OPTIMAL CONTROL OF THE VISCOUS WEAKLY DISPERSIVE BENJAMIN-BONA-MAHONY EQUATION

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Abstract. This paper is concerned with the optimal control problem for the viscous weakly dispersive Benjamin-Bona-Mahony (BBM) equation. We prove the existence and uniqueness of weak solution to the equation. The optimal control problem for the viscous weakly dispersive BBM equation is introduced, and then the existence of optimal control to the problem is proved.

1. Introduction

It is well known that the Benjamin-Bona-Mahony (BBM) equation

$$(1.1) \qquad u_t - u_{xxt} + u_x + uu_x = 0$$

was initiated by Benjamin, Bona, Mahony in [2], which is motivated by the Korteweg-de Vries (KdV) equation [4]

$$(1.2) \qquad u_t + u_{xxx} + u_x + uu_x = 0$$

for modeling water waves of small amplitude and large wavelength. In all these equations, u denotes a wave amplitude or velocity, x is proportional to the physical distance and t is proportional to the elapsed time.

When the non-linear terms of Eq.(1.1) are replaced by $u^p u_x$ for $p > 0$, the resulting equations reads

$$(1.3) \qquad u_t - u_{xxt} + u_x + u^p u_x = 0,$$

which is called a generalized BBM equation. It has been found that (1.1) possesses global solutions for smooth initial data [2] for $p = 1$, which can be easily extended to the general case (1.3), and the solitary wave solutions of (1.3) are stable when $p < 4$ [2].

A great deal of research has been devoted to the BBM equation. For example, Avrin and Goldstein [1] studied the global existence for the BBM equation

in arbitrary dimensions. Chen [3] discussed the periodic initial-value problem for BBM equation. Nickel [8] obtained the elliptic solutions to a generalized BBM equation. Zeng [18] obtained the existence and stability of solitary wave solutions of equations of BBM type. Moreover, we also remark that some studies are concerned with the control of BBM equation, such as [7, 9]. In [9], Rosier and Zhang studied the unique continuation property and control for the BBM equation on a periodic domain. And in [7], Micu discussed the controllability of the linearized BBM equation.

On the other hand, many results have also been obtained for the optimal control of partial differential equations, we refer to [6, 10] and references therein. Specially, here it is worth mentioning about the optimal control problems for the viscous weakly dispersive equations. For example, Shen and Gao [11] discussed the optimal control of the viscous weakly dispersive Degasperis-Procesi equation. Vedantham [17] studied the optimal control of the viscous Burgers equation using an equivalent index method. Sun [13] obtained the maximum principle for optimal distributed control of the viscous Dullin-Gottwald-Holm equation. Tian [16] considered the optimal control of the viscous Camassa-Holm equation. Lenells and Wunsch [5] discussed the weakly dissipative Camassa-Holm, Degasperis-Procesi and Novikov equations. Smaoui [12] studied the boundary and distributed control of the viscous Burgers equation. L. Tian and C. Shen [15] considered the optimal control of the viscous Degasperis-Procesi equation.

To the best of our knowledge, few papers can be found in the literature for the optimal control problem of the viscous weakly dispersive BBM equation. So, being inspired by the above mentioned, the purpose of this paper is to study the optimal control problem for the following viscous weakly dispersive BBM equations

$$(1.4) \qquad u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx} + u_x + uu_x + \lambda(u - u_{xx})_x = 0,$$

where $\lambda(u - u_{xx})_x$ is a dispersive item, $\varepsilon(u - u_{xx})_{xx}$ is a viscous item, $\varepsilon > 0$ and $\lambda > 0$ are two constants. More specifically, we are concerned with the following control system

$$(1.5) \quad \begin{cases} \min\{J(y, \bar{\omega})\} = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\bar{\omega}\|_{L^2(Q_0)}^2, \\ y_t - \varepsilon y_{xx} + \lambda y_x + u_x y + u_x - u_x u_{xx} = f + B^*\bar{\omega}, \\ y(x, 0) = u(x, 0) - u_{xx}(x, 0) = \psi \in H, \\ u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \end{cases}$$

where $y = u_x - u_{xx}$. The aim is to match the given desired state z by adjusting the body force, $\bar{\omega}$ is a control belongs to the Hilbert space $L^2(Q_0)$ with minimal energy and work, the first term in cost functional measures the physical objective, the second one is the size of the control, where $\delta > 0$ plays the role of a weight.

The paper is organized as follows: In Section 2, we obtain the existence and uniqueness of weak solutions to (1.5), and establish the inequality for the norm of weak solutions with initial values and control items. In Section 3, we discuss the optimal control of the viscous weakly dispersive BBM equation and prove the existence of the optimal control.

2. Existence and uniqueness of weak solution for the viscous weakly dispersive BBM equation

For fixed $T > 0$, set $\Omega = (0,1)$, $Q = (0,T) \times \Omega$ and $Q_0 \subseteq Q$ be an open set with positive measure. Let $V = H_0^1(0,1)$, $H = L^2(0,1)$, $V^* = H^{-1}(0,1)$ and $H^* = L^2(0,1)$ are dual space respectively. It is well known that V is dense in H, and

$$V \hookrightarrow H = H^* \hookrightarrow V^*,$$

moreover, the embedding being dense. The extended operator $B^* \in \mathcal{L}(L^2(Q_0), L^2(V^*))$ is introduced as

$$B^*q = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \backslash Q_0. \end{cases}$$

We also supply H with the inner product $(\cdot, \cdot)_H$ and the norm $\|\cdot\|_H$, and define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, here $D^m = \partial^m/\partial x^m$ $(m = 0, 1, 2, \ldots)$. A new space $W(0, T; V)$ is introduced as

$$W(0, T; V) = \{f : f \in L^2(0, T; V), \ f_t \in L^2(0, T; V^*)\},$$

which is a Hilbert space endowed with common inner product.

Now, consider the following viscous weakly dispersive BBM equation

$$(2.1) \quad \begin{cases} u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx} + u_x + uu_x + \lambda(u - u_{xx})_x = f + B^*\bar{\omega}, \\ u(x,0) = u_0(x), \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \end{cases}$$

where $x \in (0,1)$, $t \in [0,T]$, $f + B^*\bar{\omega} \in L^2(0,T;V^*)$. Setting $y = u - u_{xx}$, from (2.1), we obtain a quasi-linear evolution equation:

$$(2.2) \quad \begin{cases} y_t - \varepsilon y_{xx} + \lambda y_x + u_x y + u_x - u_x u_{xx} = f + B^*\bar{\omega}, \\ y(x,0) = u(x,0) - u_{xx}(x,0) = \psi, \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0. \end{cases}$$

In order to prove the existence of weak solution to the viscous weakly dispersive BBM equation, we give the definition of the weak solution in the space $W(0, T; V)$.

Definition 2.1. A function $y(x,t) \in W(0,T;V)$ is called a weak solution to Eqs.(2.2), if
(2.3)
$$\frac{d}{dt}(y,v)_H + \varepsilon(y_x,v_x)_H + \lambda(y_x,v)_H + (u_x y,v)_H + (u_x,v)_H - (u_x u_{xx},v)_H$$
$$= \langle f + B^*\bar{\omega}, v\rangle_{V^*,V}$$

for all $v \in V$, *a.e.* $t \in [0,T]$ and $y(x,0) = \psi$.

Remark 2.1. Since $W(0,T;V)$ is compactly embedded into $C(0,T;H)$ ([6], [19]), then $y(x,0) = \psi \in H$ is valid.

Theorem 2.1. *Let $\psi \in H$, $f \in L^2(0,T;V^*)$. Then for any $\bar{\omega} \in L^2(Q_0)$, there exists a unique weak solution to the Eqs.(2.2) in the interval $[0,T]$.*

Proof. Since $-\partial_x^2$ is a Laplacian operator with one dimension, we can choose the base functions $\{w_j\}_{j \in N}$ in V with w_j being the eigenfunctions subject to the Dirichlet condition:

(2.4)
$$\begin{cases} -\partial_x^2 \, w_j = \lambda_j \, w_j, & \text{in } \Omega, \\ w_j = 0, & \text{on } \partial\Omega. \end{cases}$$

We also normalize w_j such that $\|w_j\|_H = 1$ [19]. By the elliptic operator theory, $\{w_j\}$ forms base functions in V. Now we use the *Faedo-Galerkin* method to find the approximate solution.

For $\forall m \in N$, define the *ansatz* space by

$$V_m = span\{w_1, w_2, \dots, w_m\} \subseteq V.$$

Set

$$y_m = \sum_{i=1}^{m} y_i^m(t) w_i(x),$$

which satisfied the following identities:
(2.5)
$$\begin{cases} y_{mt} - \varepsilon y_{mxx} + \lambda y_{mx} + u_{mx} y_m + u_{mx} - u_{mx} u_{mxx} = f + B^*\bar{\omega}, \\ y_m(x,0) = \psi_m, \\ u_m(0,t) = u_m(1,t) = u_{mx}(0,t) = u_{mx}(1,t) = u_{mxx}(0,t) = u_{mxx}(1,t) = 0, \end{cases}$$

where $y_m = u_m - u_{mxx}$ and $y_m(0) = \psi_m \to \psi$ strongly in H as $m \to \infty$.

Now, we prove the existence of weak solution by analyzing the limiting behavior of sequences of smooth functions $\{u_m\}$ and $\{y_m\}$.

Since Eqs.(2.5) are first-order differential equations, thus by the usual *Picard iteration* method used in ODE, one can conclude that there exists a $t_m > 0$ such that, the problem Eqs.(2.5) admits a unique local solution in $[0,t_m]$. We will show that the solution is uniformly bounded as $t_m \to T$.

Multiplied Eqs.(2.5) both sides by u_m and integrating with respect to x over Ω, we get

$$(2.6) \quad \frac{1}{2}\frac{d}{dt}(\|u_m\|_H^2 + \|u_m\|_V^2) + \varepsilon(\|u_m\|_V^2 + \|u_m\|_{H^2}^2) = \langle f + B^*\bar\omega, u_m\rangle_{V^*,V}.$$

Integrating (2.6) from 0 to t, we have

$$\frac{1}{2}(\|u_m\|_H^2 + \|u_m\|_V^2) + \varepsilon\int_0^t (\|u_m\|_V^2 + \|u_m\|_{H^2}^2)ds$$

$$(2.7) \qquad = \int_0^t \langle f + B^*\bar\omega, u_m\rangle_{V^*,V}ds + \frac{1}{2}(\|u_m(0)\|_H^2 + \|u_m(0)\|_V^2).$$

Since $f + B^*\bar\omega \in L^2(0,T;V^*)$, so there exist a constant $C_1 \geq 0$ such that

$$(2.8) \qquad \|f + B^*\bar\omega\|_{L^2(0,T;V^*)} \leq C_1,$$

and $M_1 \geq 0$ such that

$$(2.9) \qquad \frac{1}{2}(\|u_m(0)\|_H^2 + \|u_m(0)\|_V^2) \leq M_1.$$

By Young's inequality, we obtain

$$\left|\int_0^t \langle f + B^*\bar\omega, u_m\rangle_{V^*,V}ds\right| \leq \int_0^t \|f + B^*\bar\omega\|_{V^*}\|u_m\|_V ds$$

$$\leq \frac{1}{\varepsilon}\int_0^t \|f + B^*\bar\omega\|_{V^*}^2 ds + \varepsilon\int_0^t \|u_m\|_V^2 ds$$

$$\leq \frac{1}{\varepsilon}\|f + B^*\bar\omega\|_{L^2(0,T;V^*)}^2 + \varepsilon\int_0^t \|u_m\|_V^2 ds$$

$$(2.10) \qquad \leq \frac{C_1^2}{\varepsilon} + \varepsilon\int_0^t \|u_m\|_V^2 ds.$$

It follows from (2.7)-(2.9) that

$$\frac{1}{2}(\|u_m\|_H^2 + \|u_m\|_V^2) + \varepsilon\int_0^t (\|u_m\|_V^2 + \|u_m\|_{H^2}^2)ds \leq \varepsilon\int_0^t \|u_m\|_V^2 ds + \frac{C_1^2}{\varepsilon} + M_1.$$

So,

$$(2.11) \qquad \|u_m\|_H^2 + \|u_m\|_V^2 \leq \frac{2}{\varepsilon}C_1^2 + 2M_1 = C_2^2$$

for $\forall t \in [0,T]$, and $C_2 \geq 0$ is a constant. Which implies, by (2.11),

$$\|u_m\|_H \leq C_2, \quad \|u_m\|_V \leq C_2.$$

Again, multiplied Eqs.(2.5) both sides by $-u_{mxx}$ and integrating on $(0,t)\times\Omega$, we obtain

$$\frac{1}{2}(\|u_m\|_V^2 + \|u_m\|_{H^2}^2) + \varepsilon\int_0^t (\|u_m\|_{H^2}^2 + \|u_m\|_{H^3}^2)ds$$

$$= -\int_0^t \langle f + B^*\bar\omega, u_{mxx}\rangle_{V^*,V}ds + \int_0^t (u_m u_{mx}, u_{mxx})_H ds$$

(2.12) $+ \dfrac{1}{2}(\|u_m(0)\|_V^2 + \|u_m(0)\|_{H^2}^2).$

Thus, from (2.8), we have

$$\left| \int_0^t \langle f + B^*\bar{\omega}, u_{mxx} \rangle_{V^*,V} ds \right|$$

$$\leq \int_0^t \|f + B^*\bar{\omega}\|_{V^*} \|u_m\|_{H^3} ds$$

$$\leq \dfrac{1}{\varepsilon} \|f + B^*\bar{\omega}\|_{L^2(0,T;V^*)}^2 + \varepsilon \int_0^t \|u_m\|_{H^3}^2 ds$$

(2.13) $\leq \dfrac{C_1^2}{\varepsilon} + \varepsilon \int_0^t \|u_m\|_{H^3}^2 ds.$

By Poincaré inequality and Sobolev embedding theorem, we get

$$\left| \int_0^t (u_m u_{mx}, u_{mxx})_H ds \right| \leq \int_0^t \|u_m\|_H \|u_{mx}\|_{L^\infty} \|u_{mxx}\|_H ds$$

$$\leq kC_2 \int_0^t \|u_{mx}\|_V \|u_m\|_{H^2} ds$$

(2.14) $\leq kC_2 \int_0^t \|u_m\|_{H^2}^2 ds,$

where k is an embedding constant. From (2.12)-(2.14), we obtain

(2.15) $\|u_m\|_V^2 + \|u_m\|_{H^2}^2 \leq 2kC_2 \int_0^t \|u_m\|_{H^2}^2 ds + \dfrac{2C_1^2}{\varepsilon} + 2M_2,$

where M_2 satisfies

$$\dfrac{1}{2}(\|u_m(0)\|_V^2 + \|u_m(0)\|_{H^2}^2) \leq M_2.$$

By using Gronwall's inequality, we get from (2.15) that

(2.16) $\|u_m\|_{H^2}^2 \leq \left(\dfrac{2C_1^2}{\varepsilon} + 2M_2 \right) e^{2kTC_2} = C_3^2.$

Thus we know that $\|u_m\|_{H^2} \leq C_3$.

Now, we will give a uniform $L^2(0,T;V)$ bound on the sequence $\{y_m\}$.

Multiplied Eqs.(2.5) both sides by y_m and integrating over Ω, yields that

(2.17) $\dfrac{1}{2}\dfrac{d}{dt}\|y_m\|_H^2 + \varepsilon\|y_m\|_V^2 + \int_0^1 u_{mx} y_m^2 dx - \int_0^1 u_{mx} u_{mxx} y_m dx$
$= \langle f + B^*\bar{\omega}, y_m \rangle_{V^*,V}.$

By Poincaré inequality and Sobolev embedding theorem, we have

$$\left| \int_0^1 u_{mx} \, y_m^2 \, dx \right| = 2\left| \int_0^1 u_m \, y_{mx} \, y_m \, dx \right|$$

$$\leq 2\|u_m\|_{L^\infty} \, \|y_m\|_H \, \|y_m\|_V$$

$$\leq 2k \, \|u_m\|_V \, \|y_m\|_H \, \|y_m\|_V$$

(2.18)
$$\leq 2kC_2 \, \|y_m\|_H \, \|y_m\|_V,$$

and

$$\left| \int_0^1 u_{mx} \, u_{mxx} \, y_m \, dx \right| = \frac{1}{2} \left| \int_0^1 u_{mx}^2 y_{mx} \, dx \right|$$

$$\leq \frac{1}{2} \|u_{mx}\|_{L^\infty} \, \|u_{mx}\|_H \, \|y_{mx}\|_H$$

$$\leq \frac{k}{2} \|u_{mx}\|_V \, \|u_m\|_V \, \|y_m\|_V$$

(2.19)
$$\leq \frac{kC_2 C_3}{2} \, \|y_m\|_V.$$

It then follows from (2.17)-(2.19) and Young's inequality that

$$\frac{d}{dt}\|y_m\|_H^2 + 2\varepsilon\|y_m\|_V^2 \leq (2\|f + B^*\bar{\omega}\|_{V^*} + kC_2 C_3)\|y_m\|_V + 4kC_2\|y_m\|_H\|y_m\|_V$$

$$\leq \frac{2}{\varepsilon}(2\|f + B^*\bar{\omega}\|_{V^*} + kC_2 C_3)^2 + \frac{\varepsilon}{2}\|y_m\|_V^2$$

$$+ \frac{32}{\varepsilon}k^2 C_2^2\|y_m\|_H^2 + \frac{\varepsilon}{2}\|y_m\|_V^2.$$

In view of $(a + b)^2 \leq 2(a^2 + b^2)$, from above inequality, we have

(2.20) $\quad \dfrac{d}{dt}\|y_m\|_H^2 + \varepsilon\|y_m\|_V^2 \leq \dfrac{4}{\varepsilon}(4\|f + B^*\bar{\omega}\|_{V^*}^2 + k^2 C_2^2 C_3^2) + \dfrac{32}{\varepsilon}k^2 C_2^2\|y_m\|_H^2$.

Integrating (2.20) on interval $[0, t]$, we obtain

(2.21)
$$\|y_m\|_H^2 + \varepsilon \int_0^t \|y_m\|_V^2 ds \leq M_3 + M_4 \int_0^t \|y_m\|_H^2 ds,$$

where $M_3 = \frac{4}{\varepsilon}(4C_1^2 + k^2 T C_2^2 C_3^2) + \|\psi\|_H^2$, $M_4 = \frac{32}{\varepsilon}k^2 C_2^2$ are positive constants.

By using Gronwall's inequality to (2.21), there exists a $M_5 > 0$ such that

(2.22)
$$\|y_m\|_H \leq M_5.$$

Thus, combine (2.21) with (2.22), there exists a $M_6 > 0$ such that

(2.23)
$$\|y_m\|_{L^2(0,T;V)} \leq M_6.$$

In what follows, we shall give a uniform $L^2(0, T; V^*)$ bound on the sequence $\{y_{mt}\}$.

By Eqs.(2.5) and Sobolev embedding theorem, we obtain

$$\|y_{mt}\|_{V^*} = \sup_{\|\phi\|_V = 1} \langle y_{mt}, \phi \rangle_{V^*, V}$$

$$\leq \|f + B^*\bar{\omega}\|_{V^*} + \varepsilon\|y_m\|_V + \lambda\|y_m\|_H + \|u_m\|_H\|y_m\|_V + \|u_m\|_H$$

$$+ \frac{k}{2}\|u_m\|_{H^2}^2$$

(2.24)
$$\leq \|f + B^*\bar{\omega}\|_{V^*} + (\varepsilon + C_2)\|y_m\|_V + C_2 + \frac{kC_3^2}{2} + \lambda M_5.$$

It follows from (2.24) that

$$(2.25) \quad \|y_{mt}\|_{V^*}^2 \leq 3\|f + B^*\bar{\omega}\|_{V^*}^2 + 3(\varepsilon + C_2)^2\|y_m\|_V^2 + 3(C_2 + \frac{k}{2}C_3^2 + \lambda M_5)^2.$$

Integrating the above inequality with respect to t on $[0, T]$, we get

$$
\begin{aligned}
\|y_{mt}\|_{L^2(0,T;V^*)}^2 \leq{}& 3\|f + B^*\bar{\omega}\|_{L^2(0,T;V^*)}^2 + 3(\varepsilon + C_2)^2\|y_m\|_{L^2(0,T;V)}^2 \\
&+ 3T(C_2 + \frac{k}{2}C_3^2 + \lambda M_5)^2 \\
\leq{}& 3C_1^2 + 3M_6^2(\varepsilon + C_2)^2 + 3T(C_2 + \frac{k}{2}C_3^2 + \lambda M_5)^2 \\
={}& M_7^2.
\end{aligned}
$$

$$(2.26)$$

Thus, from (2.23) and (2.26), we have

$$(2.27) \quad \|y_m\|_{W(0,T;V)}^2 = \|y_m\|_{L^2(0,T;V)}^2 + \|y_{mt}\|_{L^2(0,T;V^*)}^2 \leq M_6^2 + M_7^2 < \infty.$$

So, there exists a subsequence of $\{y_m\}$, without loss of generality, denote the subsequence by $\{y_m\}$, such that

$$(2.28) \qquad\qquad y_m \to y \quad \text{weakly in } L^2(0,T;V),$$

and

$$(2.29) \qquad\qquad \frac{d}{dt}y_m \to \frac{d}{dt}y \quad \text{weakly in } L^2(0,T;V^*).$$

Since $W(0,T;V)$ is compactly embedded into $L^2(0,T;L^\infty(\Omega))$ [14], we have

$$(2.30) \qquad\qquad y_m \to y \quad \text{strongly in } L^2(0,T;L^\infty(\Omega)).$$

By Remark 2.1, it is shown that

$$(2.31) \qquad\qquad y_m \to y \quad \text{strongly in } C(0,T;H).$$

Moreover,

$$u_m \to u, \quad u_{mx} \to u_x, \quad u_{mxx} \to u_{xx}, \quad \text{strongly in the space } C(0,T;H).$$

In following, we are going to verify the y is a solution of (2.2). Setting $\eta = \xi(t)\nu \in \mathscr{D}(0,T;V)$, here $\xi(t) \in \mathscr{D}(0,T)$, $\nu \in V$. From (2.5), we obtain

$$
\begin{aligned}
&\int_0^T \langle f + B^*\bar{\omega}, \eta\rangle_{V^*,V}\,dt \\
={}& \int_0^T \langle y_{mt}, \eta\rangle_{V^*,V}\,dt + \varepsilon\int_0^T (y_{mx}, \eta_x)_H\,dt + \lambda\int_0^T (y_{mx}, \eta)_H\,dt \\
&+ \int_0^T (u_{mx}y_m, \eta)_H\,dt + \int_0^T (u_{mx}, \eta)_H\,dt - \int_0^T (u_{mx}u_{mxx}, \eta)_H\,dt.
\end{aligned}
$$

$$(2.32)$$

Again, from (2.28)-(2.31), we have

$$\left| \int_0^T (u_{mx}y_m - u_xy, \varphi)_H\,dt \right|$$

$$\leq \Big| \int_0^T (u_{mx}(y_m - y), \varphi)_H \, dt \Big| + \Big| \int_0^T ((u_{mx} - u_x)y, \varphi)_H \, dt \Big|$$

$$\leq \int_0^T \|u_m\|_H \|\varphi\|_V \|y_m - y\|_{L^\infty} \, dt + \int_0^T \|u_m - u\|_H \|\varphi\|_V \|y\|_{L^\infty} \, dt$$

$$\leq \|u_m\|_{C(0,T;H)} \|\varphi\|_{L^2(0,T;V)} \|y_m - y\|_{L^2(0,T;L^\infty)}$$

$$+ \|y\|_{L^2(0,T;L^\infty)} \|\varphi\|_{L^2(0,T;V)} \|u_m - u\|_{C(0,T;H)}$$

$$\longrightarrow 0, \quad m \to +\infty, \ \forall \varphi \in L^2(0,T;V)$$

and

$$\Big| \int_0^T (u_{mx}u_{mxx} - u_x u_{xx}, \varphi)_H \, dt \Big| = \frac{1}{2} \Big| \int_0^T ((u_{mx})^2 - (u_x)^2, \varphi)_H \, dt \Big|$$

$$\leq \frac{1}{2} \Big| \int_0^T \|(u_{mx})^2 - (u_x)^2\|_H \|\varphi\|_H \, dt \Big|$$

$$\leq \frac{1}{2} T^{\frac{1}{2}} \|(u_{mx})^2 - (u_x)^2\|_{C(0,T;H)} \|\varphi\|_{L^2(0,T;H)}$$

$$\longrightarrow 0, \quad m \to +\infty, \ \forall \varphi \in L^2(0,T;V).$$

Letting $m \to +\infty$ in (2.32), yields that

$$\int_0^T \langle f + B^* \bar{\omega}, \eta \rangle_{V^*, V} \, dt$$

$$= \int_0^T \langle y_t, \eta \rangle_{V^*, V} \, dt + \varepsilon \int_0^T (y_x, \eta_x)_H \, dt + \lambda \int_0^T (y_x, \eta)_H \, dt$$

(2.33) $$\quad + \int_0^T (u_x y, \eta)_H \, dt + \int_0^T (u_x, \eta)_H \, dt - \int_0^T (u_x u_{xx}, \eta)_H \, dt.$$

Since $\mathscr{D}(0,T;V)$ is dense in $L^2(0,T;V)$, the above equality holds for all functions $\eta \in L^2(0,T;V)$. Hence, we have specially

$$\frac{d}{dt}(y, \eta)_H + \varepsilon(y_x, \eta_x)_H + \lambda(y_x, \eta)_H + (u_x y, \eta)_H + (u_x, \eta)_H - (u_x u_{xx}, \eta)_H$$

$$= \langle f + B^* \bar{\omega}, \eta \rangle_{V^*, V}$$

for each $\eta \in V$ and *a.e.* $t \in [0, T]$.

From (2.31), we have $y_m(0) \to y(0)$ strongly in H. So, it also weakly convergence in H, by the uniqueness of the limit, we have

(2.34) $$y(x, 0) = \psi.$$

On the other hand, it is clear that the uniqueness of the weak solution follows from the inequality (2.20) immediately.

Thus the proof is therefore complete. $\qquad \square$

In the following, we shall establish the inequality for the norm of weak solution with initial value and control item, which is necessary in discussing the exitance of optimal control.

Theorem 2.2. *Let $\psi \in H$, $f \in L^2(0,T;V^*)$. Then for $\forall \bar{\omega} \in L^2(Q_0)$, there exists constants $L_1, L_2 \geq 0$, such that*

$$\|y\|^2_{W(0,T;V)} \leq L_1(\|f + B^*\bar{\omega}\|^2_{L^2(0,T;V^*)} + \|\psi\|^2_H) + L_2.$$

Proof. Multiplied Eqs.(2.2) by u, and integrating over Ω, which implies

$$(2.35) \qquad \frac{1}{2}\frac{d}{dt}(\|u\|^2_H + \|u\|^2_V) + \varepsilon(\|u\|^2_V + \|u\|^2_{H^2}) = \langle f + B^*\bar{\omega}, u\rangle_{V^*,V}.$$

Similar to the argument in the Theorem 2.1, we have

$$(2.36) \qquad\qquad\qquad \|u\|_H \leq C_2, \qquad \|u\|_V \leq C_2,$$

where C_2 is a constant the same as in Theorem 2.1.

Similarly, multiplied Eqs.(2.2) by $-u_{xx}$, and integrating over Ω, we obtain

$$(2.37) \qquad\qquad\qquad \|u_m\|_{H^2} \leq C_3,$$

where C_3 is also a constant from (2.16).

Again, multiplied Eqs.(2.2) by y and integrating over Ω, which yields

$$(2.38) \quad \frac{1}{2}\frac{d}{dt}\|y\|^2_H + \varepsilon\|y\|^2_V + \int_0^1 u_x y^2 dx - \int_0^1 u_x u_{xx} y dx = \langle f + B^*\bar{\omega}, y\rangle_{V^*,V}$$

since

$$(2.39) \qquad \left|\int_0^1 u_x y^2 dx\right| \leq 2k \,\|u\|_V \,\|y\|_H \,\|y\|_V \leq 2kC_2\|y\|_H \,\|y\|_V,$$

$$(3.40)$$
$$\left|\int_0^1 u_x u_{xx} y dx\right| \leq \frac{1}{2}\|u_x\|_{L^\infty}\|u_x\|_H\|y_x\|_H \leq \frac{k}{2}\|u\|_{H^2}\|u\|_V\|y\|_V \leq \frac{kC_2C_3}{2}\|y\|_V.$$

From (2.38)-(2.40), we get

$$(2.41) \quad \frac{1}{2}\frac{d}{dt}\|y\|^2_H + \varepsilon\|y\|^2_V \leq (\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})\|y\|_V + 2kC_2\|y\|_H\|y\|_V.$$

Using Young's inequality yields

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_H + \varepsilon\|y\|^2_V \leq (\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})\|y\|_V + 2kC_2\|y\|_H\|y\|_V$$
$$\leq \frac{2}{\varepsilon}(\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})^2 + \frac{8}{\varepsilon}k^2 C_2^2\|y\|^2_H + \varepsilon\|y\|^2_V,$$

then

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_H \leq \frac{2}{\varepsilon}(\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})^2 + \frac{8}{\varepsilon}k^2 C_2^2\|y\|^2_H.$$

Using Gronwall' inequality to the last inequality, we have

$$\|y\|^2_H \leq \frac{\varepsilon e^{\frac{16}{\varepsilon}k^2 C_2^2 T}}{8k^2 C_2^2}[\frac{2}{\varepsilon}(\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})^2 + \frac{8}{\varepsilon}k^2 C_2^2\|\psi\|^2_H]$$

$$\leq \frac{\varepsilon e^{\frac{16}{\varepsilon}k^2 C_2^2 T}}{8k^2 C_2^2}[\frac{4}{\varepsilon}(\|f + B^*\bar{\omega}\|^2_{V^*} + \frac{k^2 C_2^2 C_3^2}{4}) + \frac{8}{\varepsilon}k^2 C_2^2\|\psi\|^2_H]$$

$$(2.42) \qquad \leq C_4(\|f + B^*\bar{\omega}\|_{V^*}^2 + \|\psi\|_H^2) + C_5,$$

where

$$C_4 = \max\left\{\frac{e^{\frac{16}{\varepsilon}k^2C_2^2T}}{2k^2C_2^2}, e^{\frac{16}{\varepsilon}k^2C_2^2T}\right\}, \quad C_5 = \frac{\varepsilon e^{\frac{16}{\varepsilon}k^2C_2^2T}C_3^2}{32}.$$

In addition, by using Young's inequality, (2.41) and (2.42), which yields

$$\frac{d}{dt}\|y\|_H^2 + \varepsilon\|y\|_V^2$$

$$\leq \frac{8}{\varepsilon}[(\|f + B^*\bar{\omega}\|_{V^*} + \frac{kC_2C_3}{2})^2 + 4k^2C_2^2\|y\|_H^2]$$

$$\leq \frac{8}{\varepsilon}(2\|f + B^*\bar{\omega}\|_{V^*}^2 + \frac{k^2C_2^2C_3^2}{2} + 4k^2C_2^2\|y\|_H^2)$$

$$\leq (\frac{16}{\varepsilon} + \frac{32k^2C_2^2C_4}{\varepsilon})\|f + B^*\bar{\omega}\|_{V^*}^2 + \frac{32k^2C_2^2C_4}{\varepsilon}\|\psi\|_H^2$$

$$+ \frac{32k^2C_2^2C_5}{\varepsilon} + \frac{4k^2C_2^2C_3^2}{\varepsilon}$$

$$(2.43) \qquad \leq C_6(\|f + B^*\bar{\omega}\|_{V^*}^2 + \|\psi\|_H^2) + C_7,$$

where $C_6 = \max\left\{\frac{16}{\varepsilon} + \frac{32k^2C_2^2C_4}{\varepsilon}, \frac{32k^2C_2^2C_4}{\varepsilon}\right\}$, $C_7 = \frac{32k^2C_2^2C_5}{\varepsilon} + \frac{4k^2C_2^2C_3^2}{\varepsilon}$.

Hence, integrating (2.43) over $[0, t]$, we have

$$\|y\|_{L^2(0,T;V)}^2 \leq \frac{C_6}{\varepsilon}\|f + B^*\bar{\omega}\|_{L^2(0,T;V^*)}^2 + \frac{1 + C_6T}{\varepsilon}\|\psi\|_H^2 + \frac{C_7T}{\varepsilon}$$

$$(2.44) \qquad \leq C_8(\|f + B^*\bar{\omega}\|_{L^2(0,T;V^*)}^2 + \|\psi\|_H^2) + C_9,$$

where $C_8 = \max\left\{\frac{C_6}{\varepsilon}, \frac{1+C_6T}{\varepsilon}\right\}$, and $C_9 = \frac{C_7T}{\varepsilon}$.

Meantime, from Eqs.(2.2), we get

$$\|y_t\|_{V^*} = \sup_{\|\phi\|_V=1} \langle y_t, \phi\rangle_{V^*,V}$$

$$\leq \|f + B^*\bar{\omega}\|_{V^*} + \varepsilon\|y\|_V + \lambda\|y\|_H + \|u\|_H\|y\|_V + \|u\|_H$$

$$+ \frac{k}{2}\|u\|_{H^2}^2$$

$$(2.45) \qquad \leq \|f + B^*\bar{\omega}\|_{V^*} + (\varepsilon + C_2)\|y\|_V + \lambda\|y\|_H + C_2 + \frac{k}{2}C_3^2.$$

By using Schwarz inequality and (2.37), we obtain

$$\|y_t\|_{V^*}^2 \leq (\|f + B^*\bar{\omega}\|_{V^*} + (\varepsilon + C_2)\|y\|_V + \lambda\|y\|_H + C_2 + \frac{k}{2}C_3^2)^2$$

$$\leq 4\|f + B^*\bar{\omega}\|_{V^*}^2 + 4(\varepsilon + C_2)^2\|y\|_V^2 + 4\lambda^2\|y\|_H^2 + 4(C_2 + \frac{k}{2}C_3^2)^2$$

$$\leq (4 + 4\lambda^2C_4)(\|f + B^*\bar{\omega}\|_{V^*}^2 + \|\psi\|_H^2) + 4(\varepsilon + C_2)^2\|y\|_V^2$$

$$+ 4(C_2 + \frac{k}{2}C_3^2)^2 + 4\lambda^2C_5.$$

Integrating last integrating over $[0, T]$, we have

$$\|y_t\|^2_{L^2(0,T;V^*)}$$
$$\leq (4 + 4\lambda^2 C_4)\max\{T, 1\}(\|f + B^*\bar{\omega}\|^2_{L^2(0,T;V^*)} + \|\psi\|^2_H)$$
$$+ 4(\varepsilon + C_2)^2\|y\|^2_{L^2(0,T;V)} + [4(C_2 + \frac{k}{2}C_3^2)^2 + 4\lambda^2 C_5]T$$
$$\leq [(4 + 4\lambda^2 C_4)\max\{T, 1\} + 4C_8(\varepsilon + K_2)^2](\|f + B^*\bar{\omega}\|^2_{L^2(0,T;V^*)}$$
$$+ \|\psi\|^2_H) + 4T[(C_2 + \frac{k}{2}C_3^2)^2 + \lambda^2 C_5] + 4(\varepsilon + C_2)^2$$
$$(2.46) \qquad = C_{10}(\|f + B^*\bar{\omega}\|^2_{L^2(0,T;V^*)} + \|\psi\|^2_H) + C_{11},$$

where

$$C_{10} = (4 + 4\lambda^2 C_4)\max\{T, 1\} + 4C_8(\varepsilon + K_2)^2,$$
$$C_{11} = 4T[(C_2 + \frac{k}{2}C_3^2)^2 + \lambda^2 C_5] + 4(\varepsilon + C_2)^2$$

are positive constants.

Combine (2.36) and (2.38), we obtain

$$\|y\|^2_{W(0,T;V)} = \|y\|^2_{L^2(0,T;V)} + \|y\|^2_{L^2(0,T;V^*)}$$
$$\leq L_1(\|f + B^*\bar{\omega}\|^2_{L^2(0,T;V^*)} + \|\psi\|^2_H) + L_2,$$

where $L_1 = C_8 + C_{10}, L_2 = C_9 + C_{11}$.

This completes the proof of the theorem. $\qquad\square$

3. Optimal control problem

In this section, we discuss the optimal control problem associated with the viscous weakly dispersive BBM equation.

Consider the following control system:

$$(3.1) \quad \begin{cases} \min\{J(y, \bar{\omega})\} = \frac{1}{2}\|Cy - z\|^2_S + \frac{\delta}{2}\|\bar{\omega}\|^2_{L^2(Q_0)}, \\ y_t - \varepsilon y_{xx} + \lambda y_x + u_x y + u_x - u_x u_{xx} = f + B^*\bar{\omega}, \\ y(x, 0) = u(x, 0) - u_{xx}(x, 0) = \psi \in H, \\ u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \end{cases}$$

where $y = u_x - u_{xx}$ and $\bar{\omega}$ is a control in $L^2(Q_0)$, $C \in \mathcal{L}(W(0,T;V), S)$ is a given continuous observation operator, S is a real Hilbert space, and

$$\min\{J(y, \bar{\omega})\} = \frac{1}{2}\|Cy - z\|^2_S + \frac{\delta}{2}\|\bar{\omega}\|^2_{L^2(Q_0)},$$

is performance index of tracking type, here $z \in S$ is desired state and $\delta > 0$ is fixed.

The optimal control problem for the viscous weakly dispersive BBM equation is

$$\min\{J(y, \bar{\omega})\},$$

where the optimal control pair $(y, \bar\omega)$ satisfies Eqs.(2.2) with the given initial value and boundary condition.

The following theorem is presented to demonstrate the existence of the optimal control to the viscous weakly dispersive BBM equation.

Theorem 3.1. *Let $\psi \in H$, $f \in L^2(0, T; V^*)$. Then there exists an optimal control $\bar\omega$ for the control system* (3.1).

Proof. For each $\bar\omega$, from Theorem 2.1, there exists a unique weak solution y of Eqs.(2.2). In view of (3.1), we get

$$(3.2) \qquad J(y, \bar\omega) \geq \frac{\delta}{2} \|\bar\omega\|_{L^2(Q_0)}^2 \geq 0$$

for each control pair $(y, \bar\omega) \in W(0, T; V) \times L^2(Q_0)$. Thus there exists a constant $\gamma \geq 0$, such that

$$\gamma = \inf_{W(0,T;V) \times L^2(Q_0)} J(y, \bar\omega).$$

Choosing the minimizing sequence $\{(y^n, \bar\omega^n)\}_{n \in N}$ satisfy Eqs.(2.2), such that

$$(3.3) \qquad \gamma = \lim_{n \to +\infty} J(y^n, \bar\omega^n).$$

From Theorem 2.2, we know that

$$(3.4) \qquad \|y\|_{W(0,T;V)} \to +\infty \Longrightarrow \|\bar\omega\|_{L^2(Q_0)} \to +\infty.$$

So, it follows from (3.2)-(3.4) that $\{(y^n, \bar\omega^n)\}_{n \in N}$ is bounded in the Hilbert space $W(0, T; V) \times L^2(Q_0)$, and there exists $(y^*, \bar\omega^*)$, such that

$$(3.5) \qquad \begin{cases} y^n \to y^*, & \text{weakly in } W(0, T; V) \\ \bar\omega^n \to \bar\omega^*, & \text{weakly in } L^2(Q_0). \end{cases}$$

Thus, from (3.5), we have

$$\lim_{n \to +\infty} \int_0^T \langle y_t^n - y^*, \varphi(t) \rangle_{V^*, V} dt = 0, \quad \forall \varphi \in L^2(0, T; V).$$

Again, from Remark 2.1, we know that $y^n \to y^*$ strongly in $C(0, T; H)$. Furthermore, $u^n \to u^*$, $u_x^n \to u_x^*$, $u_{xx}^n \to u_{xx}^*$ strongly in the space $C(0, T; H)$. Since $W(0, T; V)$ is compactly embedded into $L^2(0, T; L^\infty(\Omega))$ [14], then $y^n \to y^*$ strongly in the space $L^2(0, T; L^\infty(\Omega))$. Moreover, in view of $\{y^n\}$ convergence weakly in $W(0, T; V)$, then $\|y^*\|_{W(0,T;V)}$ is bounded, and so is $\|y^*\|_{L^2(0,T;L^\infty)}$.

We remark that the above estimates are sufficient to pass to the limit in the *linear terms* of Eqs.(2.2). As far as *nonlinear term* of Eqs.(2.2), by using Holder's inequality, we get

$$\left| \int_0^T (u_x^n y^n - u_x^* y^*, \varphi)_H dt \right|$$

$$\leq \left| \int_0^T (u_x^n (y^n - y^*), \varphi)_H dt \right| + \left| \int_0^T ((u_x^n - u_x^*) y^*, \varphi)_H dt \right|$$

$$\leq \int_0^T \|u^n\|_H \|\varphi\|_V \|y^n - y^*\|_{L^\infty} dt + \int_0^T \|u^n - u^*\|_H \|\varphi\|_V \|y^*\|_{L^\infty} dt$$

$$\leq \|u^n\|_{C(0,T;H)} \|\varphi\|_{L^2(0,T;V)} \|y^n - y^*\|_{L^2(0,T;L^\infty)}$$

$$+ \|y^*\|_{L^2(0,T;L^\infty)} \|\varphi\|_{L^2(0,T;V)} \|u^n - u^*\|_{C(0,T;H)}$$

$$\longrightarrow 0, \quad n \to +\infty, \ \forall \varphi \in L^2(0,T;V),$$

$$\left| \int_0^T (u_x^n u_{xx}^n - u_x^* u_{xx}^*, \varphi)_H dt \right| = \frac{1}{2} \left| \int_0^T ((u_x^n)^2 - (u_x^*)^2, \varphi)_H dt \right|$$

$$\leq \frac{1}{2} \left| \int_0^T \|(u_x^n)^2 - (u_x^*)^2\|_H \|\varphi\|_H dt \right|$$

$$\leq \frac{1}{2} T^{\frac{1}{2}} \|(u_x^n)^2 - (u_x^*)^2\|_{C(0,T;H)} \|\varphi\|_{L^2(0,T;H)}$$

$$\longrightarrow 0, \ n \to +\infty, \ \forall \varphi \in L^2(0,T;V).$$

From (3.5), we know that

$$\int_0^T \int_0^1 (B^* \bar{\omega}^n - B^* \bar{\omega}^*) \varphi \, dx \, dt \to 0, \quad n \to +\infty, \ \forall \varphi \in L^2(0,T;V).$$

Since $y^n \to y^*$ strongly in $C(0,T;H)$, we can infer that $y^n(0) \to y^*(0)$ weakly in H as $n \to +\infty$. Then, we have

$$y^*(0) = \psi.$$

From the analysis above, we deduce that $(y^*, \bar{\omega}^*)$ satisfies Eqs.(2.2), and such that

$$J(y^*, \bar{\omega}^*) = \min\{J(y, \bar{\omega})\}.$$

Hence, there exists an optimal control to the control system (3.1). $\qquad \square$

Remark 3.1. In view of the relation $u = (1 - \partial_x^2)^{-1} y$, there exists an optimal control $(u^*, \bar{\omega}^*)$ for the viscous weakly dispersive BBM equation (2.1).

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