = (-CTC + \overline{\lambda})^n x \\ &= [-C(T - \lambda)C]^n x = C[-(T - \lambda)]^n Cx \end{aligned}$$

for some $x \in \mathcal{H}$. Since $Cy = [-(T - \lambda)]^n Cx \in \operatorname{ran}(T - \lambda)^n = \operatorname{ran}(T - \lambda)^{n+1}$, there exists a vector $z \in \mathcal{H}$ such that $Cy = (T - \lambda)^{n+1} z$. This forces that

$$\begin{aligned} y &= C(T - \lambda)^{n+1} z = C[-C(T^* + \overline{\lambda})C]^{n+1} z \\ &= [-(T^* + \overline{\lambda})]^{n+1} Cz \ \in \ \operatorname{ran}(T^* + \overline{\lambda})^{n+1}. \end{aligned}$$

Hence, $\operatorname{ran}(T^* + \overline{\lambda})^n \subset \operatorname{ran}(T^* + \overline{\lambda})^{n+1}$. The converse implication holds by using a similar way. Hence $T^* + \overline{\lambda}$ has finite descent. Since $T^* + \overline{\lambda}$ has finite ascent and finite descent, it ensures that $T^* + \overline{\lambda}$ is Drazin invertible. The converse implication holds by a similar method. Hence the proof is completed.

(iii) If T satisfies Browder's theorem, then $\sigma_b(T) = \sigma_w(T)$. Note that if $T \in S_C(\mathcal{H})$, then $\sigma_w(T) = -\sigma_w(T^*)^*$. Indeed, If T is Weyl, then T is Fredholm with $\operatorname{ind}(T) = 0$. By the argument (i), T^* is also Fredholm. Since $\dim \ker(T) = \dim \ker(T^*)$, we get $\dim \ker(T^*) = \dim \ker(T)$. Hence $\operatorname{ind}(T^*) = 0$ and so T^* is Weyl. The converse implication holds by using a similar argument. By the similar proof of [20], we have $\sigma_b(T) = -\sigma_b(T^*)^*$. Hence we obtain $\sigma_b(T^*) = \sigma_w(T^*)$ which means T^* satisfies Browder's theorem. Similarly, the converse statement holds. Hence we complete the proof.

Assume that a-Browder's theorem holds for T. Then, by Lemma 3.10, we have $\sigma_{ea}(T^*) = -\sigma_{ea}(T)^* = -\sigma_{ab}(T)^* = \sigma_{ab}(T^*)$. Then a-Browder's theorem holds for T^*. The converse statement holds by a similar method.

(iv) Since $\sigma_w(T) \subset \sigma_b(T)$ holds for every $T \in \mathcal{L}(\mathcal{H})$, it suffices to show that the inclusion $\sigma_b(T) \subset \sigma_w(T)$ holds. If $\lambda \notin \sigma_w(T)$, then $T - \lambda$ is Fredholm of index 0. Since $T - \lambda$ has finite descent, $\operatorname{ran}(T-\lambda)^n = \operatorname{ran}(T-\lambda)^{n+1}$ holds for some $n \in \mathbb{N}$. By taking the orthogonal complement, we have $\ker(T^* - \overline{\lambda})^n = \ker(T^* - \overline{\lambda})^{n+1}$ for some $n \in \mathbb{N}$. Moreover, since $\operatorname{ind}(T-\lambda) = 0$, we get that $0 = n \cdot \operatorname{ind}(T-\lambda) = \operatorname{ind}(T-\lambda)^n = \dim\ker(T-\lambda)^n - \dim\ker(T^* - \overline{\lambda})^n$ and $0 = (n+1)\operatorname{ind}(T-\lambda) = \operatorname{ind}(T-\lambda)^{n+1} = \dim\ker(T-\lambda)^{n+1} - \dim\ker(T^* - \overline{\lambda})^{n+1}$. Hence we have $\dim\ker(T-\lambda)^{n+1} = \dim\ker(T-\lambda)^n$ for some $n \in \mathbb{N}$. Since $\ker(T-\lambda)^n \subset \ker(T-\lambda)^{n+1}$ and $\dim\ker(T-\lambda)^n = \dim\ker(T-\lambda)^{n+1} < \infty$, we conclude that $\ker(T-\lambda)^n = \ker(T-\lambda)^{n+1}$ for some $n \in \mathbb{N}$. Therefore, $T - \lambda$ has finite ascent and so $\lambda \notin \sigma_b(T)$. Hence T satisfies Browder's theorem and so T^* satisfies Browder's theorem from (iii).

(v) If T satisfies Weyl's theorem, then $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$ holds. Note that if $T \in S_C(\mathcal{H})$, $\sigma_w(T)^* = -\sigma_w(T^*)$ as in the proof of (iii). Since $T \in S_C(\mathcal{H})$, it follows from [25] and Lemma 3.10 that

$$-\sigma(T^*)^* + \sigma_w(T^*)^* = -\pi_{00}(T^*)^* \text{ implies } \sigma(T^*) - \sigma_w(T^*) = \pi_{00}(T^*).$$

Therefore T^* satisfies Weyl's theorem. Conversely, if T^* satisfies Weyl's theorem, then $\sigma(T^*) - \sigma_w(T^*) = \pi_{00}(T^*)$. By [25] and Lemma 3.10, we obtain $-\sigma(T)^* + \sigma_w(T)^* = -\pi_{00}(T)^*$ and hence $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$ holds. Therefore, T satisfies Weyl's theorem.

Assume that T satisfies a-Weyl's theorem. From Lemma 3.10, we get that

$$\sigma_a(T^*)\backslash\sigma_{ea}(T^*) = -(\sigma_a(T)\backslash\sigma_{ea}(T))^* = -\pi_{00}^a(T)^* = \pi_{00}^a(T^*).$$

Hence T^* satisfies a-Weyl's theorem. The converse statement holds by a similar method.

(vi) Suppose that T satisfies the property (w). From Lemma 3.10, we get that

$$\sigma_a(T^*)\backslash\sigma_{ea}(T^*) = -(\sigma_a(T)\backslash\sigma_{ea}(T))^* = -\pi_{00}(T)^* = \pi_{00}(T^*).$$

Hence T^* satisfies the property (w). The converse statement holds by a similar way.

Assume that T satisfies the property (aw). From Lemma 3.10 and [25], we get that

$$\sigma(T^*)\backslash\sigma_w(T^*) = -(\sigma(T)\backslash\sigma_w(T))^* = -\pi_{00}^a(T)^* = \pi_{00}^a(T^*).$$

Hence T^* satisfies the property (aw). The converse statement holds by a similar method.

If T satisfies the property (b), then Browder's theorem holds for T by [8] and so $\sigma_w(T)^* = \sigma_b(T)^*$. By the proof of (iii) and Corollary 3.13, T^* satisfies the Browder's theorem and $\sigma_w(T^*) = \sigma_b(T^*)$. Hence $\sigma_{ea}(T^*) = \sigma_a(T^*) \cap \sigma_b(T^*)$ by [28] and [5]. From Lemma 3.10, we get that

$$\sigma(T^*)\backslash\sigma_{ea}(T^*) = -(\sigma(T)\backslash\sigma_{ea}(T))^* = -p_{00}(T)^* = p_{00}(T^*).$$

Hence T^* satisfies the property (b). The converse statement holds by a similar method.

If T satisfies the property (ab), then Browder's theorem holds for T by [8] and $\sigma_w(T)^* = \sigma_b(T)^*$. By the proof of (iii) and Corollary 3.13, T^* satisfies the Browder's theorem and $\sigma_w(T^*) = \sigma_b(T^*)$. Hence $\sigma_{ab}(T^*) = \sigma_a(T^*) \cap \sigma_w(T^*)$ by [5, Theorem 3.1]. From Lemma 3.10, we get that

$$\sigma(T^*)\backslash\sigma_w(T^*) = -(\sigma(T)\backslash\sigma_w(T))^* = -p_{00}^a(T)^* = p_{00}^a(T^*).$$

Hence T^* satisfies the property (ab). The converse statement holds by a similar method. □

Corollary 3.12. *Let S and T be which have complex symmetric operators with a conjugation C on $\mathcal{H}$. If $[S,T] := ST - TS$, then $[S,T]$ satisfies Weyl's theorem* (*respectively, a-Weyl's theorem, Browder's theorem, and a-Browder's theorem*) *if and only if $[S,T]^*$ does.*

Proof. Since $[S,T]$ is skew symmetric from [25], the proof follows from Theorem 3.11. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G.

Corollary 3.13. *Assume that $T \in S_C(\mathcal{H})$. If $T^* + \overline{\lambda}$ has finite ascent for all $\lambda \in \mathbb{C}$, then both T and T^* have the single-valued extension property. In this case, $\sigma(T) = \sigma_{su}(T) = \sigma_a(T)$.*

Proof. Since $T^* + \overline{\lambda}$ has finite ascent for all $\lambda \in \mathbb{C}$, it follows from Theorem 3.11(ii) that $T - \lambda$ has finite ascent. Hence both T and T^* have single-valued extension property from [23]. In this case, since T has the single-valued extension property, we obtain $\sigma(T) = \sigma_{su}(T)$ by [24]. Moreover, by Lemma 3.2, [24], and [25], $\sigma(T) = \sigma_{su}(T)$ implies $\sigma(T)^* = \sigma(T^*) = \sigma_{su}(T^*) = -\sigma_{su}(T)^* = -\sigma_a(T^*) = \sigma_a(T)^*$. Thus we have $\sigma(T) = \sigma_a(T)$. Hence we conclude that $\sigma(T) = \sigma_{su}(T) = \sigma_a(T)$. □

For an operator $T \in \mathcal{L}(\mathcal{H})$, the *quasinilpotent part* of T is defined by

$$H_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $H_0(T)$ is a linear (not necessarily closed) subspace of $\mathcal{H}$.

Corollary 3.14. *For an operator $T \in S_C(\mathcal{H})$, then the following arguments hold*:

(i) *If $\ker T = \ker T^*$ and $H_0(T-\lambda) = \ker(T-\lambda)^{p_\lambda}$ for some integer $p_\lambda \in \mathbb{N}$, then these operators T, T^*, $T\mid_{(\ker T)^\perp}$, and $(T\mid_{(\ker T)^\perp})^*$ satisfy Weyl's theorem.*

(ii) *If $H_0(T-\lambda)$ is closed for every $\lambda \in \pi_{0f}(T)$, then Weyl's theorem holds for T and T^*.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$ with $\ker T = \ker T^*$, we first claim that $T \in S_C(\mathcal{H})$ if and only if $T \mid_{(\ker T)^\perp} \in S_C(\mathcal{H})$. Assume that T is skew complex symmetric. Since $CTC = -T^*$ for some conjugation C, it ensures that $C(\ker T) = \ker T^* = \ker T$ and $C(\ker T)^\perp = (\ker T)^\perp$. Then C admits the matrix representation $C = C_1 \oplus C_2$ on $\ker T \oplus (\ker T)^\perp$ where C_1 is a conjugation on $\ker T$ and C_2 is a conjugation on $(\ker T)^\perp$. This implies that $C_2(T \mid_{(\ker T)^\perp})C_2 = -(T \mid_{(\ker T)^\perp})^*$. Hence $T \mid_{(\ker T)^\perp}$ is skew complex symmetric. Conversely, we assume $T \mid_{(\ker T)^\perp} \in S_C(\mathcal{H})$ and set $T_1 = T \mid_{(\ker T)^\perp}$. Since $\ker T = \ker T^*$ by hypothesis, we conclude that $T = 0 \oplus T_1$ is also skew complex symmetric. Since $H_0(T-\lambda) = \ker(T-\lambda)^{p_\lambda}$ for some integer $p_\lambda \in \mathbb{N}$, it follows from Theorem 3.11 and [1, Theorems 3.91 and 3.99] that Weyl's theorem hold for T and T^*. Moreover, in this case, $T \mid_{(\ker T)^\perp}$ and $(T \mid_{(\ker T)^\perp})^*$ satisfy Weyl's theorem by [1, Theorem 3.99].

(ii) If $H_0(T-\lambda)$ is closed for every $\lambda \in \pi_{0f}(T)$, then T and T^* satisfy Weyl's theorem from Theorem 3.11 and [1, Theorem 3.91]. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to a-polaroid if $iso\sigma_a(T) \subset p_0(T)$ where $p_0(T)$ denotes the set of poles of the resolvent of T.

Corollary 3.15. *Let $T \in S_C(\mathcal{H})$ have the property (b) and let T be a-polaroid. Then T^* satisfies generalized a-Browder's theorem.*

Proof. Suppose that T has the property (b) and T is a-polaroid. From Lemmas 3.2 and 3.10, we know that $\sigma_a(T) = -\sigma_a(T^*)$ and $\pi_0(T)^* = -\pi_0(T^*)$. Since $T \in S_C(\mathcal{H})$, it ensures from Theorem 3.11(v) that T^* has the property (b). Moreover, since T is a-polaroid, it follows that $iso\sigma_a(T) \subset p_0(T)$. By Lemma 3.10, we know that $\sigma_a(T^*) = -\sigma_a(T)^*$ and $p_0(T^*) = -p_0(T)^*$. This gives that $iso\sigma_a(T^*) \subset p_0(T^*)$. Hence T^* is also a-polaroid. Since T^* has the property (b) from Theorem 3.11, it holds that $\sigma_a(T^*) \setminus \sigma_{ea}(T^*) = p_{00}(T^*)$. The relation $p_{00}(T^*) \subset p_{00}^a(T^*)$ for any $T \in \mathcal{L}(\mathcal{H})$ implies $\sigma_a(T^*) \setminus \sigma_{ea}(T^*) \subset p_{00}^a(T^*)$. Since $\sigma_a(T^*) \setminus \sigma_{ea}(T^*) \supset p_{00}^a(T^*)$, we get that T^* satisfies a-Browder's theorem. Therefore, we conclude that T^* satisfies generalized a-Browder's theorem from [2]. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ satisfies the property (R) if the equality $p_{00}^a(T) = \pi_{00}(T)$ holds. Finally, we provide equivalence statements of a-Weyl's theorem for skew complex symmetric operators.

Theorem 3.16. *Suppose that $T \in S_C(\mathcal{H})$ has the single-valued extension property. Then the following arguments are equivalent*:

(i) *T has the property (R).*
(ii) *T has the property (w).*
(iii) *Weyl's theorem holds for T.*
(iv) *a-Weyl's theorem holds for T.*

Proof. (i) $\Leftrightarrow$ (iii): Suppose that T has the property (R), i.e., $p_{00}^a(T) = \pi_{00}(T)$. Since $T \in S_C(\mathcal{H})$ has the single-valued extension property, we can easily show that T^* has the single-valued extension property by a similar method of [18]. This implies that $p_{00}(T) = p_{00}^a(T)$. Moreover, since T has the property (R), we get that $p_{00}(T) = \pi_{00}(T)$. Since T has the single-valued extension property, it follows that T satisfies Browder's theorem. Hence we conclude that Weyl's theorem holds for T from [3, Theorem 2.4].

Conversely, we assume that T satisfies Weyl's theorem. From [3], we have $p_{00}(T) = \pi_{00}(T)$. Since T^* has the single-valued extension property, we know that $p_{00}(T) = p_{00}^a(T)$. So, $p_{00}^a(T) = \pi_{00}(T)$. Hence T has the property (R).

(ii) $\Leftrightarrow$ (iv): Since (iv) $\Rightarrow$ (ii) is clear, it suffices to show the converse implication. Assume that T has the property (w). Since $T \in S_C(\mathcal{H})$ has the single-valued extension property, it follows that T^* has the single-valued extension property by [20]. This ensures that $\sigma_a(T) = \sigma(T)$ and $\pi_{00}(T) = \pi_{00}^a(T)$ by [1]. Since T has the property (w), it follows that $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T) = \pi_{00}^a(T)$. Hence T satisfies a-Weyl's theorem.

(iii) $\Leftrightarrow$ (iv): Since (iv) $\Rightarrow$ (iii) is obvious, it suffices to prove the converse implication. Suppose that T satisfies Weyl's theorem. Since $T \in S_C(\mathcal{H})$ has the single-valued extension property, it follows that T^* has the single-valued extension property by [18]. This ensures that $\sigma_a(T) = \sigma(T)$, $\sigma_{ea}(T) = \sigma_w(T)$, and $\pi_{00}(T) = \pi_{00}^a(T)$ by [1]. Since T satisfies Weyl's theorem, we obtain that $\sigma_a(T) \setminus \sigma_{ea}(T) = \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Hence T has the property (w). □

Corollary 3.17. *Assume that $T \in S_C(\mathcal{H})$ has the single-valued extension property. If T satisfies Weyl's theorem or T has the property (R), then T^* has the property (w).*

Proof. If T satisfies Weyl's theorem, then it has the property (w) from Theorem 3.16. Hence T^* has the property (w) from Theorem 3.11. If T has the property (R), then the above statement also holds by using a similar argument. □

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References

[1] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer Academic Pub. 2004.

[2] P. Aiena, M. T. Biondi, and C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2839–2848.

[3] P. Aiena and P. Pena, *Variations on Weyl's theorem*, J. Math. Anal. Appl. **324** (2006), no. 1, 566–579.

[4] M. Amouch and H. Zguitti, *On the equivalence of Browder's and generalized Browder's theorem*, Glasg. Math. J. **48** (2006), no. 1, 179–185.

[5] I. J. An, *Weyl type theorems for* 2×2 *operator matrices*, Kyung Hee Univ., Ph.D. Thesis. 2013.

[6] M. Berkani, *On a class of quasi-Fredholm operators*, Integral Equations Operator Theory **34** (1999), no. 2, 244–249.

[7] M. Berkani and J. J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. **69** (2003), no. 1-2, 359–376.

[8] M. Berkani and H. Zariouh, *Extended Weyl type theorems*, Math. Bohem. **134** (2009), no. 4, 369–378.
[9] J. B. Conway, *A Course in Functional Analysis*, Springer Verlag, Berlin, Heidelberg, New York, 1990.
[10] S. V. Djordjević and Y. M. Han, *Browder's theorems and spectral continuity*, Glasg. Math. J. **42** (2000), no. 3, 479–486.
[11] ______, *a-Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc **130** (2002), no. 3, 715–722.
[12] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315.
[13] ______, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931.
[14] S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077.
[15] P. R. Halmos, *A Hilbert Space Problem Book*, Second edition, Springer-Verlag, New York, 1982.
[16] R. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker Inc., New York and Basel, 1988.
[17] R. E. Harte and W. Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 2115–2124.
[18] S. Jung, E. Ko, and J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382** (2011), no. 2, 252–260.
[19] ______, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406** (2013), no. 2, 373–385.
[20] S. Jung, E. Ko, M. Lee, and J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379** (2011), no. 1, 325–333.
[21] J. J. Koliha, *A generalized Drazin inverse*, Glasg. Math J. **38** (1996), no. 3, 367–381.
[22] P. Lancaster and L. Rodman, *The Algebraic Riccati Equation*, Oxford University Press, Oxford, 1995.
[23] K. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), no. 2, 323–336.
[24] K. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.
[25] C. G. Li and S. Zhu, *Skew symmetric normal operators*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2755–2762.
[26] B. F. Madore and R. A. Martinez-Avendano, *Subspace hypercyclicity*, preprint.
[27] V. Mehrmann and H. Xu, *Numerical methods in control*, J. Comput. Appl. Math. **123** (2000), no. 1-2, 371–394.
[28] C. Sun, X. Cao, and L. Dai, *Property* (w_1) *and Weyl type theorem*, J. Math. Anal. Appl. **363** (2010), no. 1, 1–6.

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