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SYMPLECTIC 4-MANIFOLDS VIA SYMPLECTIC SURGERY ON COMPLEX SURFACE SINGULARITIES

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Abstract. We examine a family of isolated complex surface singularities whose exceptional curves consist of two complex curves with high genera intersecting transversally. Topological data of smoothings of these singularities are determined. We use these computations to construct symplectic 4-manifolds by replacing neighborhoods of the exceptional curves with smoothings of the singularities.

1. Introduction

A standard way for showing that a topological 4-manifold admits infinitely many non-diffeomorphic smooth structures involves two main steps. In the first step one constructs one smooth structure on the manifold which is compatible with a symplectic structure, that is, admits a non-degenerate closed 2-form. The existence of the symplectic structure implies that the Seiberg-Witten invariants of the smooth manifold are non-trivial [19], hence in the second step we can apply various operations (logarithmic transformation, Luttinger surgery, knot surgery) for which the change of the smooth structure (through the change of its Seiberg-Witten invariants) can be computed. Under favourable circumstances this approach can be used to show that the topological 4-manifold at hand admits infinitely many distinct smooth structures.

Many smooth cut-and-paste constructions have been shown to be compatible with symplectic structures. Most notably, Gompf [6] and McCarthy-Wolfson [9] showed that the normal sum operation along symplectic submanifolds preserves symplectic structures, and (more relevant to our present discussion) the rational blow-down procedure of Fintushel-Stern [3, 15] is also symplectic [17, 18]. In the rational blow-down construction the tubular neighbourhood of embedded spheres (with certain intersection and self-intersection patterns) is replaced by some other 4-manifold with boundary. This tubular neighbourhood can be identified with the resolution of some isolated complex surface singularity, while the other 4-manifold in the construction is diffeomorphic to a particular

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smoothing of the same singularity. This surgery operation (of replacing the resolution of a singularity with one of its smoothings) have been extended to further singularities in [4, 5] and ultimately to all singularities and any smoothings in [14].

In the original rational blow-down construction one considered singularities with resolutions involving only rational curves, while in the extension of [14] higher genus curves in the resolutions are also allowed. Very few examples of singularities are known for which the resolution admits higher genus surfaces, and the characteristic numbers of some smoothings of the singularities are described. Such examples seem to be extremally important in the study of the following two questions: (a) do symplectic 4-manifolds with characteristic numbers c_1^2 and c_2 violating the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ (established for complex surfaces, cf. [1], and for symplectic 4-manifolds admitting Einstein metrics [8]) exist? Indeed, for complex surfaces (and symplectic 4-manifolds admitting Einstein metrics) the equality $c_1^2 = 3c_2$ implies that the manifold is a quotient of the 4-ball, and hence admits infinite fundamental group. Therefore the related question is equally interesting: (b) do symplectic 4-manifolds with finite fundamental group and $c_1^2 = 3c_2$ exist?

The improvement of the Bogomolov-Miyaoka-Yau (BMY) inequality from [10] shows that complex surfaces near the BMY line (i.e., where $c_1^2 = 3c_2$ holds) do not contain rational curves. This result suggests that in constructing potential interesting symplectic 4-manifolds using the surgery operation discussed above (replacing the resolution with a smoothing) it is inevitable to consider singularities with resolutions admitting higher genus curves.

With this motivation, we have the following result:

Theorem 1.1. *Let p be a positive prime number with $p \leq 7$ and let s and t be positive integers with $s \equiv -1$ and $t \equiv 1 \pmod p$. Then there is an isolated complex surface singularity $(S_{s,t,p}, 0)$ whose minimal resolution consists of two complex curves A and B such that A and B intersect each other transversally once, $A^2 = -p-1$, $g(A) = \frac{(s-1)(p-1)}{2}$, and $B^2 = -1$, $g(B) = \frac{(t-1)(p-1)}{2}$ (where $g(A)$ and $g(B)$ denote the genera of the curves, and A^2, B^2 are the self-intersections).*

The singularity $(S_{s,t,p}, 0)$ admits a smoothing (or Milnor fiber) M with topological data as follows:

(a) $b_1(M) = 0$.
(b) *The Milnor number $\mu(M) = b_2(M)$ is given by*

$$\mu(M) = (p-1)((s+t)(s+t-1) + pt(t-1) + 1 - s - t).$$

(c) *The canonical class*

$$K = \frac{(p-1)(s+t-2)+p-2}{-p}A + \frac{(s-1)(p-1)+(t-1)(p-1)(p+1)-2}{-p}B.$$

(d) *The signature $\sigma(M)$ of the Milnor fiber M is*

$$\sigma = -\frac{1}{3}(2\mu + K^2 + 2 + 2(p-1)(s+t-2)).$$

We will also show symplectic 4-manifolds containing symplectic surfaces which intersect according to the intersection patterns of the resolutions of the singularities encountered above. The above data can be used to determine the characteristic numbers of the symplectic 4-manifolds we get by replacing the tubular neighbourhoods of the two curves with the smoothings.

The paper is organized as follows. In Section 2 we construct isolated complex surface singularities whose exceptional curves consist of two curves of high genera, and we determine the topological data of the resolutions and the smoothings. In Section 3 we construct symplectic 4-manifolds which contain the curves A and B with properties in Theorem 1.1. We then perform symplectic smoothing surgery to get other symplectic 4-manifolds and we determine the characteristic numbers of the resulting symplectic 4-manifolds.

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2. The singularities

In this section we construct the isolated complex surface singularities claimed in Theorem 1.1 and we compute the topological data of smoothings of the singularities.

The singularity

Let s and t be positive integers and p be a prime number such that $s+t \equiv 0 \pmod p$ and $t \equiv 1 \pmod p$. Consider the hypersurface singularity $(S,0) = (S_{s,t,p},0)$ in $\mathbb{C}^3$ given by the equation

$$(x^s + y^s)(x^t + y^{(p+1)t}) + z^p = 0. \tag{1}$$

The minimal resolution

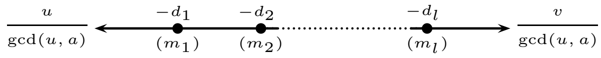
One can find the resolution graph of $(S,0)$ by the fact that $\{(x^s+y^s)(x^t + y^{(p+1)t}) + z^p = 0\}$ is a p-fold cover of $\mathbb{C}^2$ branched along $\{(x^s+y^s)(x^t+y^{(p+1)t}) = 0\}$.

We now introduce notations defined in [12, 13] to explain the process of finding the minimal resolution graph.

Definition 2.1 ([12, 13])**.** Let u, v and a be positive integers with $\gcd(u,v,a) = 1$. Find integer $0 \le x < \frac{a}{\gcd(u,a)}$ such that

$$v + x \cdot \frac{u}{\gcd(u,a)} \equiv 0, \quad (\mathrm{mod}\ \frac{a}{\gcd(u,a)}) \tag{2}$$

If $x \neq 0$, the graph $G(u, v, a)$ is defined by the following linear chain with the dual graph



where each vertex represents spheres with self-intersection numbers $-d_i$ such that

$$\frac{\frac{a}{\gcd(u,a)}}{x} = d_1 - \cfrac{1}{d_2 - \cfrac{1}{\ddots - \cfrac{1}{d_l}}}, \quad d_i \geqslant 2 \tag{3}$$

and the arrowheads represent surfaces having multiplicities $\frac{u}{\gcd(u,a)}$ and $\frac{v}{\gcd(v,a)}$. Here the multiplicity m_1 satisfies the equation

$$v + x \cdot \frac{u}{\gcd(u,a)} = m_1 \cdot \frac{a}{\gcd(u,a)} \tag{4}$$

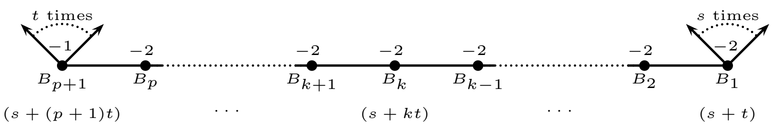
and m_i ($i \geqslant 2$) is obtained by the equations

$$-d_1 m_1 + \frac{u}{\gcd(u,a)} + m_2 = 0 \text{ and} - d_i m_i + m_{i-1} + m_{i+1} = 0 \text{ for } i \geqslant 2. \tag{5}$$

If $x = 0$, $G(u, v, a)$ is defined by only one edge with no vertex.

In [12, 13] there is an algorithm to find the minimal resolution of the singularity $(S, 0)$. Below we explain the procedure briefly in our case (see [12, 13] for further details). The procedure involves four steps.

Step 1: The dual graph of the embedded resolution of the plane curve $\{(x^s + y^s)(x^t + y^{(p+1)t}) = 0\}$ is given by the following graph

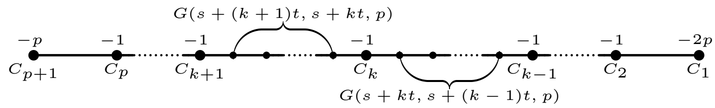


in $\mathbb{C}^2 \sharp (p+1)\overline{\mathbb{CP}^2}$ ($\mathbb{C}^2$ blown-up $(p+1)$ times), where vertices B_k represent exceptional curves and arrowheads represent proper transforms.

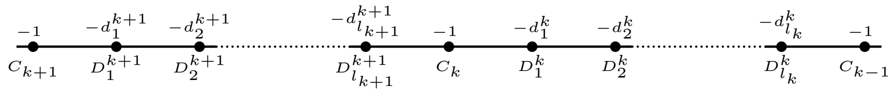
Step 2: Consider a ramified p-fold cover of $\mathbb{C}^2 \sharp (p+1)\overline{\mathbb{CP}^2}$ whose branch locus is the total transform of $\{(x^s + y^s)(x^t + y^{(p+1)t}) = 0\}$. Then there are singularities given by $x^{s+kt} y^{s+(k+1)t} = z^p$ on the ramification locus in the p-fold covering space.

Step 3: In [12, 13], there is an algorithm for finding a resolution graph of the singularities given by $x^{s+kt} y^{s+(k+1)t} = z^p$. The resolution corresponding to the singularity $x^{s+kt} y^{s+(k+1)t} = z^p$ is $G(s + kt, s + (k+1)t, p)$: Vertices and arrowheads in $G(s + kt, s + (k+1)t, p)$ represent exceptional curves and the proper transform of $x^{s+kt} y^{s+(k+1)t} = z^p$ respectively if $x \neq 0$, and there is no exceptional curve if $x = 0$. Thus a resolution of the singularity $(S, 0)$ is the union of the proper transforms C_k of the ramification locus and curves D_j^{k+1} from vertices of $G(s+kt, s+(k+1)t, p)$. Therefore we get the following lemma.

Lemma 2.2. *The dual graph of a resolution of* $(S, 0)$ *is equal to the following graph*:



In more detail,



where

(a) *the genus of* C_1 *and* C_{p+1} *are* $\frac{(s-1)(p-1)}{2}$ *and* $\frac{(t-1)(p-1)}{2}$ *respectively, and* $C_1^2 = -2p$ *and* $C_{p+1}^2 = -p$,

(b) *the genera of* C_k *are zero and* $C_k^2 = -1$ $(2 \leq k \leq p)$,

(c) *the genera of* D_i^j *are zero and the self-intersection numbers of* D_i^j, *say* $-d_i^j$, *satisfy the following equations*:

$$\frac{p}{k^* - 1} = d_1^k - \cfrac{1}{d_2^k - \cfrac{1}{\ddots - \cfrac{1}{d_{l_k}}}} \quad (3 \leq k \leq p)$$

and

$$\frac{p}{p - 1 - (k-1)^*} = d_{l_{k+1}}^{k+1} - \cfrac{1}{d_{l_{k+1}-1}^{k+1} - \cfrac{1}{\ddots - \cfrac{1}{d_1^{k+1}}}} \quad (2 \leq k \leq p-1),$$

where c^* *is the number such that* $0 \leq c^* \leq p-1$ *and* $c^* c \equiv 1 \pmod{p}$.

Proof. Since the multiplicities of B_1 and B_{p+1} are divisible by p, C_1 and C_{p+1} are p-fold coverings of B_1 and B_{p+1} with $s+1$ and $t+1$ branch points respectively. Then, by the Riemann-Hurwitz formula, (a) is proved. The curves C_k are proper transforms of ramification loci corresponding to the branch loci B_k $(2 \leq k \leq p)$ because the multiplicities of B_k are relatively prime to p $(2 \leq k \leq p)$. Thus the genera of C_k are zero. Since $s + t \equiv 0 \pmod{p}$ and $s + (p+1)t \equiv 0 \pmod{p}$, $G(s+t, s+2t, p)$ and $G(s+pt, s+(p+1)t, p)$ are only edges with no vertices, respectively. On the other hand, $G(s+kt, s(k+1)t, p)$ gives exceptional curves D_i^{k+1} for $2 \leq k \leq p-1$. The self-intersection numbers $-d_i^j$ of D_i^j can be computed by Equation (3). Consider Equation (2) for

$G(s+kt, s+(k-1)t, p)$ for $3 \le k \le p$:

$$s+(k-1)t + x_k \cdot (s+kt) \equiv 0 \pmod{p}.$$

Since $s+t \equiv 0 \pmod{p}$ and $t \equiv 1 \pmod{p}$, we have $(k-1)x_k + k - 2 \equiv 0 \pmod{p}$. By Fermat's theorem, we have $(k-1)^{p-1} \equiv 1 \pmod{p}$. Multiply $(k-1)x_k + k - 2 \equiv 0 \pmod{p}$ with $(k-1)^{p-2}$ and get $x_k \equiv (k-1)^{p-2} - 1 \pmod{p}$. Consequently, we have $x_k = (k-1)^* - 1$, where c^* is the number such that $0 \le c^* \le p-1$ and $c^* c \equiv 1 \pmod{p}$. Consider $G(s+kt, s+(k+1)t, p)$ for $2 \le k \le p-1$ to find d_i^{k+1}:

$$s+(k+1)t + x_{k+1} \cdot (s+kt) \equiv 0 \pmod{p}.$$

With the same process, we get $x_{k+1} = p-1-(k-1)^*$, providing the proof of (c).

To prove (b), let $(\widetilde{S}, E) \xrightarrow{\pi} (S, 0)$ be the resolution given by the above process, where $E = \cup E_i$ are the exceptional curves, that is, $E = \cup E_i = \cup C_i \cup (\cup D_i)$. Define a polynomial map $f : (S, 0) \to (\mathbb{C}, 0)$ by $f(x,y,z) = (x^s + y^s)(x^t + y^{(p+1)t}) + z^p$. The pullback of $f \circ \pi$ determines a principal divisor, denoted by $(f \circ \pi)$. Then

$$(f \circ \pi) = \Sigma_i n_i E_i + \overline{\pi^{-1}(\{f=0\} - \{0\})},$$

where n_i is multiplicity of $f \circ \pi$ along E_i. Since the homology class $[(f \circ \pi)]$ in $H_2(\widetilde{S}, \partial\widetilde{S}; \mathbb{Z})$ is zero, $[(f \circ \pi)] \cdot C_i = 0$. This implies that

$$(s+kt)C_k^2 + m_1^k + m_{l_{k+1}}^{k+1} = 0,$$

where m_i^j is multiplicity of D_i^j. Therefore

$$C_k^2 = \frac{-m_1^k - m_{l_{k+1}}^{k+1}}{s+kt}.$$

Furthermore we get

$$m_1^k = \frac{s+(k-1)t + ((k-1)^* - 1)(s+kt)}{p},$$
$$m_{l_{k+1}}^{k+1} = \frac{s+(k+1)t + (p-1-(k-1)^*)(s+kt)}{p}$$

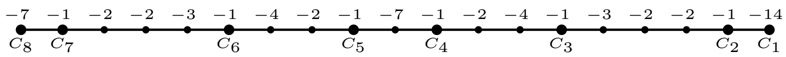
from Equation (4) on $G(s+kt, s+(k-1)t, p)$ and $G(s+kt, s+(k+1)t, p)$. Therefore we get $C_k^2 = -1$, completing the proof of (b). □

Step 4: Finally blowing down (-1)-curves, we get the minimal resolution of the singularity $(S, 0)$.

Proposition 2.3. *For $p \le 7$ the minimal resolution graph has the the following properties*:

(a) *The resolution consists of the union of two curves A and B, intersecting each other transversally once.*
(b) *$A^2 = -p-1$ and $g(A) = \frac{1}{2}(s-1)(p-1)$ while $B^2 = -1$ and $g(B) = \frac{1}{2}(t-1)(p-1)$.*

Proof. Consider the case $p = 7$. A resolution graph of $(S, 0)$ is the following:



It is easy to check that after blowing down the (rational) (-1)-curves in the above graph, the resulting graph will consist of two curves A and B with the stated properties. The further cases of $p < 7$ and prime work similarly. $\square$

Remark 2.4. We expect that the blow-down of the graph of Lemma 2.2 has properties (a) and (b) of Proposition 2.3 for every prime number p and positive integers s and t such that $s + t \equiv 0 \pmod{p}$ and $t \equiv 1 \pmod{p}$. (Indeed, we expect that the minimal resolution has the above properties (a) and (b) in the Proposition 2.3 for every positive numbers p, s, and t with $s + t \equiv 0 \pmod{p}$ and $\gcd(p, t) = 1$.)

Topological data of the smoothing

We start with a short generic discussion about the computation of topological data of the Milnor fiber of a hypersurface singularity. Suppose therefore that $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defines the isolated singularity $(S, 0)$ and $p : (\tilde{S}, E) \to (S, 0) = (f^{-1}(0), 0)$ is its minimal good resolution. We write the exceptional divisor $p^{-1}(0) = E$ as the union of irreducible components: $E = E_1 \cup \cdots \cup E_m$. Let $h = \operatorname{rank} H_1(E)$ and $p_g = \dim_{\mathbb{C}} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$. The canonical class K of $\tilde{S}$ can be written as $\sum r_i E_i$, where the r_i are rational numbers, determined by the adjunction formula $2g(E_i) - 2 = E_i^2 + K \cdot E_i$. The Milnor number and the signature of the Milnor fiber of the singularity of $f^{-1}(0)$ can be computed as follows:

Proposition 2.5 ([2]). *The Milnor number* $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ *is equal to* $\mu = K^2 - h + m + 12p_g$. *The signature of the Milnor fiber is equal to* $\sigma = -\frac{1}{3}(2\mu + K^2 + m + 2h)$, *where m is the number of irreducible components of exceptional curves E in the minimal good resolution and* $h = \operatorname{rank} H_1(E)$.

The singularity given by Equation (1) is given as a ramified cover along a singular plane curve. For an isolated plane curve singularity the Milnor number satisfies the following equation.

Proposition 2.6 ([11]). *For an isolated plane curve singularity* $(C, 0) \subset (\mathbb{C}^2, 0)$

$$\mu(C, 0) = d(d-1) + \sum_{x \in Sing(\tilde{C})} \mu(\widetilde{C}, x) + 1 - r,$$

where $\widetilde{C}$ is the proper transform of C after one blow-up at the singular point 0, the sum runs through all singular points x of $\widetilde{C}$ lying over 0, r is the number of different tangent lines of $(C, 0)$, and d is the multiplicity of C at 0.

Regarding the first Betti number of an isolated singularity, we have:

Proposition 2.7 ([7]). *Let X_t be the Milnor fiber of a smoothing of a pure-dimensional isolated normal surface singularity $(X_0, 0)$, then $b_1(X_t) = 0$.*

Using the above formulae, we can compute topological invariants of the smoothings of $(S, 0)$.

Lemma 2.8. *For the singularity $(S, 0) = (S_{s,t,p}, 0)$ specified by the function of Equation (1) we have*

(a) *The Milnor number*

$$\mu = (p-1)((s+t)(s+t-1) + pt(t-1) + 1 - s - t).$$

(b) *The canonical class*

$$K = \frac{(p-1)(s+t-2)+p-2}{-p}A + \frac{(s-1)(p-1)+(t-1)(p-1)(p+1)-2}{-p}B.$$

(c) *The signature σ of the Milnor fiber is*

$$\sigma = -\frac{1}{3}(2\mu + K^2 + m + 2h)$$

with $m = 2$, $h = (p-1)(s+t-2)$. So we can calculate the signature σ explicitly by (a) *and* (b).

Proof. We have

$$\begin{aligned}\mu(S,0) &= \dim_{\mathbb{C}} \mathbb{C}\{x,y,z\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \\ &= (p-1)\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\ &= (p-1)\mu((x^s+y^s)(x^t+y^{(p+1)t}) = 0, 0).\end{aligned}$$

Let H_0 be the plane curve singularity $((x^s+y^s)(x^t+y^{(p+1)t}) = 0, 0)$ in $\mathbb{C}^2$. Let H_1 be the proper transform of H_0 in $\mathbb{C}^2\sharp\overline{\mathbb{CP}^2}$ after one blow-up at the singular point 0 and let H_{k+1} be the proper transform in $\mathbb{C}^2\sharp(k+1)\overline{\mathbb{CP}^2}$ of H_k after one blow-up at the unique infinitely near singular point, say h_k, for $0 \leq k \leq p$. We can check that 0 in H_0 is a singular point with multiplicity $s+t$ and has $s+1$ different tangent line. Thus we have

$$\mu(H_0, 0) = (s+t)(s+t-1) + \Sigma_{h \in Sing(H_1)}\mu(H_1, x) + 1 - s - 1$$

from Proposition 2.6. It is not difficult to show that the curve H_k has only one infinitely near singular point h_k lying over 0 with multiplicity t and one tangent line for $1 \leq k \leq p-1$, and H_p has only one infinitely near singular point h_p with multiplicity t and t different tangent lines, and the singularity $(H_0, 0)$ is resolved after $p+1$ times blowing up. Therefore by Proposition 2.6

$$\mu(H_k, h_k) = t(t-1) + \mu(H_{k+1}, h_{k+1}) \text{ for } 1 \leq k \leq p-1$$

and

$$\mu(H_p, h_p) = t(t-1) + 1 - t.$$

Thus we have

$$\mu(S,0) = (p-1)((s+t)(s+t-1) + pt(t-1) + 1 - s - t),$$

hence (a) is proved. By the adjunction formula (b) follows easily. Since the exceptional curve E consists of two curves A and B with genera $\frac{(s-1)(p-1)}{2}$ and $\frac{(t-1)(p-1)}{2}$ respectively, we have $m = 2$, $h = (p-1)(s+t-2)$. Then K^2 can be calculated explicitly by (b). We can compute σ by Proposition 2.5, completing the proof of (c). □

Proof of Theorem 1.1. Composing the results of Proposition 2.3 and Proposition 2.8, the claim of the theorem follows at once. □

3. An example of symplectic surgery on the singularity

In this section we give examples of symplectic manifolds which contain the curve configuration (A, B) described in Section 2. Let s and t be positive integers and p be a prime number such that $s+t \equiv 0 \pmod p$, $t \equiv 0 \pmod p$, and $p \leq 7$. Consider the projective line $\mathbb{CP}^1$ and a complex curve Σ with genus $\frac{(s-1)(p-1)}{2}$, and fix points $p_1, \dots, p_{2(p+1)}$ in Σ and $q_1, \dots, q_{tp-t-p+3}$ in $\mathbb{CP}^1$. Define the complex curve $C = (\bigcup_{i=1}^{tp-t-p+3}(\Sigma \times \{q_i\})) \bigcup (\bigcup_{j=1}^{2p+2}(\{p_j\} \times \mathbb{CP}^1))$ and take the branched double covering of $\Sigma \times \mathbb{CP}^1$ along C. After desingularization, we get a genus $\frac{(t-1)(p-1)}{2}$ Lefschetz fibration $X \to \Sigma$ admitting sections with self-intersection number $-p-1$. Let M denote the blow-up of X in a regular fiber. The fiber passing through the blown-up point, together with a section now provides the configuration of two curves (A, B) with intersection patterns as in the resolution graph of the singularity given by Equation (1) in Section 2.

Applying the symplectic smoothing surgery operation of replacing the neighbourhood $\nu(A \cup B)$ with the smoothing W of the corresponding singularity, we get a symplectic 4-manifold M_W (where the existence of the symplectic structure follows from [14]).

The topological invariants (the first betti number, Euler characteristic, and signature) of the symplectic manifold M_W can be computed from the topological invariants of M as follows.

Proposition 3.1. (a) *The first betti number $b_1(M_W)$ is zero.*

(b) *The topological Euler characteristic of M_W is equal to*

$$\begin{aligned} e(M_W) = {} & 9 + 5p - 3p^2 + 2s - ps - p^2 s - s^2 + ps^2 - 2t + 2p^2 t \\ & - st + p^2 st - t^2 + p^2 t^2. \end{aligned}$$

(c) *The signature is*

$$\begin{aligned} \sigma(M_W) = {} & \frac{17}{3} - \frac{38p}{3} + p^2 - 2p^3 - p^4 - \frac{16s}{3} + \frac{10ps}{3} - p^2 s + 3p^3 s \\ & - \frac{s^2}{3} + \frac{ps^2}{3} + p^2 s^2 - p^3 s^2 - \frac{10t}{3} + \frac{8pt}{3} - \frac{10p^2 t}{3} + 2p^3 t + 2p^4 t \end{aligned}$$

$$-\frac{2st}{3}-\frac{pst}{3}+4p^2st-3p^3st-\frac{t^2}{3}+\frac{4p^2t^2}{3}-p^4t^2.$$

Proof. Since the embedding map $A\cup B\to X$ is onto on the first homology, Proposition 2.7 implies that $b_1(M_W)=0$. We have $e(W)$ and $\sigma(W)$ by Lemma 2.8. Thus the topological Euler characteristic

$$\begin{aligned}
e(M_W) &= e(M\setminus\nu(A\cup B))+e(W)=e(M)-e(A\cup B)+e(W)\\
&= e(M)+e(W)-3+(p-1)(s+t-2)\\
&= (10+6p-3p^2+s-p^2s-3t+3p^2t+st-2pst+p^2st)\\
&\quad +(p+2s-2ps-s^2+ps^2+2t-pt-p^2t-2st+2pst-t^2+p^2t)\\
&\quad -3+(p-1)(s+t-2)\\
&= 9+5p-3p^2+2s-ps-p^2s-s^2\\
&\quad +ps^2-2t+2p^2t-st+p^2st-t^2+p^2t^2,
\end{aligned}$$

and the signature

$$\begin{aligned}
\sigma(M_W) &= \sigma(M\setminus\nu(A\cup B))+\sigma(W)=\sigma(M)-\sigma(A\cup B)+\sigma(W)\\
&= \sigma(M)+\sigma(W)+2\\
&= (-7-4p+2p^2+2t-2p^2t)+(\frac{32}{3}-\frac{26p}{3}-p^2-2p^3-p^4-\frac{16s}{3}\\
&\quad +\frac{10ps}{3}-p^2s+3p^3s-\frac{s^2}{3}+\frac{ps^2}{3}+p^2s^2-p^3s^2-\frac{16t}{3}+\frac{8pt}{3}\\
&\quad -\frac{4p^2t}{3}+2p^3t+2p^4t-\frac{2st}{3}-\frac{pst}{3}+4p^2st-3p^3st-\frac{t^2}{3}\\
&\quad +\frac{4p^2t^2}{3}-p^4t^2)+2\\
&= \frac{17}{3}-\frac{38p}{3}+p^2-2p^3-p^4-\frac{16s}{3}+\frac{10ps}{3}-p^2s+3p^3s-\frac{s^2}{3}\\
&\quad +\frac{ps^2}{3}+p^2s^2-p^3s^2-\frac{10t}{3}+\frac{8pt}{3}-\frac{10p^2t}{3}+2p^3t+2p^4t-\frac{2st}{3}\\
&\quad -\frac{pst}{3}+4p^2st-3p^3st-\frac{t^2}{3}+\frac{4p^2t^2}{3}-p^4t^2.
\end{aligned}$$

□

Symplectic 4-manifolds containing similar configurations of symplectic submanifolds can be found near the Bogomolov-Miyaoka-Yau (BMY) line $c_1^2=3c_2$. (For 4-manifold near the BMY line, see [16].) We hope that using the symplectic smoothing surgery, one will be able to construct symplectic manifolds with $b_1=0$ (or even with $\pi_1=0$) on the BMY line. We hope to return to this question in a future project.

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