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SOME MONOMIAL SEQUENCES ARISING FROM GRAPHS

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Abstract. s-sequences and d-sequences are fundamental sequences intensively studied in many fields of algebra. In this paper we are interested in dealing with monomial sequences associated to graphs in order to establish conditions for which they are s-sequences and/or d-sequences.

1. Introduction

In this work we consider monomial sequences establishing conditions for which they are s-sequences and d-sequences in order to deduce some properties of the symmetric algebra of monomial ideals, in particular of some monomial ideals arising from graphs.

In [2] the notion of d-sequence was firstly given by Huneke for the study of Rees rings. In [1] the notion of s-sequence is employed to compute the invariants of the symmetric algebra of finitely generated modules and it is proved that any d-sequence is a strong s-sequence.

Some properties about monomial s-sequences are studied. In [1] it is given a necessary and sufficient condition for monomial sequences of length three to be s-sequences. Afterwards, in [6] it is shown a necessary and sufficient condition for monomial squarefree sequences of length four, and in [5] a more general statement for monomial sequences of any length related to forests. Our aim is to find necessary and sufficient conditions for monomial sequences of length greater than four to be s-sequences. Some results are obtained for edge sequences associated to very important classes of graphs and components of them.

The d-sequences have been intensively studied by many algebraists, the monomial d-sequences are characterized in [6]. Our aim is to investigate the notion of d-sequence for the monomial ideals arising from simple graphs in order to compute standard algebraic invariants of their symmetric algebra in terms of the corresponding invariants of special quotients of the polynomial ring related to the graphs.

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The paper is organized as follows. Section 2 is devoted to discuss monomial s-sequences. We introduce some classes of acyclic graphs for which, using the theory of Gröbner bases, we prove conditions in order that their edge ideals are generated by s-sequences. In Section 3 we deduce some properties of the symmetric algebra of such graph ideals generated by d-sequences. More precisely, we compute the following invariants of the symmetric algebra of such path ideals generated by a d-sequence: the dimension, the multiplicity and the Castelnuovo-Mumford regularity.

2. Monomial s-sequences associated to graphs

Main definitions and notations principally come from [4], [6] and [7].

Let $\mathcal{G}$ be a graph, $V(\mathcal{G})$ and $E(\mathcal{G})$ be the sets of its vertices and edges respectively. $\mathcal{G}$ is said to be *simple* if, for all $\{v_i, v_j\} \in E(\mathcal{G})$, it is $v_i \neq v_j$. $\mathcal{G}$ is *connected* if it has no isolated subgraphs.

Let $\mathcal{G}$ be a graph with vertex set $[n] = \{v_1, \ldots, v_n\}$. Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K with one variable X_i for each vertex v_i and $I(\mathcal{G}) = \big(\{X_iX_j \,|\, \{v_i, v_j\} \in E(\mathcal{G}),\, i \neq j\}\big)$ be the *edge ideal* associated to $\mathcal{G}$ generated by degree two squarefree monomials of R.

Let's recall the theory of s-sequences in order to apply it to some classes of edge ideals.

Let M be a finitely generated module on a Noetherian ring R, and $f_1, \ldots, f_t$ be the generators of M. Let (a_{ij}), for $i = 1, \ldots, t$, $j = 1, \ldots, p$, be the relation matrix of M. Let $Sym_R(M)$ be the symmetric algebra of M, then $Sym_R(M) = R[T_1, \ldots, T_t]/J$, where $R[T_1, \ldots, T_t]$ is a polynomial ring in the variables $T_1, \ldots, T_t$ and J the *relation ideal*, namely the ideal generated by $g_j = \sum_{i,j} a_{ij}T_i$ for $i = 1, \ldots, t$, $j = 1, \ldots, p$.

If we assign degree 1 to each variable T_i and degree 0 to the elements of R, then J is a graded ideal and $Sym_R(M)$ is a graded algebra on R.

Set $S = R[T_1, \ldots, T_t]$ and let $\prec$ be a term order on the monomials of S. With respect to it, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_t^{\alpha_t}$ and $\alpha = (\alpha_1, \ldots, \alpha_t) \in \mathbb{N}^t$, we put $\mathrm{in}_\prec(f) = a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$. So we can define the monomial ideal $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) \,|\, f \in J\})$.

For every $i = 1, \ldots, t$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and let $I_i = M_{i-1} :_R f_i$ be the colon ideal. Since $M_i/M_{i-1} \simeq R/I_i$, I_i is the annihilator of the cyclic module R/I_i. I_i is called an *annihilator* ideal of the sequence $f_1, \ldots, f_t$.

It is $(I_1T_1, I_2T_2, \ldots, I_tT_t) \subseteq \mathrm{in}_\prec(J)$, and the two ideals coincide in degree 1.

Definition 2.1. The sequence $f_1, \ldots, f_t$ is said to be an *s-sequence* for M if

$$(I_1T_1, I_2T_2, \ldots, I_tT_t) = \mathrm{in}_\prec(J).$$

When $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$, $f_1, \ldots, f_t$ is said to be a *strong* s-sequence.

We can use the Gröbner bases theory to compute $\mathrm{in}_\prec(J)$. Let $\prec$ be any term order on $S = K[X_1, \ldots, X_n, T_1, \ldots, T_t]$ with $X_i \prec T_j$ for all i, j. Then for any Gröbner basis B for $J \subset S$ with respect to $\prec$, we have $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) \,|\, f \in$

$B\})$. If the elements of B are linear in the T_i, then $f_1, \ldots, f_t$ is an s-sequence for M.

Let $M = I = (f_1, \ldots, f_t)$ be a monomial ideal of R. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$, $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leq i < j \leq t$. The monomial sequence $f_1, \ldots, f_t$ is an s-sequence if and only if g_{ij} is a Gröbner basis for J with respect to the term order $\prec$.

Notice that the annihilator ideals of the monomial sequence $f_1, \ldots, f_t$ are the ideals $I_i = (f_{1i}, f_{2i}, \ldots, f_{i-1,i})$ for $i = 1, \ldots, t$ (see [1]).

Remark 2.1 ([1, Lemma 1.4]). From the theory of Gröbner bases, if $f_1, \ldots, f_t$ is a monomial s-sequence with respect to some admissible term order $\prec$, then $f_1, \ldots, f_t$ is an s-sequence for any other admissible term order.

In [1, Proposition 1.7], it is shown that a monomial sequence $f_1, \ldots, f_t$ is an s-sequence if $[f_{ij}, f_{kl}] = 1$ for all $i, j, k, l \in \{1, \ldots, t\}$ with $i < j$, $k < l$, $i \neq k$, and $j \neq l$.

Moreover, for monomial sequences of length 3, the above condition is necessary, see [1, Proposition 1.8].

Afterwards, in [6, Theorem 4.1], it was proved that a monomial sequence $f_1, \ldots, f_t$ is an s-sequence if $[f_{ij}, f_{kl}] = 1$ or $f_{jl}[f_{ij}, f_{kl}] \mid f_{kl}f_{ji}$, or in case $i > k$, $f_{ki}[f_{ij}, f_{kl}] \mid f_{kl}f_{ji}$ for any $i < j$, $k < l$, $j < l$ and $k \neq i$.

Furthermore, for monomial *squarefree* sequences of length 4, the above condition is also necessary, see [6, Proposition 4.7].

Using [1, Proposition 1.7], and [6, Theorem 4.1], we introduce some classes of graphs $\mathcal{G}$ for which necessary and sufficient conditions hold in the case that their edge ideals are generated by s-sequences of length $t > 4$.

Theorem 2.1. *Let $\mathcal{G}$ be the connected acyclic graph whose edge ideal $I(\mathcal{G})$ is generated by $f_1 = X_1X_n, f_2 = X_2X_n, \ldots, f_{n-1} = X_{n-1}X_n$ in $R = K[X_1, \ldots, X_n]$ for $n - 1 = t$. Then $f_1, f_2, \ldots, f_{n-1}$ is an s-sequence if and only if $[f_{ij}, f_{kl}] = 1$ for all $i < j, k < l, i \neq k$ and $j \neq l$, $i, j, k, l \in \{1, \ldots, n-1\}$.*

Proof. Let $f_1 = X_1X_n$, $f_2 = X_2X_n, \ldots, f_{n-1} = X_{n-1}X_n$ be an s-sequence. We show that $[f_{ij}, f_{kl}] = 1$ for all $i < j$, $k < l$, $i \neq k$ and $j \neq l$, $i, j, k, l \in \{1, \ldots, n-1\}$. The s-sequence property implies that $B = \{g_{ij} = f_{ij}T_j - f_{ji}T_i\}$ for $1 \leq i < j \leq n-1$ is a Gröbner basis of J. Hence the S-pairs

$$S(g_{ij}, g_{kl}) = \frac{f_{ij}f_{lk}}{[f_{ij}, f_{kl}]}T_jT_k - \frac{f_{kl}f_{j\,i}}{[f_{ij}, f_{kl}]}T_iT_l$$

with $i, j, k, l \in \{1, \ldots, n-1\}$, $i < j$, $i < k < l$, have a standard expression with respect to B with remainder 0. Note that, to get a standard expression of $S(g_{ij}, g_{kl})$ is equivalent to find some $g_{st} \in B$ whose initial term divides the initial term of $S(g_{ij}, g_{kl})$ and substitute a multiple of g_{st} such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. By [4, Theorem 3.1], one has the standard expression $S(g_{ij}, g_{kl}) = g_{ij}T_l - g_{kl}T_j$. Then there exists $g_{ij} \in B$ whose initial term divides the initial term of $S(g_{ij}, g_{kl})$. It follows that $f_{ij}T_j \;\Big|\; \frac{f_{ij}f_{lk}}{[f_{ij},f_{kl}]}T_jT_k$, then $f_{ij} \;\Big|\; \frac{f_{ij}f_{lk}}{[f_{ij},f_{kl}]}$ and $[f_{ij}, f_{kl}] \mid f_{lk}$. Being $[[f_{ij}, f_{kl}], f_{kl}] = 1$, one has $[f_{ij}, f_{kl}] = 1$.

Conversely, we have $[f_{ij}, f_{kl}] = [X_i, X_k] = 1$ for all $i < j$, $k < l$, $i \neq k$ and $j \neq l$, $i, j, k, l \in \{1, \ldots, n-1\}$. Hence, by [1, Proposition 1.7], one can state that $I(\mathcal{G})$ is generated by an s-sequence (see also [3, Theorem 2.2]). □

Theorem 2.2. *Let $\mathcal{G}$ be the connected acyclic graph whose edge ideal $I(\mathcal{G})$ is generated by $f_1 = X_1X_2, f_2 = X_2X_3, \ldots, f_{n-1} = X_{n-1}X_n$ in $R = K[X_1, \ldots, X_n]$, for $n-1 = t$. Then $f_1, f_2, \ldots, f_{n-1}$ is an s-sequence if and only if $[f_{ij}, f_{kl}] = 1$ or $f_{jl}[f_{ij}, f_{kl}] \mid f_{kl}f_{ji}$, or $f_{ki}[f_{ij}, f_{kl}] \mid f_{kl}f_{ji}$ in case $i > k$ for any $i < j$, $k < l$, $j < l$ and $i \neq k$, $i, j, k, l \in \{1, \ldots, n-1\}$.*

Proof. Let $f_1 = X_1X_2$, $f_2 = X_2X_3$, $\ldots$, $f_{n-1} = X_{n-1}X_n$ be an s-sequence. We show that $[f_{ij}, f_{kl}] = 1$ or $\frac{f_{kl}f_{ji}}{[f_{ij},f_{kl}]}$ is divided by f_{jl}, or by f_{ki} in case $i > k$, for any $i < j$, $k < l$, $j < l$ and $i \neq k$, $i, j, k, l \in \{1, \ldots, n-1\}$. The s-sequence property implies that $B = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq n-1\}$ is a Gröbner basis of J. Hence the S-pairs

$$S(g_{ij}, g_{kl}) = \frac{f_{ij}f_{lk}}{[f_{ij}, f_{kl}]}T_jT_k - \frac{f_{kl}f_{ji}}{[f_{ij}, f_{kl}]}T_iT_l$$

with $i, j, k, l \in \{1, \ldots, n-1\}$, $i < j$, $i < k < l$, have a standard expression with respect to B with remainder 0. First note that $[f_{ij}, f_{k,l}] = 1$ for $j = k$, because $f_1, \ldots, f_{n-1}$ are squarefree monomials. Moreover, by the structure of the monomials $f_1, \ldots, f_{n-1}$, one has $[f_{ij}, f_{kl}] = 1$ for $j \neq k$ and $i \neq k-1$. Otherwise, by [4, Theorem 3.1], one has the following standard expression:

$$S(g_{ij}, g_{kl}) = [f_{ji}, f_{lk}]\left(\frac{f_{jl}}{[f_{ik}, f_{jl}]}g_{ik}T_l - \frac{f_{ik}}{[f_{ik}, f_{jl}]}g_{jl}T_k\right).$$

Then there exists $g_{jl} \in B$, or, in case $i > k$, $g_{ik} \in B$, whose initial term divides the initial term of $S(g_{ij}, g_{kl})$. So, it follows that $f_{jl} \;\Big|\; \frac{f_{kl}f_{ji}}{[f_{ij},f_{kl}]}$ or, in case $i > k$, $f_{ki} \;\Big|\; \frac{f_{kl}f_{ji}}{[f_{ij},f_{kl}]}$. The thesis follows.

Conversely, because $f_1, \ldots, f_{n-1}$ are squarefree monomials, $[f_{ij}, f_{jl}] = 1$ and, by the structure of these monomials, one has $[f_{ij}, f_{kl}] = 1$ for $j \neq k$ and $k \neq i+1$. Otherwise one has $[f_{ij}, f_{kl}] = X_i$ for $i > k$, and $[f_{ij}, f_{kl}] = X_k$ for $i < k$. Then for $i > k$: $f_{jl}[f_{ij}, f_{kl}] | f_{kl}f_{ji}$, in fact $X_jX_{j+1}X_i | X_iX_kX_jX_{j+1}$, or $f_{ki}[f_{ij}, f_{kl}] | f_{kl}f_{ji}$, in fact $X_kX_i | X_iX_kf_{ji}$. For $i < k$: $f_{jl}[f_{ij}, f_{kl}] | f_{kl}f_{ji}$, in fact $X_jX_{j+1}X_k | X_iX_kX_jX_{j+1}$. Hence, by [6, Theorem 4.1], one can state that $I(\mathcal{G})$ is generated by an s-sequence (see also [3, Theorem 2.2]). □

3. Monomial *d*-sequences and symmetric algebras of graph ideals

The concept of d-sequence was firstly given by Huneke [2] for the study of Rees rings. We are interested in monomial d-sequences.

Let R be a Noetherian ring, $f_1, \dots, f_t \in R$ be a monomial sequence. We say that $f_1, \dots, f_t$ is *minimal* if it is a minimal system of generators of the ideal $I = (f_1, \dots, f_t)$, which is equivalent to f_i does not divide f_j for all $i \neq j$.

Definition 3.1. A monomial (minimal) sequence $f_1, \dots, f_t \in R$ is called a *d-sequence* if $(f_1, \dots, f_{i-1}) : f_i f_j = (f_1, \dots, f_{i-1}) : f_j$ for all i, j with $1 \leq i \leq j \leq t$, or, equivalently, if and only if

$$\Big(\frac{f_1}{[f_1, f_i f_j]}, \dots, \frac{f_{i-1}}{[f_{i-1}, f_i f_j]}\Big) = \Big(\frac{f_1}{[f_1, f_j]}, \dots, \frac{f_{i-1}}{[f_{i-1}, f_j]}\Big).$$

Further definitions and notations are in [6] and [4].

Lemma 3.1 ([6, Theorem 2.1])**.** *Let $f_1, \dots, f_t$ be a squarefree monomial sequence. Then $f_1, \dots, f_t$ is a d-sequence if and only if there is no $i \neq j$ such that $f_i \mid f_j$ and $[f_i, f_j] \mid f_k, \forall\, 1 \leq i < j < k \leq t$.*

Corollary 3.1 ([1, Corollary 3.3])**.** *Any d-sequence is a strong s-sequence.*

Corollary 3.2. *If $f_1, \dots, f_t$ is a d-sequence, then $(f_1, \dots, f_t)$ is an ideal of linear type.*

We are interested to give a classification of graphs $\mathcal{G}$ having generalized graph ideals generated by d-sequences.

Let's recall the definitions of certain examined graphs.

Cycle graphs on vertex set $[n]$, denoted by $\mathcal{C}_n$, consist of a unique cycle of length n, that is an alternating sequence of $n+1$ distinct vertices and n edges that begins and ends at the same vertex.

Complete graphs on vertex set $[n]$, denoted by $\mathcal{K}_n$, are those for which there exists an edge for all the possible pairs $\{v_i, v_j\}$ of vertices of it.

Star graphs on vertex set $[n] = \big\{\{v_i\}, \{v_1, \dots, v_{i-1}, v_{i+1}, v_n\}\big\}$ with center v_i, denoted by $\mathrm{star}_i(n), i = 1, \dots, n$, are complete bipartite graphs of the type $K_{1,n-1}$.

Definition 3.2. The *generalized graph ideal* of $\mathcal{G}$, denoted by $I_q(\mathcal{G})$, is the ideal of R generated by all the squarefree monomials $X_{i_1} \cdots X_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$ for all $1 \leq j \leq q-1$.

Definition 3.3. A *path* of length $q-1$ in $\mathcal{G}$, or $(q-1)$-*path*, is an alternating sequence of vertices and edges $\{v_1, f_1, v_2, \dots, v_{q-1}, f_{q-1}, v_q\}$, where $f_i = \{v_i, v_{i+1}\}$ is the edge joining v_i and v_{i+1}, and all the vertices are distinct.

Paths consisting of the same elements, different only on the order, are equal.

Remark 3.1. $I_q(\mathcal{G})$ is associated to the paths of length $q-1$ in $\mathcal{G}$. More precisely, the generators of $I_q(\mathcal{G})$ correspond to the $(q-1)$-paths in $\mathcal{G}$.

In particular, $I_2(\mathcal{G})$ is the generalized graph ideal generated by degree two squarefree monomials corresponding to the edges of $\mathcal{G}$. $I_2(\mathcal{G})$ is the *edge ideal* of $\mathcal{G}$, simply denoted by $I(\mathcal{G})$.

Now we expose a classification of graphs $\mathcal{G}$ having generalized graph ideals generated by d-sequences, for fixed $q \geq 3$.

Proposition 3.1. *Let $\mathcal{G}$ be a graph with $n \geq 3$ vertices, $I_3(\mathcal{G}) = (f_1, \ldots, f_t)$. Then $f_1, \ldots, f_t$ form a d-sequence if and only if $\mathcal{G}$ is one of the following*:
$\mathcal{C}_3 = \mathcal{K}_3$, the triangle;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$ and $I_3(\mathcal{G}) = (X_1X_2X_3)$;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ and $I_3(\mathcal{G}) = (X_1X_2X_3, X_2X_3X_4)$;
$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i=1,2,3,4\}$ and $I_3(\mathcal{G}) = (X_1X_2X_3, X_3X_4X_5, X_2X_3X_4)$;
any other one consisting of the union of two or more among them.

Proof. Let $\mathcal{G}$ be a graph on $[n]$ vertices. Suppose that $I_3(\mathcal{G}) = (f_1, \ldots, f_t)$ and $f_1, \ldots, f_t$ form a d-sequence, for any t. Then it is necessary that $\mathcal{G}$ is one of the graphs described in the statement.

Suppose by contradiction that $\mathcal{G}$ is the graph on [6] vertices with edge set $E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, 2, 3, 4, 5\}$ and consider $I_3(\mathcal{G}) = (X_1X_2X_3, X_2X_3X_4, X_3X_4X_5, X_4X_5X_6)$; regardless of the order, by Lemma 3.1, the sequence of the generators of $I_3(\mathcal{G})$ will never be a d-sequence. For such graphs it is possible to generalize to $[n]$ vertices.

The same reason can be done for graphs on $[n]$ vertices with edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \ldots, \{v_2, v_n\}\}$ and for any other graph.

Conversely, suppose that $\mathcal{G}$ is one of the graphs in the statement; in particular, without lack of generality, if it is the graph on [5] vertices with edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$ and $I_3(\mathcal{G}) = (f_1, f_2, f_3)$ such that $f_1 = X_1X_2X_3, f_2 = X_3X_4X_5, f_3 = X_2X_3X_4$, then f_1, f_2, f_3 is a d-sequence because $[f_1, f_2] = X_3$ divides f_3. $\square$

Proposition 3.2. *Let $\mathcal{G}$ be a graph with $n \geq 4$ vertices, $I_4(\mathcal{G}) = (f_1, \ldots, f_t)$. Then $f_1, \ldots, f_t$ form a d-sequence if and only if $\mathcal{G}$ is one of the following*:
$\mathcal{C}_4$, the square;
$\mathcal{S}_4 = \mathcal{C}_4 \cup \{$one diagonal$\}$;
$\mathcal{K}_4$, the square together with its diagonals;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ and $I_4(\mathcal{G}) = (X_1X_2X_3X_4)$;
$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, 2, 3, 4\}$ and $I_4(\mathcal{G}) = (X_1X_2X_3X_4, X_2X_3X_4X_5)$;
$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, 2, 3, 4, 5\}$ and $I_4(\mathcal{G}) = (X_1X_2X_3X_4, X_3X_4X_5X_6, X_2X_3X_4X_5)$;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \ldots, \{v_3, v_n\}\}$ and $I_4(\mathcal{G}) = (X_1X_2X_3X_4, \ldots, X_1X_2X_3X_n)$;
$\mathcal{C}_3 \cup star_\alpha(n-2)$, the union of $\mathcal{C}_3$ and the star graph with center a vertex v_α of $\mathcal{C}_3$;
any other one consisting of the union of two or more among them.

Proof. Let $\mathcal{G}$ be a graph on $[n]$ vertices. Suppose that $I_4(\mathcal{G}) = (f_1, \ldots, f_t)$ and $f_1, \ldots, f_t$ form a d-sequence, for any t. Then it is necessary that $\mathcal{G}$ is one of the graphs described in the statement.

Suppose by contradiction that $\mathcal{G}$ is the graph on $[7]$ vertices with edge set $E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, \ldots, 6\}$ and $I_4(\mathcal{G}) = (X_1X_2X_3X_4, X_2X_3X_4X_5, X_3X_4X_5X_6, X_4X_5X_6X_7)$; regardless of the order, by Lemma 3.1, the sequence of the generators of $I_4(\mathcal{G})$ never will be a d-sequence. For such graphs it is possible to generalize to $[n]$ vertices.

A similar reasoning can be made for any other graph.

Conversely, suppose that $\mathcal{G}$ is one of the graphs in the statement; in particular, without lack of generality, if it is the graph on $[n]$ vertices with edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \ldots, \{v_3, v_n\}\}$ and $I_4(\mathcal{G}) = (f_1, f_2, \ldots, f_t)$ such that $f_1 = X_1X_2X_3X_4, f_2 = X_1X_2X_3X_5, \ldots, f_t = X_1X_2X_3X_n$, then $f_1, \ldots, f_t$ is a d-sequence because $[f_i, f_j] = X_1X_2X_3$ divides $f_k, \forall\ 1 \leq i < j < k \leq t$. $\square$

Proposition 3.3. *Let $\mathcal{G}$ be a graph with $n \geq 5$ vertices, $I_5(\mathcal{G}) = (f_1, \ldots, f_t)$. Then $f_1, \ldots, f_t$ form a d-sequence if and only if $\mathcal{G}$ is one of the following*:

$\mathcal{C}_5$, the pentagon;

$\mathcal{S}_5 = \mathcal{C}_5 \cup \{$any proper subset of the diagonals of $\mathcal{C}_5\}$;

$\mathcal{K}_5$, the pentagon together with all its diagonals;

$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$ and $I_5(\mathcal{G}) = (X_1X_2X_3X_4X_5)$;

$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, 2, 3, 4, 5\}$ and
$I_5(\mathcal{G}) = (X_1X_2X_3X_4X_5, X_2X_3X_4X_5X_6)$;

$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, \ldots, 6\}$ and
$I_5(\mathcal{G}) = (X_1X_2X_3X_4X_5, X_3X_4X_5X_6X_7, X_2X_3X_4X_5X_6)$;

$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \ldots, \{v_4, v_n\}\}$ and
$I_5(\mathcal{G}) = (X_1X_2X_3X_4X_5, \ldots, X_1X_2X_3X_4X_n)$;

$\mathcal{C}_4 \cup star_\alpha(n-3)$, the center of the star graph is any vertex v_α of $\mathcal{C}_4$;

$\mathcal{S}_4 \cup star_\alpha(n-3)$, v_α vertex of $\mathcal{S}_4$;

$\mathcal{K}_4 \cup star_\alpha(n-3)$, v_α vertex of $\mathcal{K}_4$;

$\mathcal{C}_3 \cup star_\alpha(n-3) \cup star_\beta(2)$, $v_\alpha \neq v_\beta$ vertices of $\mathcal{C}_3$;

$\mathcal{C}_3 \cup \mathcal{C}'_3$ such that a vertex of $\mathcal{C}_3$ is in common with one of $\mathcal{C}'_3$;

any other one consisting of the union of two or more among them.

Proof. Let $\mathcal{G}$ be a graph on $[n]$ vertices. Suppose that $I_5(\mathcal{G}) = (f_1, \ldots, f_t)$ and $f_1, \ldots, f_t$ form a d-sequence, for any t. Then it is necessary that $\mathcal{G}$ is one of the graphs described in the statement.

Suppose by contradiction that $\mathcal{G}$ is the graph on $[8]$ vertices with edge set $E(\mathcal{G}) = \{\{v_i, v_{i+1}\},\ i = 1, \ldots, 7\}$ and consider $I_5(\mathcal{G}) = (X_1X_2X_3X_4X_5, X_2X_3X_4X_5X_6,\ X_3X_4X_5X_6X_7,\ X_4X_5X_6X_7X_8)$; regardless of the order, by Lemma 3.1, the sequence of the generators of $I_5(\mathcal{G})$ never will be a d-sequence. For such graphs it is possible to generalize to $[n]$ vertices. A similar reasoning can be made for any other graph.

Conversely, suppose that $\mathcal{G}$ is one of the graphs in the statement; in particular, without lack of generality, if it is the graph on $[n]$ vertices with edge set $\{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\},\ \{v_3, v_4\}, \{v_4, v_5\}, \ldots, \{v_4, v_n\}\}$, namely the graph $\mathcal{C}_4 \cup \mathrm{star}_4(n-3)$, and $I_5(\mathcal{G}) = (f_1, \ldots, f_t)$ such that $f_1 = X_1X_2X_3X_4X_5, f_2 =$

$X_1X_2X_3X_4X_6, \dots, f_t = X_1X_2X_3X_4X_n$, then $f_1, \dots, f_t$ is a d-sequence because $[f_i, f_j] = X_1X_2X_3X_4$ divides $f_k, \forall\, 1 \le i < j < k \le t$. $\square$

Proposition 3.4. *Let $\mathcal{G}$ be a graph with $n \ge 6$ vertices, $I_6(\mathcal{G}) = (f_1, \dots, f_t)$. Then $f_1, \dots, f_t$ form a d-sequence if and only if $\mathcal{G}$ is one of the following:*
$\mathcal{C}_6$, the hexagon;
$\mathcal{S}_6 = \mathcal{C}_6 \cup \{$any proper subset of the diagonals of $\mathcal{C}_6\}$;
$\mathcal{K}_6$, the hexagon together with all its diagonals;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}\}$ and
$I_6(\mathcal{G}) = (X_1X_2X_3X_4X_5X_6)$;
$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, \dots, 6\}$ and
$I_6(\mathcal{G}) = (X_1X_2X_3X_4X_5X_6, X_2X_3X_4X_5X_6X_7)$;
$E(\mathcal{G}) = \{\{v_i, v_{i+1}\}, i = 1, \dots, 7\}$ and
$I_6(\mathcal{G}) = (X_1X_2X_3X_4X_5X_6, X_3X_4X_5X_6X_7X_8, X_2X_3X_4X_5X_6X_7)$;
$E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \dots, \{v_5, v_n\}\}$ and
$I_6(\mathcal{G}) = (X_1X_2X_3X_4X_5X_6, \dots, X_1X_2X_3X_4X_5X_n)$;
$\mathcal{C}_5 \cup star_\alpha(n-4)$, the center of the star graph is any vertex v_α of $\mathcal{C}_5$;
$\mathcal{S}_5 \cup star_\alpha(n-4)$, v_α vertex of $\mathcal{S}_5$;
$\mathcal{K}_5 \cup star_\alpha(n-4)$, v_α vertex of $\mathcal{K}_5$;
$\mathcal{C}_4 \cup star_\alpha(n-4) \cup star_\beta(2)$, v_α, v_β adjacent vertices of $\mathcal{C}_4$;
$\mathcal{S}_4 \cup star_\alpha(n-4) \cup star_\beta(2)$, v_α, v_β vertices of $\mathcal{S}_4$ not belonging to its diagonal;
$\mathcal{C}_3 \cup \mathcal{C}_4$ such that a vertex of $\mathcal{C}_4$ is in common with one of $\mathcal{C}_3$;
$\mathcal{C}_3 \cup \mathcal{S}_4$ such that the vertex $v_\alpha \notin \{$diagonal of $\mathcal{S}_4\}$ is in common with one of $\mathcal{C}_3$;
$\mathcal{C}_3 \cup \mathcal{C}'_3 \cup star_\beta(n-4)$ with v_α common vertex of $\mathcal{C}_3, \mathcal{C}'_3$; v_β $(\beta \ne \alpha)$ vertex of $\mathcal{C}_3 \cup \mathcal{C}'_3$;
any other one consisting of the union of two or more among them.

Proof. Let $\mathcal{G}$ be a graph on $[n]$ vertices. Suppose that $I_6(\mathcal{G}) = (f_1, \dots, f_t)$ and $f_1, \dots, f_t$ form a d-sequence, for any t. Then it is necessary that $\mathcal{G}$ is one of the graphs described in the statement.

Suppose by contradiction that $\mathcal{G}$ is the graph on [9] vertices with edge set $E(\mathcal{G}) = \{\{v_i, v_{i+1}\},\ i = 1, \dots, 8\}$ and $I_6(\mathcal{G}) = (X_1X_2X_3X_4X_5X_6,\ X_2X_3X_4X_5X_6X_7,\ X_3X_4X_5X_6X_7X_8,\ X_4X_5X_6X_7X_8X_9)$; regardless of the order, by Lemma 3.1, the sequence of the generators of $I_6(\mathcal{G})$ never will be a d-sequence. For such graphs it is possible to generalize to $[n]$ vertices. A similar reasoning can be made for any other graph.

Conversely, suppose that $\mathcal{G}$ is one of the graphs in the statement; in particular, without lack of generality, if it is the graph on $[n]$ vertices with edge set $\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\},\ \{v_3, v_4\},\ \{v_3, v_5\},\ \{v_4, v_5\},\ \{v_5, v_6\}, \dots, \{v_5, v_n\}\}$, namely the graph $\mathcal{C}_3 \cup \mathcal{C}'_3 \cup \mathrm{star}_5(n-4)$, and $I_6(\mathcal{G}) = (f_1, f_2, \dots, f_t)$ such that $f_1 = X_1X_2X_3X_4X_5X_6, f_2 = X_1X_2X_3X_4X_5X_7, \dots, f_t = X_1X_2X_3X_4X_5X_n$, then $f_1, \dots, f_t$ is a d-sequence because $[f_i, f_j] = X_1X_2X_3X_4X_5$ divides $f_k, \forall\, 1 \le i < j < k \le t$. $\square$

Let's consider a significant class of generalized graph ideals generated by a d-sequence.

Theorem 3.1. *Let $\mathcal{G}$ be the graph on vertex set $\{v_1, \dots, v_n\}$ whose edge set is*

$$E(\mathcal{G}) = E(C_m) \cup \{\{v_i, v_{m+1}\}, \{v_{m+1}, v_{m+2}\}, \dots, \{v_{m+k-1}, v_{m+k}\}\} \\ \cup \{\{v_j, v_{m+k+1}\}, \{v_j, v_{m+k+2}\}, \dots, \{v_j, v_n\}\},$$

where C_m is the cycle on vertices $\{v_1, \dots, v_m\}$, $m < n$, k an integer, v_i, v_j are vertices of C_m with $i \neq j$. Then the generalized graph ideal $I_{m+k+1}(\mathcal{G}) \subset R = K[X_1, \dots, X_n]$ is generated by a d-sequence.

Proof. The generalized graph ideal $I_{m+k+1}(\mathcal{G})$ is generated by the squarefree monomials

$$\begin{aligned} f_1 &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_{m+k+1}, \\ f_2 &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_{m+k+2}, \\ &\vdots \\ f_{n-m-k} &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_n. \end{aligned}$$

One has $[f_i, f_j] = X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k}$ divides f_k for all $1 \leq i < j < k \leq n - m - k$. Hence by [6, Theorem 2.1], $f_1, f_2, \dots, f_{n-m-k}$ is a monomial d-sequence. $\square$

Corollary 3.3. *Let $\mathcal{G}$ be the graph on vertex set $\{v_1, \dots, v_n\}$ of the above theorem. Then the generalized graph ideal $I_{m+k+1}(\mathcal{G}) \subset R = K[X_1, \dots, X_n]$ is generated by a strong s-sequence.*

Proof. It descends from [1, Corollary 3.3]. $\square$

Finally, let's use the theory of s-sequences for computing standard algebraic invariants of the symmetric algebra of the generalized graph ideal $I_{m+k+1}(\mathcal{G})$ in terms of their annihilator ideals.

Proposition 3.5. *Let $\mathcal{G}$ be the graph on vertex set $\{v_1, \dots, v_n\}$ of Theorem 3.1. The annihilator ideals of the generators of $I_{m+k+1}(\mathcal{G})$ are*

$$I_1 = (0), \quad I_i = (X_{m+k+1}, X_{m+k+2}, \dots, X_{m+k+i-1}) \text{ for } i = 2, \dots, n - m - k.$$

Proof. Let $I(\mathcal{G}) = (f_1, \dots, f_t)$, where

$$\begin{aligned} f_1 &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_{m+k+1}, \\ f_2 &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_{m+k+2}, \\ &\vdots \\ f_t &= X_1 X_2 \cdots X_m X_{m+1} \cdots X_{m+k} X_n \text{ and } t = n - m - k. \end{aligned}$$

Set $f_{hk} = \frac{f_h}{[f_h, f_k]}$ for $h < k$, $h, k = 1, \dots, t$. Then the annihilator ideals of the monomial sequence $f_1, \dots, f_t$ are $I_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, t$.

For $i = 1$ we have $I_1 = (0)$ and by the structure of these monomials it follows $I_2 = (f_{12}) = (X_{m+k+1})$, $I_3 = (f_{13}, f_{23}) = (X_{m+k+1}, X_{m+k+2}), \ldots,$ $I_t = (f_{1,t}, f_{2,t}, \ldots, f_{t-1,t}) = (X_{m+k+1}, X_{m+k+2}, \ldots, X_{n-1})$. Hence

$$I_i = (X_{m+k+1}, X_{m+k+2}, \ldots, X_{m+k+i-1}) \text{ for } i = 2, \ldots, n-m-k. \qquad \square$$

Remark 3.2. By Proposition 3.5 one has

$$\begin{aligned}\mathrm{in}_{\prec}(J) = ((X_{m+k+1})\,T_2, (X_{m+k+1}, X_{m+k+2})\,T_3, \ldots,\\ (X_{m+k+1}, X_{m+k+2}, \ldots, X_{n-1})\,T_{n-m-k}).\end{aligned}$$

Theorem 3.2. *Let $\mathcal{G}$ be as in Theorem 3.1. Then*:
(a) *$Sym_R(I_{m+k+1}(\mathcal{G}))$ is Cohen-Macaulay of dimension $n+1$*;
(b) *$e(Sym_R(I_{m+k+1}(\mathcal{G}))) = n-m-k$*;
(c) *$reg(Sym_R(I_{m+k+1}(\mathcal{G}))) = 1$.*

Proof. The s-sequence that generates $I_{m+k+1}(\mathcal{G})$ is strong.
(a) It descends from [6, Theorem 4.8].
(b) By [1, Proposition 2.4], it follows that

$$\mathrm{e}(Sym_R(I_{m+k+1}(\mathcal{G}))) = \sum_{i=1}^{n-m-k} \mathrm{e}(R/I_i).$$

By Proposition 3.5 the annihilator ideals I_i are generated by a regular sequence, then $\mathrm{e}(R/I_i) = 1$ for $i = 2, \ldots, n-m-k$, and $\mathrm{e}(R/(0)) = 1$. Hence $\mathrm{e}(Sym_R(I_{m+k+1}(\mathcal{G}))) = n-m-k$.
(c) By [6, Theorem 4.8], one has

$$\mathrm{reg}(Sym_R(I_{m+k+1}(\mathcal{G}))) \le \max_{2\le j\le t}\{\sum_{i=1}^{j-1} \deg(f_{ij}) - (j-2)\},$$

where $t = n-m-k$. Then

$$\begin{aligned}\mathrm{reg}(Sym_R(I_{m+k+1}(\mathcal{G}))) &\le \max_{2\le j\le t}\{\sum_{i=1}^{j-1} \deg(X_{m+k+i}) - (j-2)\}\\ &= (j-1) - (j-2) = 1.\end{aligned}$$

Moreover J is generated by linear forms of degree two. Then

$$\mathrm{reg}(Sym_R(I_{m+k+1}(\mathcal{G}))) \ge 1.$$

It follows that $\mathrm{reg}(Sym_R(I(\mathcal{G}))) = 1$. $\square$

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