

KAPLANSKY-TYPE THEOREMS IN GRADED INTEGRAL DOMAINS

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ABSTRACT. It is well known that an integral domain D is a UFD if and only if every nonzero prime ideal of D contains a nonzero principal prime. This is the so-called Kaplansky's theorem. In this paper, we give this type of characterizations of a graded PvMD (resp., G-GCD domain, GCD domain, Bézout domain, valuation domain, Krull domain, π -domain).

0. Introduction

This is a continuation of our works on Kaplansky-type theorems [13, 20]. It is well known that an integral domain D is a UFD if and only if every nonzero prime ideal of D contains a nonzero principal prime [19, Theorem 5]. This is the so-called *Kaplansky's theorem*. A generalization of this type of theorems was first studied by Anderson and Zafrullah in [5], where they gave several characterizations of this type for GCD domains, valuation domains, and Prüfer domains. Also, characterizations for PvMDs and Krull domains were given in [15] and [6], respectively.

Later, in [20], the second-named author gave a Kaplansky-type characterization of G-GCD domains and PvMDs and gave an ideal-wise version of Kaplansky-type theorems. This ideal-wise version is then used to give characterizations of UFDs, π -domains, and Krull domains. Let D be an integral domain with quotient field K , X be an indeterminate over D , and $D[X]$ be the polynomial ring over D . A nonzero prime ideal Q of $D[X]$ is called an *upper to zero in $D[X]$* if $Q \cap D = (0)$. Clearly, Q is an upper to zero in $D[X]$ if and only if $Q = fK[X] \cap D[X]$ for some nonzero polynomial $f \in D[X]$. For $f \in D[X]$, let $c(f)$ be the ideal of D generated by the coefficients of f . In [13], the first two authors of this paper studied an integral domain D such that every upper to zero in $D[X]$ contains a prime (resp., primary) element,

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a prime (resp., primary) element f with $c(f) = D$, and an invertible (resp., a t -invertible) primary ideal.

Throughout this paper, R will denote a (nontrivial) graded (integral) domain $\bigoplus_{\alpha \in \Gamma} R_\alpha$, where Γ is a nontrivial torsion-free grading monoid. Following [2, Definition 4.1], R is a *graded UFD* if each nonzero nonunit homogeneous element of R is a product of (necessarily homogeneous) principal primes. In [2, Proposition 4.2], it was shown that R is a graded UFD if and only if each nonzero homogeneous prime ideal of R contains a nonzero homogeneous principal prime ideal. The purpose of this paper is to give several characterizations like this for graded GCD domains, graded valuation domains, graded PvMDs, etc. which can be considered as generalizations of the results in [13, 20]. Precisely, in Section 1, we show that (i) a graded domain R is a (graded) GCD domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous extractor; (ii) R is a graded valuation domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous comparable element; (iii) R is a graded G-GCD domain if and only if every nonzero homogeneous prime ideal contains a d -locally extractor; (iv) R is a graded PvMD if and only if every nonzero homogeneous prime ideal contains a t -locally extractor, if and only if each nonzero homogeneous prime ideal of R contains an h - t -locally comparable element, if and only if each nonzero homogeneous prime ideal of R contains an h - t -valuation element; (v) R is a graded Krull domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous Krull element; (vi) R is a graded UFD (resp., graded π -domain, graded Krull domain) if and only if every homogeneous prime t -ideal contains a homogeneous principal (resp., invertible, t -invertible) prime ideal.

In Section 2, we study a graded integral domain R with a unit of nonzero degree whose homogeneous quotient field is a UFD (e.g., the Laurent polynomial ring $D[X, X^{-1}]$ over an integral domain D). We first introduce the notion of “upper to zero in R ”, and then we prove that (i) R is a graded GCD domain if and only if every upper to zero in R contains a prime element; (ii) R is a graded PvMD if and only if R is integrally closed and each upper to zero in R is a maximal t -ideal; and (iii) the set of nonzero homogeneous elements in R is an almost splitting set in R if and only if each upper to zero in R contains a primary element.

Let D be an integral domain with quotient field K , and let $F(D)$ be the set of nonzero fractional ideals of D . For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup \{J_v \mid J \in F(D), J \subseteq I, \text{ and } J \text{ is finitely generated}\}$, and $I_w = \{x \in K \mid xJ \subseteq I \text{ for some finitely generated } J \in F(D) \text{ with } J_v = D\}$. An $I \in F(D)$ is called a t -ideal (resp., v -ideal) if $I_t = I$ (resp., $I_v = I$), while a v -ideal I is of *finite type* if there is a finitely generated ideal J so that $I = J_v$. A proper integral t -ideal is a *maximal t -ideal* if it is maximal among proper integral t -ideals. Let $t\text{-Max}(D)$ be the set of maximal t -ideals of D . Then $t\text{-Max}(D) \neq \emptyset$ when D is not a field and each proper integral t -ideal is contained in a maximal t -ideal. An $I \in F(D)$ is said to be *t -invertible* if $(II^{-1})_t = D$.

and I is said to be *strictly v -finite* if there is a finitely generated subideal J of I such that $I_v = J_v$. Let $T(D)$ be the group of t -invertible fractional ideals of D , and let $\text{Prin}(D)$ be its subgroup of principal fractional ideals. The (t) -class group of D is the Abelian group $\text{Cl}(D) = T(D)/\text{Prin}(D)$.

Let Γ be a nontrivial torsion-free grading monoid, i.e., a nonzero commutative cancellative monoid (written additively) such that its quotient group $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$ is a torsion-free Abelian group, and let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain graded by Γ . Since R is an integral domain, we may assume that $R_\alpha \neq \{0\}$ for every $\alpha \in \Gamma$. A nonzero $x \in R_\alpha$ for $\alpha \in \Gamma$ is called a *homogeneous element* of $\deg(x) = \alpha$, and hence each $f \in R$ can be written as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$, where $\alpha_i \in \Gamma$, x_{α_i} is homogeneous, and $\alpha_1 < \cdots < \alpha_n$. Let H be the set of nonzero homogeneous elements in R . Then H is a saturated multiplicative subset of R , and R_H is called the *homogeneous quotient field* of R . For $f \in R_H$, let $C(f)$ be the fractional ideal of R generated by the homogeneous components of f . An ideal A of R is said to be *homogeneous* if $A = \bigoplus_{\alpha \in \Gamma} (A \cap R_\alpha)$; so A is homogeneous if and only if A is generated by homogeneous elements. A homogeneous ideal of R is a *maximal homogeneous ideal* if it is maximal among proper homogeneous ideals. It is easy to show that every maximal homogeneous ideal is a prime ideal and every homogeneous ideal is contained in a maximal homogeneous ideal. Clearly, if A is homogeneous, then A_v and A_t are also homogeneous.

1. General graded integral domains

The following theorem of Kaplansky is well known: An integral domain D is a UFD if and only if every nonzero prime ideal of D contains a nonzero principal prime. While Kaplansky's theorem appears to apply only to UFD's, the method of proof yields a more general result [5, Theorem 1] that is used to characterize GCD domains, valuation domains, and Prüfer domains. The following is a graded domain analogue of [5, Theorem 1].

Theorem 1.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. Let (\mathbf{p}) be a property of nonzero homogeneous elements of R . Suppose that the set S of nonzero homogeneous elements of R with property (\mathbf{p}) is a nonempty saturated multiplicative subset. Then every nonzero homogeneous element of R has property (\mathbf{p}) if and only if every nonzero homogeneous prime ideal of R contains a nonzero homogeneous element with property (\mathbf{p}) .*

Proof. The sufficiency is trivial. We prove the necessity. If some nonzero homogeneous element a of R does not have property (\mathbf{p}) , then $(a) \cap S = \emptyset$ since S is saturated. Enlarging (a) to an ideal P maximal with respect to being disjoint from S gives us the prime ideal P which contains no nonzero element with property (\mathbf{p}) . Let P^* be the ideal of R generated by the homogeneous elements of P . Then P^* is a homogeneous prime ideal of R [21, p. 124]. Hence P^* is a nonzero homogeneous prime ideal of R such that $P^* \cap S = \emptyset$, a contradiction. \square

We first give a Kaplansky-type theorem for graded GCD domains. As in [2], a graded domain R is called a *graded GCD domain* if every pair of nonzero homogeneous elements of R has a GCD, equivalently, if $xR \cap yR$ is principal for every $x, y \in H$. It is known that R is a graded GCD domain if and only if R is a GCD domain [2, Theorem 3.4]. We call a nonzero homogeneous element e of R a *homogeneous extractor* if $(e) \cap (x)$ is principal for all $x \in H$. Of course, R is a (graded) GCD domain if and only if every nonzero homogeneous element of R is a homogeneous extractor.

Theorem 1.2. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a (graded) GCD domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous extractor.*

Proof. Let S be the set of all homogeneous extractors. By Theorem 1.1 it suffices to show that S is a saturated multiplicative subset of R . Let a and b be homogeneous extractors and let $x \in H$. Then $(a) \cap (x)$ is principal, say $(a) \cap (x) = (ar)$ for some $r \in R$. Clearly, r is homogeneous because $ax \in (ar) \cap H$. Then $(ab) \cap (x) = (ab) \cap ((a) \cap (x)) = (ab) \cap (ar) = (a)((b) \cap (r))$ is principal because $(b) \cap (r)$ is principal. Moreover, S is saturated. For if c is not a homogeneous extractor, then $(c) \cap (y)$ is not principal for some $y \in H$. Thus $(cd) \cap (yd) = ((c) \cap (y))(d)$ is not principal for all $d \in H$. \square

We next give a Kaplansky-type theorem for graded valuation domains. Recall from [17, Definition 1.1] that a graded domain R is a graded valuation ring if for each homogeneous element $x \in R_H$, either x or $1/x$ is in R . Equivalently if $a, b \in H$, then either $(a) \subseteq (b)$ or $(b) \subseteq (a)$. We define a nonzero homogeneous element a of R to be a *homogeneous comparable element* if for each $b \in H$, either $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Of course, R is a graded valuation domain if and only if every nonzero homogeneous element of R is a homogeneous comparable element.

Theorem 1.3. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded valuation domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous comparable element.*

Proof. Let S be the set of homogeneous comparable elements of R . By Theorem 1.1 it suffices to show that S is a saturated multiplicative subset of R . Let $a, b \in S$ and let $t \in H$. If $(b) \subseteq (t)$, then $(ab) \subseteq (b) \subseteq (t)$. If $(t) \subsetneq (b)$, then $t/b \in H$. So if $(a) \subseteq (t/b)$, then $(ab) \subseteq (t)$, while if $(t/b) \subseteq (a)$, then $(t) \subseteq (ab)$. Thus $ab \in S$. If c is a nonzero homogeneous element of R with $c \notin S$, then there is an $x \in H$ with $(c) \not\subseteq (x)$ and $(x) \not\subseteq (c)$. Then for any $d \in H$, $(cd) \not\subseteq (xd)$ and $(xd) \not\subseteq (cd)$. Thus S is a saturated multiplicative subset of R . \square

In [1], Anderson and Anderson defined an integral domain D to be a *G-GCD domain* (generalized GCD domain) if the intersection of two invertible (equivalently, principal) ideals of D is invertible. It is also shown that D is a G-GCD domain if and only if every v -ideal of finite type of D is invertible

[1, Theorem 1]. They defined a graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ to be a *graded G-GCD domain* if the intersection of two homogeneous invertible ideals of R is invertible. As in [3], we call a graded domain R a *graded PvMD* if the monoid of homogeneous v -ideals of finite type forms a group under v -multiplication $I * J = (IJ)_v$. In [3, Proposition 6.6], it is shown that a graded domain R is a G-GCD domain (resp., PvMD) if and only if R is a graded G-GCD domain (resp., graded PvMD). It is mentioned that a graded domain R is a PvMD if and only if every nonzero ideal of R generated by two homogeneous elements is t -invertible [9]. The following result is the graded version of [1, Theorem 1] (resp., [20, Theorem 2.2]).

Theorem 1.4. *The following are equivalent for a graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

- (1) *R is a graded G-GCD domain (resp., graded PvMD).*
- (2) *For $a, b \in H$, $(a) \cap (b)$ is invertible (resp., t -invertible).*
- (3) *For $a, b \in H$, $(a) : (b)$ is invertible (resp., t -invertible).*
- (4) *$hInv(R)$ (resp., $hInv_t(R)$), the group of homogeneous invertible (resp., t -invertible) ideals of R , is a lattice-ordered group under the partial order $A \leq B \Leftrightarrow B \subseteq A$.*
- (5) *Every homogeneous v -ideal of finite type of R is invertible (resp., t -invertible).*

Proof. This can be proved along the lines of the proof of [1, Theorem 1] with the fact that the sum, product, and intersection of homogeneous ideals are homogeneous. \square

We say that a nonzero homogeneous element x of a graded domain R is a *homogeneous d -locally extractor* (resp., *homogeneous t -locally extractor*) if $xR \cap yR$ is invertible (resp., t -invertible) for any nonzero homogeneous element $y \in R$. Then by Theorem 1.4, a graded domain R is a graded G-GCD domain (resp., graded PvMD) if and only if every nonzero homogeneous element of R is a homogeneous d -locally extractor (resp., homogeneous t -locally extractor). We next give a graded version of Kaplansky-type theorems for graded G-GCD domains and graded PvMDs.

Theorem 1.5. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded G-GCD domain (resp., a graded PvMD) if and only if every nonzero homogeneous prime ideal contains a homogeneous d -locally extractor (resp., homogeneous t -locally extractor).*

Proof. Let $*$ = d or $*$ = t . Let S be the set of all homogeneous $*$ -locally extractors of R . Then by Theorem 1.1, it suffices to show that S is a saturated multiplicative subset of R . Let $a, b \in S$ and $x \in H$. Then $I := (a) \cap (x)$ is $*$ -invertible. Note that $(a) \cap (x) = aJ$, where $J = a^{-1}I$ is a $*$ -invertible ideal of R . Let M be a maximal $*$ -ideal of R . Then $JR_M = cR_M$ for some $c \in J \cap H$, and hence $((b) \cap J)_M = bR_M \cap JR_M = ((b) \cap (c))_M$ is principal because $b \in S$. Also, if $J = (a_1, \dots, a_n)_v$, then $JR_M = (a_1, \dots, a_n)_v R_M = ((a_1, \dots, a_n)R_M)_v = a_i R_M$

for some i , and hence $bR_M \cap JR_M = ((b) \cap (a_i))R_M \subseteq (\sum(b) \cap (a_i))R_M \subseteq ((b) \cap J)R_M$. Hence, $(b) \cap J = (\sum(b) \cap (a_i))_v$, and since $(b) \cap (a_i)$ is $*$ -invertible, $(b) \cap J$ is of finite type. Thus, $(b) \cap J$ is $*$ -invertible. Then $(ab) \cap (x) = (ab) \cap ((a) \cap (x)) = (ab) \cap aJ = (a)((b) \cap J)$ is $*$ -invertible. Thus $ab \in S$. Moreover S is saturated. Indeed, let $cd \in S$ with $c, d \in H$ and assume to the contrary that $c \notin S$. Then $(c) \cap (y)$ is not $*$ -invertible for some $y \in H$. Thus $(cd) \cap (yd) = ((c) \cap (y))(d)$ is not $*$ -invertible, a contradiction. \square

Let R be a graded domain and $d \in H$. Then we say that d is h - t -locally comparable if d is a comparable element in R_M for each maximal homogeneous t -ideal M of R .

Lemma 1.6. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. If A is a strictly v -finite homogeneous ideal of R such that A contains an h - t -locally comparable element d , then A is t -invertible.*

Proof. Suppose that M is a maximal homogeneous t -ideal of R containing A and let B be a finitely generated homogeneous ideal contained in A such that $A_v = B_v$. Without loss of generality, we assume that $d \in B$. Since B is finitely generated, we note that $(A_v R_M)_v = (B_v R_M)_v = (BR_M)_v$ by [22, Lemma 4]. Since $d \in B$ and B is finitely generated, $BR_M = bR_M$ for some $b \in B$ by [15, Theorem 2.4]. Hence $bR_M \subseteq AR_M \subseteq (A_v R_M)_v = (BR_M)_v = bR_M$. So $AR_M = bR_M$ is principal for each maximal homogeneous t -ideal M containing A .

For any multiplicative subset S of R , $A^{-1}R_S = B^{-1}R_S = (BR_S)^{-1} \supseteq (AR_S)^{-1} \supseteq A^{-1}R_S$ and thus $A^{-1}R_S = (AR_S)^{-1}$. We claim that $(AA^{-1})_t = R$. Otherwise, there exists a maximal homogeneous t -ideal M such that $AA^{-1} \subseteq M$ because AA^{-1} is homogeneous. So $AA^{-1}R_M = (AR_M)(AR_M)^{-1} = R_M$ since AR_M is principal. Hence $R_M \subseteq MR_M$, a contradiction. \square

It is known that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a PvMD if and only if R_M is a valuation domain for each maximal homogeneous t -ideal M of R [12, Lemma 2.7].

Theorem 1.7. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a PvMD if and only if each nonzero homogeneous prime ideal of R contains an h - t -locally comparable element.*

Proof. If R is a PvMD, then each nonzero homogeneous element of R is h - t -locally comparable, and so the condition holds. Conversely, suppose that each nonzero homogeneous prime ideal of R meets S , the set of all h - t -locally comparable elements of R . Note that S is a saturated multiplicative subset of R ; hence each nonzero homogeneous ideal of R meets S . Now let A be a homogeneous v -ideal of finite type of R . Then $A = B_v = B_t$ for some finitely generated homogeneous ideal B of R . Since A is strictly v -finite, it is t -invertible by Lemma 1.6. Thus by Theorem 1.4, R is a PvMD. \square

We say that a nonzero homogeneous element d of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is an h - t -valuation element if R_M is a valuation domain for each maximal homogeneous

t -ideal M of R containing d . Clearly every h - t -valuation element is h - t -locally comparable, and since each nonzero homogeneous element of a graded PvMD is an h - t -valuation element, Theorem 1.7 has the following:

Corollary 1.8. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a PvMD if and only if each nonzero homogeneous prime ideal of R contains an h - t -valuation element.*

Let \mathcal{S} be a multiplicative system of homogeneous ideals of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, that is, a family of nonzero homogeneous integral ideals of R closed under multiplication. We denote by $\text{Sat}(\mathcal{S})$ the saturation of \mathcal{S} , that is the set of homogeneous ideals of R containing some (homogeneous) ideal in \mathcal{S} . We say that \mathcal{S} is *saturated* if $\mathcal{S} = \text{Sat}(\mathcal{S})$. Then $\text{Sat}(\mathcal{S})$ is a saturated multiplicative system of homogeneous ideals of R .

Let $*$ be a finite character star operation on R , i.e., for each fractional ideal A of R , $A_* = \bigcup \{F_* \mid 0 \neq F \text{ is a finitely generated subideal of } A\}$. The v -operation is a star operation, while the t - and w -operations are finite character star operations. The reader can refer to [14, Sections 32 and 34] for some basic properties of star operations. Let us denote by $*\text{-Sat}(\mathcal{S})$ the set of all homogeneous $*$ -ideals in $\text{Sat}(\mathcal{S})$. Note that $*\text{-Sat}(\mathcal{S})$ is closed under $*$ -multiplication (for, if $I, J \in \text{Sat}(\mathcal{S})$, then $(IJ)_* \in *\text{-Sat}(\mathcal{S})$) and is $*$ -saturated (that is, each homogeneous $*$ -ideal containing an ideal in $*\text{-Sat}(\mathcal{S})$ is in $*\text{-Sat}(\mathcal{S})$). We say that \mathcal{S} is $*$ -finite if each ideal $I \in *\text{-Sat}(\mathcal{S})$ contains a finitely generated homogeneous ideal J such that $J_* \in \text{Sat}(\mathcal{S})$. It is immediate that \mathcal{S} is $*$ -finite if and only if $\text{Sat}(\mathcal{S})$ is $*$ -finite, and that, if each homogeneous $*$ -ideal in \mathcal{S} is $*$ -finite, then \mathcal{S} is $*$ -finite.

The proof of following lemma is easy, and so we omit it.

Lemma 1.9. *Let $*$ be a finite character star operation on a graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. Let $\{I_\alpha\}$ be a directed family of homogeneous $*$ -ideals. If $I = \bigcup I_\alpha$ is a fractional homogeneous ideal of R , then I is a homogeneous $*$ -ideal.*

The proof of Kaplansky's Theorem depends on the following Theorem of Krull: Let S be a multiplicative subset in a ring D and let I be an ideal in D maximal with respect to the exclusion of S . Then I is prime. The following is the homogeneous ideal-wise version of Krull's Theorem.

Theorem 1.10. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain, $*$ be a finite character star operation on R , and \mathcal{S} be a $*$ -finite $*$ -saturated multiplicative system of homogeneous ideals of R . Assume that there is a homogeneous integral $*$ -ideal I such that $J \not\subseteq I$ for any $J \in \mathcal{S}$. Then I may be extended to a homogeneous $*$ -ideal P maximal with respect to $J \not\subseteq P$ for any $J \in \mathcal{S}$.*

Proof. Let $\mathcal{C} = \{A \mid A \subseteq R \text{ is a homogeneous } * \text{-ideal and } A \not\supseteq J \text{ for any } J \in \mathcal{S}\}$. Since $I \in \mathcal{C}$, \mathcal{C} is a nonempty partially ordered set with respect to set inclusion. By Lemma 1.9, every nonempty totally ordered subset of \mathcal{C} has an upper bound in \mathcal{C} since $*$ is a finite character star operation on R and \mathcal{S} is

$*$ -finite and $*$ -saturated. Hence by Zorn's Lemma, there is a maximal element $P \in \mathcal{C}$. Now it is shown in [20, Theorem 3.2] that P is a prime ideal of R . \square

Now we are ready to state the ideal-wise version of a Kaplansky-type theorem for graded domains.

Theorem 1.11. *Let $*$ be a finite character star operation on $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. Let (\mathbf{p}) be a property of proper $*$ -ideals of R . Suppose the set \mathcal{S} of homogeneous $*$ -ideals with property (\mathbf{p}) is a $*$ -finite $*$ -saturated multiplicative system of homogeneous ideals of R . Then every proper homogeneous $*$ -ideal of R satisfies property (\mathbf{p}) if and only if every nonzero homogeneous prime $*$ -ideal of R contains a homogeneous $*$ -ideal with property (\mathbf{p}) .*

Proof. Suppose that a proper homogeneous $*$ -ideal I of R does not satisfy property (\mathbf{p}) . Then $J \not\subseteq I$ for any $J \in \mathcal{S}$. Applying Theorem 1.10, I may be extended to a homogeneous prime $*$ -ideal P such that $J \not\subseteq P$ for any $J \in \mathcal{S}$. By hypothesis, P contains a homogeneous $*$ -ideal with property (\mathbf{p}) . Since \mathcal{S} is $*$ -saturated, $P \in \mathcal{S}$, a contradiction. The other implication is trivial. \square

For a graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, let $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$. Then $N(H)$ is a saturated multiplicative subset of R [9, Lemma 1.1(2)]. Following [9], we say that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ satisfies property $(\#)$ if, for any nonzero ideal I of R , $(\sum_{g \in I} C(g))_t = R$ implies that there exists an $f \in I$ such that $C(f)_v = R$. Note that the Laurent polynomial ring $D[X, X^{-1}] = \bigoplus_{n \in \mathbb{Z}} DX^n$ over an integral domain D satisfies property $(\#)$ [9, p. 172].

It is shown in [18] that a domain D is a UFD (resp, π -domain, Krull domain) if and only if every t -ideal of D is a t -product of principal (resp., invertible, t -invertible) prime ideals. The following two results are the graded domain analogues of these results.

Theorem 1.12. *The following are equivalent for a graded domain R .*

- (1) R is a graded Krull domain.
- (2) Every proper homogeneous principal ideal is a t -product of (t -invertible) prime ideals.
- (3) Every proper homogeneous t -ideal is a t -product of (t -invertible) prime ideals.
- (4) Every proper homogeneous t -invertible t -ideal is a t -product of (t -invertible) prime ideals.

Proof. (1) \Rightarrow (3). Let R be a graded Krull domain. By [9, Theorem 2.3], $R_{N(H)}$ is a PID. Let I be a proper homogeneous t -ideal of R . Then $IR_{N(H)} = Q_1 \cdots Q_n$, where each Q_i is a principal prime ideal of $R_{N(H)}$. Note that R satisfies property $(\#)$ by [2, Propositions 5.5 and 5.6] and [9, Lemma 2.2]. Since R is a (graded) PvMD, every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R [9, Corollary 1.10]. Let $Q_i = P_i R_{N(H)}$, where P_i is a homogeneous prime ideal of R . Then $IR_{N(H)} = Q_1 \cdots Q_n = (P_1 R_{N(H)}) \cdots (P_n R_{N(H)}) =$

$(P_1 \cdots P_n)R_{N(H)}$. Note that $t = w$ in a PvMD. Hence by [9, Lemma 1.7], we have $I = I_t = I_w = (P_1 \cdots P_n)_w = (P_1 \cdots P_n)_t$.

(3) \Rightarrow (4) \Rightarrow (2). This is obvious.

(2) \Rightarrow (1). Let P be a nonzero homogeneous prime ideal of R . Choose a nonzero homogeneous element $a \in P$. By assumption $(a) = (P_1 \cdots P_n)_t$ for some prime ideals P_1, \dots, P_n of R . Then P contains a t -invertible prime ideal. Thus every nonzero homogeneous prime ideal of R contains a t -invertible prime ideal. Hence by [7, Theorem 2.4], R is a graded Krull domain. \square

Following [6], a nonzero element x of an integral domain D is called a *Krull element* if (x) is a t -product of prime t -ideals. Then it is shown that D is a Krull domain if and only if every nonzero prime ideal of D contains a Krull element [6, Theorem 7]. The following is the graded domain analogue of this result.

Corollary 1.13. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Krull domain if and only if every nonzero homogeneous prime ideal of R contains a homogeneous Krull element.*

Proof. If R is a graded Krull domain, then by Theorem 1.12, every nonzero homogeneous prime ideal of R contains a homogeneous Krull element. Conversely, suppose that every nonzero homogeneous prime ideal of R contains a homogeneous Krull element. Let S be the set of all homogeneous Krull elements of R . As in the proof of [6, Theorem 7], we can see that R is a graded Krull domain. \square

Following [2, Definition 6.1], we say that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded π -domain* if each nonzero principal homogeneous ideal is a product of (necessarily homogeneous invertible) prime ideals.

Theorem 1.14. *The following are equivalent for a graded domain R .*

- (1) *R is a graded π -domain.*
- (2) *Every nonzero homogeneous prime ideal of R contains an invertible prime ideal.*
- (3) *Every proper homogeneous t -ideal of R is a finite product of (invertible) prime ideals.*
- (4) *Every proper homogeneous invertible ideal of R is a finite product of prime ideals.*

Proof. (1) \Rightarrow (2). The proof is similar to that of (1) \Rightarrow (3) in Theorem 1.12.

(2) \Rightarrow (3). Suppose that every nonzero homogeneous prime ideal contains an invertible prime ideal. Then by [7, Theorem 2.4], R is a graded Krull domain, and hence if I is a proper homogeneous t -ideal of R , then $I = (P_1 \cdots P_n)_t$ for some homogeneous t -invertible prime ideals P_1, \dots, P_n by Theorem 1.12. By the given assumption, each P_i contains some invertible prime ideal Q_i . We may assume that $(P_i)_t \neq R$ for any i by discarding the P_i 's with $(P_i)_t = R$ from $I = (P_1 \cdots P_n)_t$ since $I \neq R$. Then by [18, Theorem 2.2(4)], $P_i = Q_i$. Thus

P_i is invertible for every $i = 1, \dots, n$. Hence the ideal $P_1 \cdots P_n$ is invertible. Therefore $I = (P_1 \cdots P_n)_t = P_1 \cdots P_n$ and every P_i is invertible.

(3) \Rightarrow (4) \Rightarrow (1). This is clear. \square

In [18], it is shown that D is a UFD (resp., π -domain, Krull domain) if and only if every prime t -ideal of D contains a principal (resp., invertible, t -invertible) prime ideal. The following is the graded domain analogue of this result.

Corollary 1.15. *A graded domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded UFD (resp., graded π -domain, graded Krull domain) if and only if every homogeneous prime t -ideal of R contains a homogeneous principal (resp., invertible, t -invertible) prime ideal.*

Proof. Let \mathcal{S} be the set of all (homogeneous) t -ideals of R expressible as a t -product of homogeneous principal (resp., invertible, t -invertible) prime ideals. Then by Theorem 1.11, it suffices to show that \mathcal{S} is a t -finite t -saturated multiplicative system of R . However, with a slight modification as in [15, Theorem 1.3], we can see that \mathcal{S} is a t -saturated multiplicative system. The rest is clear. \square

2. Graded integral domains with a unit of nonzero degree

Throughout this section $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ denotes a (nontrivial) graded integral domain with a unit of nonzero degree and H is the set of nonzero homogeneous elements in R .

Lemma 2.1. *Let $0 \neq f \in R$.*

- (1) *$fR_H \cap R = fR$ if and only if $C(f)_v = R$.*
- (2) *If f is a product of primary elements in R that are not homogeneous, then $fR_H \cap R = fR$.*

Proof. (1) [9, Lemma 1.2(3)].

(2) Let $g, h \in R$ be primary elements that are not homogeneous. If $\sqrt{gR} \neq \sqrt{hR}$, then \sqrt{gR} and \sqrt{hR} are both maximal t -ideals [10, Lemma 2.1]. Hence, $(g, h)_v = R$, and so $gR \cap hR = ghR$. If $\sqrt{gR} = \sqrt{hR}$, then $\sqrt{ghR} = \sqrt{gR}$, and since \sqrt{gR} is a maximal t -ideal, ghR is a primary ideal. Hence, $f = f_1 \cdots f_n$ can be written as a product of distinct primary elements and $fR_H = f_1R_H \cap \cdots \cap f_nR_H$. So, we may assume that f is primary, and so \sqrt{fR} is a maximal t -ideal. If $C(f)_v \subsetneq R$, then $fR \subsetneq C(f)_v \subseteq Q$ for some maximal homogeneous t -ideal Q of R [7, Lemma 1.2], and hence $\sqrt{fR} = Q$. However, note that $fR_H \subsetneq R_H = QR_H$ because f is not homogeneous, a contradiction. Therefore, $C(f)_v = R$, and thus by (1), $fR_H \cap R = fR$. \square

If R_H is a UFD, as in the polynomial ring case, we will say that a nonzero prime ideal Q of R is an *upper to zero* in R if $Q = fR_H \cap R$ for some prime element $f \in R_H$. Hence, if $R = D[X, X^{-1}]$ is the Laurent polynomial ring over

an integral domain D , then R_H is a UFD, and Q is an upper to zero in R if and only if $Q \cap D[X]$ is an upper to zero in $D[X]$. Thus, our definition of “upper to zero in R ” is a natural generalization of the original definition of “upper to zero in $D[X]$ ”. Note that the assumption that R_H is a UFD is not too strong in the sense that the property arises in ‘almost all’ graded domains.

Theorem 2.2. *If R_H is a UFD, then the following statements are equivalent.*

- (1) R is a graded GCD domain.
- (2) R is a GCD domain.
- (3) $fR_H \cap R$ is principal for each $0 \neq f \in R$.
- (4) Every upper to zero in R contains a prime element.

Proof. (1) \Rightarrow (2) [2, Theorem 3.4].

(2) \Rightarrow (3) Since R is a GCD domain, there is an $\alpha \in R$ such that $C(f)_v = \alpha R$. Since $C(f)$ is homogeneous, $\alpha \in H$, and hence $fR_H \cap R = \frac{f}{\alpha}R_H \cap R = \frac{f}{\alpha}R$ by Lemma 2.1(1).

(3) \Rightarrow (4) Let Q be an upper to zero in R . Then $Q = fR_H \cap R$ for some $f \in R$, and by assumption $fR_H \cap R = gR$ for some $g \in R$. Clearly, g is a prime element of R , and thus Q contains a prime element g .

(4) \Rightarrow (1) Let $\alpha, \beta \in H$. If $u \in H$ is a unit of nonzero degree, then there is an integer $n \geq 1$ such that $\deg \alpha u^n \neq \deg \beta$. So if we let $f = \alpha u^n + \beta$, then $C(f) = (\alpha, \beta)$. Let $f = f_1^{k_1} \cdots f_m^{k_m}$ be a prime factorization of f in R_H . Then by hypothesis, each $f_i R_H \cap R$ contains a prime element g_i in R . Note that $g_i \notin H$. Hence $f_i R_H = g_i R_H$ and $f_i R_H \cap R = g_i R_H \cap R = g_i R$. By Lemma 2.1(1), $C(g_i)_v = R$. Since $f_i R_H = g_i R_H$, we have $a_i f_i = b_i g_i$ for some $a_i, b_i \in H$. Put $a = a_1^{k_1} \cdots a_m^{k_m}$ and $b = b_1^{k_1} \cdots b_m^{k_m}$. Then $af = bg_1^{k_1} \cdots g_m^{k_m}$. Note that $a, b \in H$. Clearly, $(\alpha, \beta)_v = C(f)_v = \frac{b}{a}R$. Therefore, R is a graded GCD domain. \square

We say that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded Bézout domain* if (a, b) is principal for all homogeneous elements $a, b \in R$. Clearly, graded Bézout domains are (graded) GCD domains. Let $R = D[X, X^{-1}]$ be the Laurent polynomial ring over a Bézout domain D . Then R is a graded Bézout domain, and it is easy to see that R is a Bézout domain if and only if D is a field. Thus, a graded Bézout domain need not be a Bézout domain.

Corollary 2.3. *If R_H is a UFD, then R is a graded Bézout domain if and only if every upper to zero in R contains a prime element f with $C(f) = R$.*

Proof. (\Rightarrow) Let $0 \neq g \in R$ be such that $gR_H \cap R$ is a prime ideal. Then, by Theorem 2.2, $gR_H \cap R = fR$ for some $f \in R$, and it is obvious that f is a prime element. Also, since R is a graded Bézout domain, $C(f) = C(f)_v = R$ by Lemma 2.1.

(\Leftarrow) The proof is similar to that of (4) \Rightarrow (1) of Theorem 2.2. \square

We next give a graded integral domain analogue of [16, Proposition 2.6] that D is a PvMD if and only if D is an integrally closed domain and each upper

to zero in $D[X]$ is a maximal t -ideal. We first recall a well-known result for an integrally closed graded domain.

Lemma 2.4. *If R is integrally closed, then*

- (1) ([9, Lemma 1.2(4)]) $fR_H \cap R = fC(f)^{-1}$ for $0 \neq f \in R$ and
- (2) ([3, Theorem 3.5]) $C(fg)_v = (C(f)C(g))_v$ for $0 \neq f, g \in R$.

Note that if Q is an upper to zero in R , then $\text{ht}Q = 1$ and $Q \cap H = \emptyset$, and thus Q is t -invertible if and only if Q is a maximal t -ideal [7, Corollary 2.2].

Theorem 2.5. *If R_H is a UFD, then R is a PvMD if and only if R is integrally closed and each upper to zero in R is a maximal t -ideal.*

Proof. (\Rightarrow) Since R is a PvMD, clearly R is integrally closed. Let Q be an upper to zero in R . Then $Q = fR_H \cap R$ for some $f \in R$, and since R is integrally closed, $Q = fC(f)^{-1}$ by Lemma 2.4. Also, since R is a PvMD, $Q = fC(f)^{-1}$ is t -invertible.

(\Leftarrow) Let $I = (x_1, \dots, x_n)$ be a nonzero finitely generated ideal of R , where each x_i is a nonzero homogeneous element of R , and let u be a unit of nonzero degree. Then there are positive integers k_2, \dots, k_n such that if we let $f = x_1 + x_2u^{k_2} + \dots + x_nu^{k_n}$, then $C(f) = I$. Since R_H is a UFD, we can write $f = f_1^{e_1} \dots f_m^{e_m}$ for some distinct prime elements $f_i \in R_H$ and positive integers e_i . Hence,

$$\begin{aligned} fC(f)^{-1} &= fR_H \cap R \\ &= (f_1^{e_1}R_H \cap R) \cap \dots \cap (f_m^{e_m}R_H \cap R) \\ &= f_1^{e_1}C(f_1^{e_1})^{-1} \cap \dots \cap f_m^{e_m}C(f_m^{e_m})^{-1} \\ &= ((f_1^{e_1}C(f_1^{e_1})^{-1}) \dots (f_m^{e_m}C(f_m^{e_m})^{-1}))_t \\ &= f(C(f_1^{e_1})^{-1} \dots C(f_m^{e_m})^{-1})_t \\ &= f((C(f_1)^{e_1})^{-1} \dots (C(f_m)^{e_m})^{-1})_t, \end{aligned}$$

where the last equality follows from Lemma 2.4 because R is integrally closed. Note that $f_iR_H \cap R = f_iC(f_i)^{-1}$ is an upper to zero in R ; so $C(f_i)^{-1}$ is t -invertible by assumption. Hence, $(C(f_1)^{e_1})^{-1} \dots (C(f_m)^{e_m})^{-1}$ is t -invertible, and so $fC(f)^{-1}$ is t -invertible. Thus, $I = C(f)$ is t -invertible. \square

Let S be a saturated multiplicative subset of an integral domain D . As in [4], we say that S is an *almost splitting set* if, for each $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^n = st$ for some $s \in S$ and $t \in \{x \in D \mid x \neq 0 \text{ and } (x, s)_v = D \text{ for all } s \in S\}$. It is known that S is an almost splitting set if and only if, for each $d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n D_S \cap D$ is principal [4, Proposition 2.7].

Lemma 2.6. *If H is an almost splitting set in R , then $\text{Cl}(R)$ is torsion.*

Proof. Let $N(H) = \{f \in R \mid C(f)_v = R\}$. Since R contains a unit of nonzero degree, R satisfies property (#) [9, Example 1.6]. We note that $\text{Max}(R_{N(H)}) =$

$\{Q_{N(H)} \mid Q \text{ is a maximal } t\text{-ideal of } R \text{ with } Q \cap H \neq \emptyset\}$, and $\text{Pic}(R_{N(H)}) = \text{Cl}(R_{N(H)}) = 0$ [9, Proposition 1.4, Theorem 3.3]. Note also that R_H is a GCD domain; so $\text{Cl}(R_H) = 0$. By [7, Lemma 1.1], $N(H) = \{x \in R \mid x \neq 0 \text{ and } (x, s)_v = R \text{ for all } s \in H\}$. Since H is an almost splitting set, $\text{Cl}(R)$ is torsion [11, Theorem 2.10]. \square

Theorem 2.7. *If R_H is a UFD, then the following statements are equivalent.*

- (1) *H is an almost splitting set in R .*
- (2) *Each upper to zero in R is a maximal t -ideal and $\text{Cl}(R)$ is torsion.*
- (3) *Each upper to zero in R contains a primary element.*

Proof. (1) \Rightarrow (2) By Lemma 2.6, $\text{Cl}(R)$ is torsion. Next, let $Q = fR_H \cap R$ be an upper to zero in R . Then $Q \cap H = \emptyset$ and $\text{ht}Q = 1$. Hence Q is a maximal t -ideal ([7, Corollary 2.2] and [11, Proposition 2.3]).

(2) \Rightarrow (3) Let $Q = fR_H \cap R$ be an upper to zero in R . Then Q is t -invertible [7, Corollary 2.2], and since $\text{Cl}(R)$ is torsion, there is an integer $n \geq 1$ such that $(Q^n)_t = gR$ for some $g \in R$. Clearly, $\sqrt{gR} = \sqrt{(Q^n)_t} = Q$, and since Q is a maximal t -ideal, gR is a primary ideal [10, Lemma 2.1]. Thus, Q contains a primary element g .

(3) \Rightarrow (1) Let $0 \neq f \in R$. By [4, Proposition 2.7], it suffices to show that there is an integer $n = n(f) \geq 1$ such that $f^n R_H \cap R$ is principal. Clearly, we may assume that f is not homogeneous.

Case 1. fR_H is a prime ideal of R_H . Then $fR_H \cap R$ contains a primary element, say $g \in R$, and so $gR_H = f^n R_H$ for some integer $n \geq 1$. Thus, $f^n R_H \cap R = gR_H \cap R = gR$ by Lemma 2.1.

Case 2. Let $f = f_1^{e_1} \cdots f_k^{e_k}$ be the prime factorization of f in R_H . Then $fR_H = f_1^{e_1} R_H \cap \cdots \cap f_k^{e_k} R_H$, and by Case 1, there are integers $m_i \geq 1$ and primary elements g_i such that $f_i^{m_i} R_H \cap R = g_i R_H$. Let $m = m_1 \cdots m_k$, $e = e_1 \cdots e_k$ and $n_i = \frac{mee_i}{m_i}$. Then

$$\begin{aligned} f^{me} R_H \cap R &= (f_1^{m_1})^{n_1} R_H \cap \cdots \cap (f_k^{m_k})^{n_k} R_H \cap R \\ &= ((f_1^{m_1})^{n_1} R_H \cap R) \cap \cdots \cap ((f_k^{m_k})^{n_k} R_H \cap R) \\ &= (g_1^{n_1} R_H \cap R) \cap \cdots \cap (g_k^{n_k} R_H \cap R) \\ &= g_1^{n_1} R \cap \cdots \cap g_k^{n_k} R \\ &= (g_1^{n_1} \cdots g_k^{n_k}) R, \end{aligned}$$

where the first and fifth equalities follow from the fact that $(g, h)_v = D$ if and only if $gD \cap hD = ghD$ for an integral domain D . \square

Corollary 2.8. *If R_H is a UFD, then every upper to zero in R contains a primary element f with $C(f) = R$ if and only if every upper to zero in R is a maximal t -ideal, each maximal homogeneous ideal is a t -ideal, and $\text{Cl}(R)$ is torsion.*

Proof. (\Rightarrow) By Theorem 2.7, every upper to zero in R is a maximal t -ideal and $\text{Cl}(R)$ is torsion. If there is a maximal homogeneous ideal that is not a t -ideal,

then, since R contains a unit of nonzero degree, we can find an $f \in R$ such that $C(f)_v = R$ but $C(f) \subsetneq R$. Let $f = f_1^{e_1} \cdots f_n^{e_n}$ be the prime factorization of f in R_H . Then $fR = fR_H \cap R = (f_1^{e_1}R_H \cap R) \cap \cdots \cap (f_n^{e_n}R_H \cap R)$ by Lemma 2.1. Note that if $Q_i = f_iR_H \cap R$, then Q_i is an upper to zero in R and $f_i^{e_i}R_H \cap R$ is a Q_i -primary ideal. Hence, each Q_i contains a primary element g_i with $C(g_i) = R$. Clearly, $g_i^{e_i} \in f_i^{e_i}R_H \cap R$, and so if we set $g = g_1^{e_1} \cdots g_n^{e_n}$, then $g \in fR$ and $C(g) = R$. Thus, $C(f) = R$, a contradiction.

(\Leftarrow) Let Q be an upper to zero in R . Then Q is t -invertible [7, Corollary 2.2], and since $\text{Cl}(R)$ is torsion, there is a positive integer $n = n(Q)$ such that $(Q^n)_t = fR$ for some $f \in R$. Note that $\sqrt{fR} = Q$ is a maximal t -ideal; so f is primary and $C(f)_t = R$. Hence, f is a primary element with $C(f) = R$. \square

An integral domain D is an *almost GCD domain* (AGCD domain) if, for each $a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is principal. Clearly, GCD domains are AGCD domains. Also, it is known that D is an integrally closed AGCD domain if and only if D is a PvMD with $\text{Cl}(D)$ torsion [23, Corollary 3.8].

Corollary 2.9. *Assume that R_H is a UFD and R is integrally closed. Then R is an AGCD domain if and only if H is an almost splitting set in R .*

Proof. (\Rightarrow) Let $0 \neq f \in R$. If $f \in H$, then there is nothing to prove, and so we assume that $f \notin H$. Then $C(f)$ is t -invertible, and since $\text{Cl}(R)$ is torsion, there is an integer $n \geq 1$ such that $C(f^n)_v = (C(f)^n)_v = \alpha R$ for some $\alpha \in R$ by Lemma 2.4(2). Clearly, $\alpha \in H$, and hence $C(\frac{f^n}{\alpha})_v = R$. Thus, $f^n = \alpha \cdot \frac{f^n}{\alpha}$, where $\alpha \in H$ and $\frac{f^n}{\alpha} \in N(H)$. Note that $N(H) = \{f \in R \mid C(f)_v = R\} = \{x \in R \mid x \neq 0 \text{ and } (x, s)_v = R \text{ for all } s \in H\}$ [7, Lemma 1.1]. Therefore, H is an almost splitting set.

(\Leftarrow) Let $\alpha, \beta \in H$, and let $u \in H$ be a unit of nonzero degree. Then $\deg \alpha u^k \neq \deg \beta$ for some integer $k \geq 1$, and so if we let $f = \alpha u^n + \beta$, then $C(f) = (\alpha, \beta)$. Since H is almost splitting, there is an integer $n \geq 1$ such that $f^n R_H \cap R$ is principal. Note that $f^n R_H \cap R = f^n C(f^n)^{-1}$ by Lemma 2.4(1); hence $C(f^n)^{-1}$ is principal. Since $((\alpha, \beta)^n)_v = (C(f)^n)_v = C(f^n)_v$ by Lemma 2.4(2), $((\alpha, \beta)^n)_v$ is principal, and hence (α, β) is t -invertible. Thus, R is a PvMD, and thus R is an AGCD domain because $\text{Cl}(R)$ is torsion by Theorem 2.7. \square

Remark 2.10. Let D be an integral domain with quotient field K , and let $R = D[X, X^{-1}]$ be the Laurent polynomial ring over D . Then $R = \bigoplus_{n \in \mathbb{Z}} DX^n$ is a graded integral domain such that R contains a unit of nonzero degree and $R_H = K[X, X^{-1}]$ is a UFD. Note that $\text{Cl}(D[X]) = \text{Cl}(R)$, $XD[X]$ is a principal prime ideal of $D[X]$, $c_D(X) = D$, where $c_D(f)$ is the ideal of D generated by the coefficients of $f \in D[X]$. Also, note that every upper to zero Q in $D[X]$ with $Q \neq XD[X]$ is contracted from an upper to zero in $R = D[X, X^{-1}]$. Hence, every result of [13] on uppers to zero in $D[X]$ holds for every upper to

zero in $R = D[X, X^{-1}]$. Thus, the results of this section can be considered as generalizations of those in [13] and [8, Lemma 2.3 and Theorem 2.4], and in fact, their proofs are similar to those of their counterparts.

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