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GENERALIZED CAYLEY GRAPHS OF RECTANGULAR GROUPS

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Abstract. We describe generalized Cayley graphs of rectangular groups, so that we obtain (1) an equivalent condition for two Cayley graphs of a rectangular group to be isomorphic to each other, (2) a necessary and sufficient condition for a generalized Cayley graph of a rectangular group to be (strong) connected, (3) a necessary and sufficient condition for the colour-preserving automorphism group of such a graph to be vertex-transitive, and (4) a sufficient condition for the automorphism group of such a graph to be vertex-transitive.

1. Introduction

The concept of Cayley graphs of groups ([1, 2, 5, 6, 15, 16]) was extended to that of Cayley graphs of semigroups ([9, 10, 11, 12, 13, 17, 19, 20]). The *Cayley graph* $\mathrm{Cay}(G, S)$ of a semigroup G relative to a subset S of G is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (a, b) such that $xa = b$ for some $x \in S$. This concept was further extended to that of generalized Cayley graphs of a semigroup by the author in [24]. Recall that if S is an ideal of a semigroup T, then we call T an *ideal extension* of S. Let T be an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$, where T^1 stands for the semigroup T with identity adjoined if necessary. As in [22, 24, 25, 26], the *generalized Cayley graph* $Cay(S, \rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set $E(\mathrm{Cay}(S, \rho))$ consisting of those ordered pairs (a, b) such that $xay = b$ for some $(x, y) \in \rho$. Some combinatorial properties related to generalized Cayley graphs of semigroups were investigated in [25], where the author suggested to characterize semigroups S such that $Cay(S, S_l) = Cay(S, S_r)$, where $S_l = S^1 \times \{1\}$ and $S_r = \{1\} \times S^1$ are the left and right universal relations on S^1, respectively. This problem was partially solved by Wang in [22], where it was proved that for any regular semigroup S, $Cay(S, S_l) = Cay(S, S_r)$ if and only if S is a Clifford semigroup, i.e., a semilattice of groups. This result was further extended by the author in [26].

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Recall that an element e of a semigroup is called an *idempotent* if $e^2 = e$. A semigroup is called a *band* if all elements of it are idempotents. Among all idempotents of a semigroup there is a *natural* (*partial*) *order* defined by $e \leq f$ if and only if $ef = fe = e$. An idempotent of a semigroup is called *primitive* if it is minimal with respect to the natural order within the set of all idempotents of this semigroup. A band is called a *left zero* (*right zero, rectangular*) *band* if it satisfies the identity $xy = x$ (resp. $xy = y$, $xyx = x$). A semigroup is said to be (*left, right*) *simple* if it has no proper (left, right) ideals. A semigroup S is *left* (*right*) *cancellative* if $xy = xz$ (resp. $yx = zx$) implies $y = z$ for all $x, y, z \in S$. A semigroup is called a *right* (*left*) *group* if it is right (left) simple and left (right) cancellative. A direct product of a rectangular band and a group is called a *rectangular group.* A simple semigroup is referred to as *completely simple* if it contains a primitive idempotent. Using [8, Theorem 3.3.1], one may prove easily that every rectangular group is a completely simple semigroup.

The usual Cayley graphs of some semigroups related to groups such as right (left) groups, rectangular groups and Clifford semigroups have been investigated in [17, 18, 19, 20]. Rectangular group congruences on a semigroup were characterized in [4]. Also, Cayley graphs of rectangular groups were characterized in [14, 17], and Cayley graphs of strong semilattices of rectangular groups were investigated in [7]. The present paper is devoted to generalized Cayley graphs of a rectangular group so that the transitivity, connectivity and the isomorphism problem of generalized Cayley graphs of a rectangular group are described.

The paper is organized as follows: We first introduce some related basic knowledge for preliminaries in Section 2, and then we establish some fundamental properties of generalized Cayley graphs of a rectangular group in Section 3. In Section 4, we discuss the connectivity of generalized Cayley graphs of rectangular groups so that a necessary and sufficient condition for a generalized Cayley graph of a rectangular group to be (strong) connected is given, see Theorem 4.3. In Section 5, we discuss the isomorphism problem of generalized Cayley graphs of rectangular groups so that we obtain some equivalent conditions under which two generalized Cayley graphs of a rectangular group to be isomorphic to each other, see Theorem 5.4. At last, in Section 6, we describe the vertex-transitivity of generalized Cayley graphs of a rectangular group and the main results of this section are Theorems 6.1 and 6.2.

2. Preliminaries

In this section, we introduce a few related notions, notation and simple facts on graphs, groups and semigroups, which are required for our further discussion. For more basic knowledge on Semigroup Theory, Group Theory and Graph Theory, we refer the reader to [8], [21] and [23], respectively.

Let $D(V, E)$ be a graph with vertex set V and edge set $E \subseteq V \times V$. If $U \subseteq V$, then the subgraph of D *induced* by U is defined as the graph with vertex set U and edge set $\{(u_1, u_2) \,|\, u_1, u_2 \in U \text{ and } (u_1, u_2) \in E\}$. Sometimes we equate the subgraph of D induced by U with the vertex set U. An subgraph of D induced by U is called a *strong subgraph* of D if whenever $u \in U$ and (u, v) or $(v, u) \in E$, we have $v \in U$.

For a graph $D(V, E)$, a mapping $\phi : V \longrightarrow V$ is called an *endomorphism* of the graph D if $(\phi(u), \phi(v)) \in E$ for all $(u, v) \in E$. If ϕ is not only an endomorphism of the graph D but also a bijection from V onto V itself and if ϕ^{-1} is also an endomorphism of the graph D, then we call ϕ an *automorphism* of the graph D. The automorphism group and endomorphism monoid of D are denoted by $\mathrm{Aut}(D)$ and $\mathrm{End}(D)$, respectively.

A graph $D(V, E)$ is said to be *vertex-transitive* if, for any two vertices $x, y \in V$, there exists an automorphism $\phi \in \mathrm{Aut}(D)$ such that $\phi(x) = y$ (see [1] or [11]). More generally, a subset A of $\mathrm{End}(D)$ is said to be *vertex-transitive* on D, and D is said to be *A-vertex-transitive* if, for any two vertices $x, y \in V$, there exists an endomorphism $\phi \in A$ such that $\phi(x) = y$ (see [11]).

Let G be a semigroup with a subset S. As in [11], we denote the automorphism group (endomorphism monoid) of $\mathrm{Cay}(G, S)$ by $\mathrm{Aut}_S(G)$ (resp. $\mathrm{End}_S(G)$). That is,

$$\mathrm{Aut}_S(G) = \mathrm{Aut}(\mathrm{Cay}(G, S)) \text{ and } \mathrm{End}_S(G) = \mathrm{End}(\mathrm{Cay}(G, S)).$$

An element $\phi \in \mathrm{End}_S(G)$ is called a colour-preserving endomorphism if $sx = y$ implies $s(\phi(x)) = \phi(y)$ for every $x, y \in G$ and $s \in S$. If we regard an edge (x, sx), for $s \in S$, as having 'colour' s, so that the elements of S are thought of as colours associated with the edges of the Cayley graph, then every colour-preserving endomorphism maps each edge to an edge of the same colour. Denote by $\mathrm{ColEnd}_S(G)$ (and $\mathrm{ColAut}_S(G)$) the set of all colour-preserving endomorphisms (resp. automorphisms) of $\mathrm{Cay}(G, S)$.

More generally, let T be an ideal extension of a semigroup S and let $\rho \subseteq T^1 \times T^1$. Denote the automorphism group (endomorphism monoid) of the generalized Cayley graph $\mathrm{Cay}(S, \rho)$ by $\mathrm{Aut}_\rho(S)$ (resp. $\mathrm{End}_\rho(S)$). That is,

$$\mathrm{Aut}_\rho(S) = \mathrm{Aut}(\mathrm{Cay}(S, \rho)) \text{ and } \mathrm{End}_\rho(S) = \mathrm{End}(\mathrm{Cay}(S, \rho)).$$

An element $\phi \in \mathrm{End}_\rho(S)$ will be called a *colour-preserving endomorphism* if $sxt = y$ implies $s(\phi(x))t = \phi(y)$, for every $x, y \in S$ and $(s, t) \in \rho$. Denote by $\mathrm{ColEnd}\rho(S)$ (and $\mathrm{ColAut}\rho(S)$) the sets of all colour-preserving endomorphisms (resp. automorphisms) of the generalized Cayley graph $\mathrm{Cay}(S, \rho)$. Evidently, with the usual composition of mappings, $\mathrm{ColAut}\rho(S)$ and $\mathrm{ColEnd}\rho(S)$ are a group and a monoid, and thus are called the *colour-preserving automorphism group* and the *colour-preserving endomorphism monoid* of $\mathrm{Cay}(S, \rho)$, respectively. Let U and W be subgraphs of $\mathrm{Cay}(S, \rho)$. We may define a *colour-preserving homomorphism* (*isomorphism*) of U to W in a similar way. If there exists such a homomorphism (isomorphism) of U onto W, then we say that U is *colour-preservingly homomorphic* (*isomorphic*) to W.

Let $D = (V, E)$ be a graph. The *underlying undirected graph* of D has the same vertex set V and it has an undirected edge $\{u, v\}$ for each directed edge (u, v) of D. The graph D is said to be *connected* if its underlying undirected graph is connected. If $v_0, v_1, \ldots, v_n \in V$ and $(v_{i-1}, v_i) \in E$ for $i = 1, \ldots, n$, then the sequence $(v_0, v_1, \ldots, v_n)$ is called a *directed walk* of D of length n. A directed walk $(v_0, v_1, \ldots, v_n)$ is called a *cycle* of length n (or simply, an n-*cycle*) if $v_0, v_1, \ldots, v_n$ are distinguished pairwise except that $v_0 = v_n$. If, for each pair of vertices u, v of D, there exists a directed walk from u to v, then D is said to be *strongly connected*.

Let G be a group. We always use e_G to denote the identity of G. The trivial subgroup of G is simply denoted by 1, that is, $1 = \{e_G\}$. Notice that as mentioned before, sometimes 1 represents the identity of T^1 for a semigroup T. If A is a subset of G, then the subgroup generated by A in G is denoted by $\langle A \rangle$.

If S is a semigroup with respect to the multiplication $\cdot$, then the *anti-semigroup* of S, denoted by S^*, is defined as the semigroup $(S, *)$, where the multiplication $*$ is defined by $a * b = b \cdot a$ for any $a, b \in S$. If G is a group, then the anti-semigroup G^* is also a group, called the *anti-group* of G. Let G be a group and let $\rho \subseteq G \times G$. Then by $[\rho\rangle$ (resp. $\{\rho\rangle$) we denote the subgroup (resp. subsemigroup) generated by ρ in the direct product $G^* \times G$, where G^* is the anti-group of G; dually, by $\langle\rho]$ (resp. $\langle\rho\}$) we denote the subgroup (resp. subsemigroup) generated by ρ in the direct product $G \times G^*$.

If H is a subgroup of a group G, then we write $H \leq G$. If $H \leq G$ and $a \in G$, then $H^a = a^{-1}Ha = \{a^{-1}ha \mid h \in H\} \leq G$. Let G be a group and $H, K \leq G$. Then for any $a \in G$, put $HaK = \{hak \mid h \in H, k \in K\}$, called a bi-coset of G with respect to H and K. The next two lemmas are readily verified.

Lemma 2.1. *Let G be a group and $H, K \leq G$. Then there exist $a_i \in G$, $i \in I$* (*where I is an index set*) *such that*

$$G = \dot{\bigcup}_{i \in I} Ha_iK. \tag{2.1}$$

We call equation (2.1) a *bi-coset decomposition* of the group G with respect to its subgroups H and K.

Lemma 2.2. *Let G be a group and $H, K \leq G$ such that $H^a \cap K = 1$ for some $a \in K$. Then every element x of HaK can be expressed uniquely as*

$$x = hak$$

with $h \in H$ and $k \in K$.

3. Fundamental properties of generalized Cayley graphs of rectangular groups

In the sequel, by $L\times G\times R$, we always mean a rectangular group, where L is a left zero semigroup, R is a right zero semigroup, and G is a group. When saying that a rectangular group $L\times G\times R$ has an ideal extension $U\times P\times V$, we always mean that P, U and V are ideal extensions of G, L and R, respectively. In this section, we shall establish some fundamental properties of the generalized Cayley graph $\mathrm{Cay}(S,\rho)$, where $S = L\times G\times R$ has an ideal extension $T = U\times P\times V$ and where $\rho\subseteq T^1\times T^1$.

The next two lemmas can be checked by some straightforward computations.

Lemma 3.1. *Let $S = L\times G\times R$ be a rectangular group with an ideal extension $T = U\times P\times V$ such that*

$$ul = ul', rv = r'v \tag{3.1}$$

for all $u\in U$, $v\in V$, $l,l'\in L$ and $r,r'\in R$. Let $\rho\subseteq T^1\times T^1$ and put

(3.2)

$$\rho^* = \{((ul, he_G, vr),(lu', e_Gh', rv'))\,|\,((u,h,v),(u',h',v'))\in\rho,\ l\in L,\ r\in R\}.$$

Then $\rho^*\subseteq S\times S$ and $Cay(S,\rho) = Cay(S,\rho^*)$.

Lemma 3.2. *Let $S = L\times G\times R$ be a rectangular group and let $\rho\subseteq S\times S$. Let $l_i\in L$, $r_i\in R$, $g_i\in G$ for $i = 1,2$. Then the following conditions are equivalent*:

(i) $((l_1,g_1,r_1),(l_2,g_2,r_2))\in E(Cay(S,\rho))$;
(ii) $((l,g_1,r),(l_2,g_2,r_2))\in E(Cay(S,\rho))$ *for all $l\in L$, $r\in R$*;
(iii) $((l_2,g_1,r_2),(l_2,g_2,r_2))\in E(Cay(S,\rho))$.

Let $l\in L, r\in R$ and put

$$G_{lr} = \{l\}\times G\times\{r\}.$$

Then G_{lr} is a group isomorphic to G. Let E_{lr} denote the edge set of the subgraph of $\mathrm{Cay}(S,\rho)$ induced by G_{lr} and put

$$E'_{lr} = \{((l',g',r'),(l,g,r))\,|\,((l,g',r),(l,g,r))\in E_{lr},\ l'\in L,\ r'\in R\}.$$

Then $E_{lr}\subseteq E'_{lr}$. According to Lemma 3.2, we immediately deduce the following

Lemma 3.3. *Let $S = L\times G\times R$ be a rectangular group and let $\rho\subseteq S\times S$. Then*

$$E(Cay(S,\rho)) = \bigcup_{l\in L}\bigcup_{r\in R} E'_{lr}.$$

We use π_i to denote the ith projection of a Cartesian product for any index i. Let $S = L\times G\times R$ be a rectangular group and let $\rho\subseteq S\times S$. Then we use π_L, π_G, π_R to denote the projection of $L\times G\times R$ onto L, G and R, respectively. Specifically, we have

$$\pi_1(\rho) = \{(l,g,r)\,|\,((l,g,r),(l',g',r'))\in\rho\},$$

$$\begin{aligned}\pi_2(\rho) &= \{(l', g', r') \mid ((l, g, r), (l', g', r')) \in \rho\},\\ \pi_L(\pi_1(\rho)) &= \{l \mid (l, g, r) \in \pi_1(\rho)\},\\ \pi_G(\pi_1(\rho)) &= \{g \mid (l, g, r) \in \pi_1(\rho)\},\\ \pi_R(\pi_1(\rho)) &= \{r \mid (l, g, r) \in \pi_1(\rho)\};\end{aligned}$$

and $\pi_L(\pi_2(\rho))$, $\pi_G(\pi_2(\rho))$, $\pi_R(\pi_2(\rho))$ are of similar expressions. Set

$$\begin{aligned}H &= \langle \pi_G(\pi_1(\rho)\rangle, \text{ the subgroup generated by } \pi_G(\pi_1(\rho) \text{ in G},\\ K &= \langle \pi_G(\pi_2(\rho)\rangle, \text{ the subgroup generated by } \pi_G(\pi_2(\rho) \text{ in G}.\end{aligned}$$

Since $H, K \leq G$ and by Lemma 2.1, we have the bi-coset decomposition (2.1) of G with respect to H and K. For $l \in L, r \in R$, equation (2.1) induces naturally the following bi-coset decomposition of G_{lr} with respect to H_{lr} and K_{lr}:

$$G_{lr} = \dot{\bigcup}_{i\in I} H_{lr}(l, a_i, r)K_{lr}, \tag{3.3}$$

where

$$\begin{aligned}H_{lr} &= \{l\} \times H \times \{r\} = H \cap G_{lr},\\ K_{lr} &= \{l\} \times K \times \{r\} = K \cap G_{lr}.\end{aligned}$$

Put

$$\begin{aligned}M^i_{lr} &= H_{lr}(l, a_i, r)K_{lr} \text{ for } i \in I,\\ \rho_{lr} &= \{((l, g, r'), (l', g', r)) \in \rho \mid l' \in L,\ r' \in R; g, g' \in G\}.\end{aligned}$$

By Lemma 3.2, the subgraph of $\mathrm{Cay}(S, \rho)$ induced by G_{lr} is exactly the subgraph of $\mathrm{Cay}(S, \rho_{lr})$ induced by G_{lr}. Furthermore, we obtain:

Lemma 3.4. *Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. Then for any $l \in L$ and $r \in R$,*

$$(G_{lr}, E_{lr}) = \dot{\bigcup}_{i\in I} M^i_{lr}.$$

For $i \in I$, let E^i_{lr} denote the edge set of the subgraphs induced by M^i_{lr} and put

$$(E^i_{lr})' = \{((l', g', r'), (l, g, r)) \mid ((l, g', r), (l, g, r)) \in E^i_{lr},\ l' \in L,\ r' \in R\}.$$

Then

$$E'_{lr} = \dot{\bigcup}_{i\in I} (E^i_{lr})'.$$

For $i \in I$, set

$$S_i = \bigcup_{l\in L} \bigcup_{r\in R} M^i_{lr}.$$

Then

$$S_i = L \times (Ha_iK) \times R.$$

Let E_i denote the edge set of the subgraph of $\mathrm{Cay}(S, \rho)$ induced by S_i. Then by Lemma 3.4, we can prove:

Lemma 3.5. *Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. Let $(s_1, s_2) \in E(Cay(S, \rho))$. Then $s_1, s_2 \in S_i$ for some $i \in I$. That is, (S_i, E_i) is a strong subgraph of $Cay(S, \rho)$ for any $i \in I$.*

4. Connectivity of generalized Cayley graphs of rectangular groups

Connectivity of addition Cayley graphs was studied in [6]. In this section we discuss the connectivity of generalized Cayley graphs of rectangular groups.

Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. As in the last section, we set

$$H = \langle \pi_G(\pi_1(\rho)) \rangle \text{ and } K = \langle \pi_G(\pi_2(\rho)) \rangle,$$

where $\langle X \rangle$ denotes the subgroup generated by X in the group G. Assume that $H^a \cap K = 1$ for some $a \in G$. By Lemma 2.2, every element x of HaK can be expressed uniquely as $x = hak$ with $h \in H$ and $k \in K$. The subgraph of $\mathrm{Cay}(G, \rho)$ induced by $M^i_{lr} = \{l\} \times Ha_iK \times \{r\}$ is connected if and only if, for any $\alpha, \beta \in M^i_{lr}$, there exists an undirected walk from α to β, that is, there exist (λ_k, μ_k), $k = 1, \ldots, p$ such that $(\lambda_k, \mu_k) \in \rho_{lr}$ or $(\lambda_k^{-1}, \mu_k^{-1}) \in \rho_{lr}$, and

$$\beta = \lambda_p \cdots \lambda_2 \lambda_1 \alpha \mu_1 \mu_2 \cdots \mu_p.$$

Suppose that $\alpha = (l, ha_ik, r)$ and $\beta = (l, h'a_ik', r)$. Then the above equality is equivalent to

$$\pi_G(\beta) = \pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1)\pi_G(\alpha)\pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p),$$

that is,

$$\begin{aligned} h' \cdot a_i \cdot k' &= \pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1)(ha_ik)\pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p) \\ &= \pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1)h \cdot a_i \cdot k\pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p). \end{aligned}$$

This holds if and only if

$$\begin{cases} h' = \pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1)h, \\ k' = k\pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p), \end{cases}$$

which is equivalent to

$$\begin{cases} h'h^{-1} = \pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1), \\ k^{-1}k' = \pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p), \end{cases}$$

i.e.,

$$\begin{aligned} (h'h^{-1}, k^{-1}k') &= (\pi_G(\lambda_p) \cdots \pi_G(\lambda_2)\pi_G(\lambda_1), \pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p)) \\ &= (\pi_G(\lambda_1) * \pi_G(\lambda_2) * \cdots * \pi_G(\lambda_p), \pi_G(\mu_1)\pi_G(\mu_2) \cdots \pi_G(\mu_p)) \\ &= (\pi_G(\lambda_1), \pi_G(\mu_1))(\pi_G(\lambda_2), \pi_G(\mu_2)) \cdots (\pi_G(\lambda_p), \pi_G(\mu_p)) \\ &\in H^* \times K. \end{aligned}$$

Since H and K are groups and α, β are chosen arbitrarily, we see that $(h'h^{-1}, k^{-1}k')$ runs over all elements of $H^* \times K$. Thus the subgraph of $\mathrm{Cay}(G, \rho)$

induced by M^i_{lr} is connected if and only if $H^* \times K = [\rho_{lr}\rangle$, where $[\rho_{lr}\rangle$ denotes the subgroup generated by ρ_{lr} in the group $G^* \times G$, and where G^* is the anti-group of G. Similarly, the subgraph of $\mathrm{Cay}(G, \rho)$ induced by M^i_{lr} is strong connected if and only if $H^* \times K = \{\rho_{lr}\rangle$, where $\{\rho_{lr}\rangle$ denotes the subsemigroup generated by ρ_{lr} in the group $G^* \times G$. So we have proved the following:

Lemma 4.1. *Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. Put*

$$H = \langle \pi_G(\pi_1(\rho)) \rangle \text{ and } K = \langle \pi_G(\pi_2(\rho)) \rangle.$$

Suppose that G has the bi-coset decomposition (2.1) *with respect to H and K. Assume that $i \in I$ such that $H^{a_i} \cap K = 1$. Then for $l \in L$ and $r \in R$, the subgraph of Cay(G, ρ) induced by M^i_{lr} is*

(i) *connected if and only if $H^* \times K = [\rho_{lr}\rangle$, the subgroup generated by ρ_{lr} in $G^* \times G$;*
(ii) *strong connected if and only if $H^* \times K = \{\rho_{lr}\rangle$, the subsemigroup generated by ρ_{lr} in $G^* \times G$.*

Lemma 4.1, combining with Lemmas 3.3 and 3.5, implies the following

Lemma 4.2. *Under the same assumptions as in Lemma 4.1, the subgraph of Cay(G, ρ) induced by S_i is*

(i) *connected if and only if $H^* \times K = [\rho_{lr}\rangle$ for some $l \in L$ and $r \in R$;*
(ii) *strong connected if and only if $H^* \times K = \{\rho_{lr}\rangle$ for all $l \in L$ and $r \in R$.*

According to Lemmas 3.1, 3.5 and 4.2, we immediately obtain the following theorem.

Theorem 4.3. *Let $S = L \times G \times R$ be a rectangular group with an ideal extension $T = U \times P \times V$. Assume that condition* (3.1) *is satisfied. Let $\rho \subseteq T^1 \times T^1$ and let ρ^* be defined as in equation* (3.2). *Then the generalized Cayley graph Cay(G, ρ) is*

(i) *connected if and only if $G = HK$ and $H^* \times K = [\rho^*_{lr}\rangle$ for some $l \in L$ and $r \in R$;*
(ii) *strong connected if and only if $G = HK$ and $H^* \times K = \{\rho^*_{lr}\rangle$ for all $l \in L$ and $r \in R$.*

5. Isomorphism problem of generalized Cayley graphs of rectangular groups

Many research articles studied the isomorphism problem of usual Cayley graphs of a group or a semigroup, see for example [1, 5, 16]. In this section, we discuss the same problem for generalized Cayley graphs of rectangular groups.

Lemma 5.1. *Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. Put*

$$H = \langle \pi_G(\pi_1(\rho)) \rangle \text{ and } K = \langle \pi_G(\pi_2(\rho)) \rangle.$$

Let G have the bi-coset decomposition (2.1) *with respect to H and K. If $H^{a_i} \cap K = H^{a_j} \cap K = 1$ for some $i, j \in I$, then for any $a \in Ha_iK$ and $b \in Ha_jK$, and for any $l \in L$ and $r \in R$,*

$$\phi^{ab}_{lr} : M^i_{lr} \longrightarrow M^j_{lr}, (l, hak, r) \longmapsto (l, hbk, r)$$

is a colour-preserving isomorphism of (M^i_{lr}, E^i_{lr}) to (M^j_{lr}, E^j_{lr}), which induces a colour-preserving isomorphism of (S_i, E_i) to (S_j, E_j).

Proof. Take $a \in Ha_iK$ and $b \in Ha_jK$. Then $a = h_1a_ik_1$ and $b = h_2a_jk_2$ for some $h_1, h_2 \in H$ and some $k_1, k_2 \in K$. Since $H^{a_i} \cap K = 1$, we get

$$H^a \cap K = H^{h_1a_ik_1} \cap K = H^{a_ik_1} \cap K = (H^{a_i} \cap K)^{k_1} = 1^{k_1} = 1.$$

Similarly, we get $H^b \cap K = 1$. It is clear that $Ha_iK = HaK$ and $Ha_jK = HbK$. Thus by Lemma 2.1, every element x of Ha_iK can be expressed uniquely as $x = hak$, and also, every element y of Ha_jK can be expressed uniquely as $y = hbk$ with $h \in H$ and $k \in K$. Define a mapping

$$\phi^{ab} : Ha_iK \longrightarrow Ha_jK,\ hak \longmapsto hbk.$$

Then ϕ^{ab} is well defined and $\phi^{ab}(a) = b$. It is clear that ϕ^{ab} is a bijection, which induces naturally a bijection

$$\phi^{ab}_{lr} : M^i_{lr} \longrightarrow M^j_{lr},\ (l, hak, r) \longmapsto (l, hbk, r).$$

Some computations show that ϕ^{ab}_{lr} is a colour-preserving isomorphism of M^i_{lr} to M^j_{lr}. It is easily checked that ϕ^{ab}_{lr} can be naturally extended to a colour-preserving isomorphism of S_i to S_j. This completes the proof. ☐

Theorem 5.2. *Let $S = L \times G \times R$ be a rectangular group with an ideal extension $T = U \times P \times V$. Assume that condition* (3.1) *is satisfied. For $i = 1, 2$, let $\rho_i \subseteq T^1 \times T^1$, let ρ^*_i be defined as in equation* (3.2), *and let*

$$H_i = \langle \pi_G(\pi_1(\rho^*_i)) \rangle \ \text{ and } \ K_i = \langle \pi_G(\pi_2(\rho^*_i)) \rangle$$

be such that $H^a_i \cap K_i = 1$ for all $a \in G$ and that G can be decomposed into a disjoint union of a same finite number of bi-cosets with respect to H_i and K_i. Then $Cay(S, \rho_1) \cong Cay(S, \rho_2)$ if and only if $Cay(L \times (H_1K_1) \times R, \rho_1) \cong Cay(L \times (H_2K_2) \times R, \rho_2)$.

Proof. For $i = 1, 2$, since G can be decomposed into a disjoint union of a same finite number of bi-cosets with respect to H_i and K_i, there exists a finite number t such that

$$G = \dot{\bigcup}_{\lambda=1}^{t} H_ia_{i,\lambda}K_i$$

for some $a_{i,\lambda} \in G$. We may assume that $a_{1,1} = a_{2,1} = e_G$ without loss of generality. By Lemma 5.1, we obtain that for $i = 1, 2$ and all $\lambda \in \{1, 2, \ldots, t\}$,

$$L \times (H_ia_{i,\lambda}K_i) \times R \cong L \times (H_ia_{i,1}K_i) \times R = L \times (H_iK_i) \times R.$$

By Lemma 3.5, $H_ia_{i,\lambda}K_i$ is a strong subgraph of $\mathrm{Cay}(S, \rho)$ for any i and λ. So the assertion of the theorem follows immediately. ☐

We shall use the following condition in the sequel: for any $l, l', x' \in L$, $r, r', y \in R$ and any $g, g' \in G$,

$$\begin{aligned} &((l', g', r'), (l, g, r)) \in \rho \\ (5.1) \qquad \Longrightarrow \ &\text{there exist } x \in L,\ y' \in R \text{ such that } ((x', g', y'), (x, g, y)) \in \rho. \end{aligned}$$

It is easily seen that condition (5.1) is equivalent to the following condition: for any $l, l' \in L$, $r, r' \in R$ and any $(\lambda, \mu) \in \rho_{lr}$, there exists $(\lambda', \mu') \in \rho_{l'r'}$ such that

$$\pi_G(\lambda) = \pi_G(\lambda'), \text{ and } \pi_G(\mu) = \pi_G(\mu').$$

Lemma 5.3. *Let $S = L \times G \times R$ be a rectangular group and let $\rho \subseteq S \times S$. Put*

$$H = \langle \pi_G(\pi_1(\rho)) \rangle \text{ and } K = \langle \pi_G(\pi_2(\rho)) \rangle.$$

Assume that G has the bi-coset decomposition (2.1) *with respect to H and K. If $H^{a_i} \cap K = H^{a_j} \cap K = 1$ for some $i, j \in I$ and if condition* (5.1) *is satisfied, then for any $a \in Ha_iK$ and $b \in Ha_jK$,*

$$\phi^{ab}_{lrl'r'} : M^i_{lr} \longrightarrow M^j_{l'r'},\ (l, hak, r) \longmapsto (l', hbk, r')$$

is a graph isomorphism.

Proof. Take any $a \in Ha_iK$ and $b \in Ha_jK$. As in the proof of Lemma 5.1, we get that $Ha_iK = HaK$ and $Ha_jK = HbK$, and that $H^a \cap K = H^b \cap K = 1$. By Lemma 2.1, every element x of Ha_iK can be expressed uniquely as $x = hak$, and also, every element y of Ha_jK can be expressed uniquely as $y = hbk$ with $h \in H$ and $k \in K$. We have the following bijection

$$\phi^{ab} : Ha_iK \longrightarrow Ha_jK,\ hak \longmapsto hbk.$$

This induces naturally a bijection

$$\phi^{ab}_{lrl'r'} : M^i_{lr} \longrightarrow M^j_{l'r'},\ (l, hak, r) \longmapsto (l', hbk, r').$$

Let $((l, hak, r), (l, h'ak', r)) \in E^i_{lr}$. Then there exists $(\lambda, \mu) \in \rho_{lr}$ such that

$$\lambda(l, hak, r)\mu = (l, h'ak', r),$$

from which we deduce that

$$\pi_G(\lambda) \cdot hbk \cdot \pi_G(\mu) = h'bk'.$$

According to condition (5.1), there exists $(\lambda', \mu') \in \rho_{l'r'}$ such that

$$\pi_G(\lambda) = \pi_G(\lambda'), \text{ and } \pi_G(\mu) = \pi_G(\mu').$$

It follows that

$$\pi_G(\lambda') \cdot hbk \cdot \pi_G(\mu') = h'bk',$$

which yields that

$$\lambda'(l', hbk, r')\mu' = (l', h'bk', r').$$

Hence $((l', hbk, r'), (l', h'bk', r')) \in E^j_{l'r'}$, i.e.,

$$(\phi^{ab}_{lrl'r'}(l, hak, r), \phi^{ab}_{lrl'r'}(l, h'ak', r)) \in E^j_{lr}.$$

We have proved that $\phi^{ab}_{lrl'r'}$ is an isomorphism of the induced graph M^i_{lr} to the induced graph $M^j_{l'r'}$, which completes the proof. □

Using Theorem 5.2 and Lemma 5.3, one may prove the main theorem of this section:

Theorem 5.4. *Let $S = L\times G\times R$ be a rectangular group with an ideal extension $T = U \times P \times V$. Assume that condition* (3.1) *is satisfied. For $i = 1, 2$, let $\rho_i \subseteq T^1 \times T^1$, let ρ_i^* be defined as in equation* (3.2)*, and let*

$$H_i = \langle \pi_G(\pi_1(\rho_i^*))\rangle \text{ and } K_i = \langle \pi_G(\pi_2(\rho_i^*))\rangle$$

be such that $H_i^a \cap K_i = 1$ for all $a \in G$ and that G can be decomposed into a disjoint union of a same finite number of bi-cosets with respect to H_i and K_i. Suppose that both ρ_1^* and ρ_2^* satisfy condition (5.1)*. Then the following statements are equivalent*:

(i) *$Cay(S, \rho_1) \cong Cay(S, \rho_2)$;*
(ii) *$Cay(\{l\} \times (H_1K_1) \times \{r\}, \rho_1) \cong Cay(\{l\} \times (H_2K_2) \times \{r\}, \rho_2)$ for some $l \in L$ and $r \in R$;*
(iii) *$Cay(\{l\}\times(H_1K_1)\times\{r\}, \rho_1) \cong Cay(\{l\}\times(H_2K_2)\times\{r\}, \rho_2)$ for all $l \in L$ and $r \in R$.*

6. Vertex-transitivity of generalized Cayley graphs of rectangular groups

Some kind of transitivity of Cayley graphs of a group was discussed in [2]. There were also some papers on the vertex-transitivity of Cayley graphs of semigroups ([3, 11, 19, 20]). As main results of Andrei V. Kelarev and Cheryl E. Praeger in [11], Theorems 2.1 and 2.2 of [11] characterized all semigroups G and all subsets S of G, satisfying a certain finiteness condition, such that the usual Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive and $\mathrm{Aut}_S(G)$-vertex-transitive, respectively. These results are invalid for generalized Ceyley graphs of semigroups in general. In this section, we investigate the vertex-transitivity of a generalized Cayley graph of a rectangular group. As our main results of this section, Theorems 6.1 and 6.2 characterize $\mathrm{ColAut}_\rho(S)$-vertex-transitivity and $\mathrm{Aut}_\rho(S)$-vertex-transitivity of $\mathrm{Cay}(S, \rho)$ respectively, where S is a rectangular group with a certain ideal extension T and where $\rho \subseteq T^1 \times T^1$.

Theorem 6.1. *Let $S = L\times G\times R$ be a rectangular group with an ideal extension $T = U \times P \times V$. Assume that condition* (3.1) *is satisfied. Let $\rho \subseteq T^1 \times T^1$ and let ρ^* be defined as in equation* (3.2)*. Put*

$$H = \langle \pi_G(\pi_1(\rho^*))\rangle \text{ and } K = \langle \pi_G(\pi_2(\rho^*))\rangle.$$

Assume that $H^a \cap K = 1$ for all $a \in G$. Then the generalized Cayley graph $Cay(S, \rho)$ is $ColAut_\rho(S)$-vertex-transitive if and only if S is a group.

Proof. According to Lemma 3.1, $\rho^* \subseteq S \times S$ and $\mathrm{Cay}(S, \rho) = \mathrm{Cay}(S, \rho^*)$. So we may assume that $\rho = \rho^* \subseteq S \times S$ without loss of generality.

The 'only if' part. Assume that $\mathrm{Cay}(S, \rho)$ is $\mathrm{ColAut}_\rho(S)$-vertex-transitive. Suppose in contrast that S is not a group. Then $|L \times R| > 1$. Take any edge (α, β) of $\mathrm{Cay}(S, \rho)$. Then according to Lemma 3.5, we get $\alpha, \beta \in S_i$ for some i. Set $\alpha = (x, h'a_ik', y), \beta = (l, ha_ik, r)$. Then there exists $(\lambda, \mu) \in \rho$ such that $\lambda\alpha\mu = \beta$. It follows that $(\lambda, \mu) \in \rho_{lr}$. Since $|L \times R| > 1$, there exists $(l', r') \in L \times R$ such that $(l, r) \neq (l', r')$. Set $\beta' = (l', ha_ik, r')$. Then $\beta' \in S_i$. It follows that there exists $\phi \in \mathrm{ColAut}_\rho(S)$ such that $\phi(\beta) = \beta'$. Set $\phi(\alpha) = \alpha'$. By the definition of colour-preserving automorphisms, we get that $\lambda\phi(\alpha)\mu = \phi(\beta)$, that is, $\lambda\alpha'\mu = \beta'$. Since $(\lambda, \mu) \in \rho_{lr}$, a simple computation shows that $\lambda\alpha'\mu \in M^i_{lr}$, which is a contradiction to that $\beta' \in M^i_{l'r'}$. Thus S is a group.

The 'if' part. Assume that S is a group. Then $|L| = |R| = 1$. Let $L = \{l\}$ and $R = \{r\}$. By Lemma 2.1, G has the bi-coset decomposition (2.1) with respect to H and K. Fix $\alpha, \beta \in S$. Suppose that $\alpha = (l, a, r)$ and $\beta = (l, b, r)$ with $a, b \in G$. Then there exist $i, j \in I$ such that $a \in Ha_iK$ and $b \in Ha_jK$. In view of Lemma 5.1, for any $l \in L$ and $r \in R$,

$$\phi^{ab}_{lr} : M^i_{lr} \longrightarrow M^j_{lr},\ (l, hak, r) \longmapsto (l, hbk, r)$$

is a colour-preserving isomorphism of (M^i_{lr}, E^i_{lr}) to (M^j_{lr}, E^j_{lr}); similarly, ϕ^{ba}_{lr} is a colour-preserving isomorphism of (M^j_{lr}, E^j_{lr}) to (M^i_{lr}, E^i_{lr}). Next we consider two cases.

Case 1. $i = j$. For $\gamma \in S$, we define

$$\phi(\gamma) = \begin{cases} \phi^{ab}_{lr}(\gamma) & \text{if } \gamma \in S_i (= M^i_{lr}), \\ \gamma & \text{otherwise.} \end{cases}$$

Then $\phi \in \mathrm{ColAut}_\rho(G)$ such that $\phi(\alpha) = \beta$.

Case 2. $i \neq j$. Then S_i and S_j are two disjoint strong subgraphs of $\mathrm{Cay}(S, \rho)$ by Lemma 3.5. In view of Lemma 5.1, there exists a colour-preserving isomorphism ϕ^{ab}_{lr} of $S_i = M^i_{lr}$ to $S_j = M^j_{lr}$. Also, there is a colour-preserving isomorphism ϕ^{ba}_{lr} of $S_j = M^j_{lr}$ to $S_i = M^i_{lr}$. For $\gamma \in S$, we define

$$\phi(\gamma) = \begin{cases} \phi^{ab}_{lr}(\gamma) & \text{if } \gamma \in S_i, \\ \phi^{ba}_{lr}(\gamma) & \text{if } \gamma \in S_j, \\ \gamma & \text{otherwise.} \end{cases}$$

Then $\phi \in \mathrm{ColAut}_\rho(S)$ such that $\phi(\alpha) = \beta$. Therefore, the generalized Cayley graph $\mathrm{Cay}(S, \rho)$ is $\mathrm{ColAut}_\rho(S)$-vertex-transitive and the proof is complete. □

Theorem 6.2. *Let $S = L \times G \times R$ be a rectangular group with an ideal extension $T = U \times P \times V$. Assume that condition* (3.1) *is satisfied. Let $\rho \subseteq T^1 \times T^1$ and let ρ^* be defined as in equation* (3.2). *Put*

$$H = \langle \pi_G(\pi_1(\rho^*)) \rangle \ \text{ and } \ K = \langle \pi_G(\pi_2(\rho^*)) \rangle.$$

Assume that $H^a \cap K = 1$ for all $a \in G$ and suppose that ρ^* satisfies condition (5.1). *Then the generalized Cayley graph Cay(S,ρ) is Aut$_\rho(S)$-vertex-transitive.*

Proof. First, according to Lemma 3.1, $\rho^* \subseteq S \times S$ and $\mathrm{Cay}(S,\rho) = \mathrm{Cay}(S,\rho^*)$. So we may assume that $\rho = \rho^* \subseteq S \times S$ without loss of generality. By Lemma 2.1, G has the bi-coset decomposition (2.1) with respect to H and K, that is, there exist an index set I and some $a_i \in G$ for each $i \in I$ such that

$$G = \dot{\bigcup}_{i \in I} H a_i K.$$

Since ρ^* satisfies condition (5.1), so does ρ. It follows that Lemma 5.3 applies. For any $a, b \in G$, $l, l' \in L$ and any $r, r' \in R$, define a mapping $\psi^{ab}_{lrl'r'} : S_i \longrightarrow S_j$ by

$$\psi^{ab}_{lrl'r'}(x, hak, y) = \begin{cases} \phi^{ab}_{lrl'r'}(l, hak, r) & \text{if } (x,y) = (l,r), \\ \phi^{ab}_{l'r'lr}(l', hak, r') & \text{if } (x,y) = (l', r'), \\ (x, hak, y) & \text{otherwise}, \end{cases}$$

where $\phi^{ab}_{lrl'r'}$ is defined as in Lemma 5.3. Let $((x', h'ak', y'), (x, hak, y)) \in E_i$. Then by Lemma 3.2, $((x, h'ak', y), (x, hak, y)) \in E^i_{xy}$. Consider three cases:

Case 1. $(x,y) = (l,r)$. Then $((l, h'ak', r), (l, hak, r)) \in E^i_{lr}$. By Lemma 5.3, $(\phi^{ab}_{lrl'r'}(l, hak, r), \phi^{ab}_{lrl'r'}(l, h'ak', r)) \in E^j_{l'r'}$. But by the definition of $\psi^{ab}_{lrl'r'}$, we get that

$$\psi^{ab}_{lrl'r'}(x, hak, y) = \phi^{ab}_{lrl'r'}(l, hak, r),$$
$$\psi^{ab}_{lrl'r'}(x, h'ak', y) = \phi^{ab}_{lrl'r'}(l, h'ak', r).$$

So we have $(\psi^{ab}_{lrl'r'}(x, hak, y), \psi^{ab}_{lrl'r'}(x, h'ak', y)) \in E_j$.

Case 2. $(x,y) = (l', r')$. A similar argument as in Case 1 shows that we also have $(\psi^{ab}_{lrl'r'}(x, hak, y), \psi^{ab}_{lrl'r'}(x, h'ak', y)) \in E_j$.

Case 3. It is routine to check that $(\psi^{ab}_{lrl'r'}(x, hak, y), \psi^{ab}_{lrl'r'}(x, h'ak', y)) \in E_j$.

Summing up, we have proved that for any $a, b \in G$, $l, l' \in L$ and any $r, r' \in R$, $\psi^{ab}_{lrl'r'}$ is an isomorphism of S_i to S_j such that $\psi^{ab}_{lrl'r'}(l, a, r) = (l', b, r')$.

Next, take any $\alpha, \beta \in S$. Assume that $\alpha = (l, a, r)$ and $\beta = (l, b, r)$ for some $a, b \in G$, $l, l' \in L$ and $r, r' \in R$. Then there exist $i, j \in I$ such that $a \in Ha_iK$ and $b \in Ha_jK$. We consider two cases.

Case 1. $i = j$. For $\gamma \in S$, we define

$$\phi(\gamma) = \begin{cases} \psi^{ab}_{lrl'r'}(\gamma) & \text{if } \gamma \in S_i (= M^i_{lr}), \\ \gamma & \text{otherwise}. \end{cases}$$

Then $\phi \in \mathrm{Aut}_\rho(G)$ such that $\phi(\alpha) = \beta$.

Case 2. $i \neq j$. Then S_i and S_j are two disjoint strong subgraphs of $\mathrm{Cay}(S,\rho)$ by Lemma 3.5. In view of Lemma 5.1, there exists a colour-preserving isomorphism ϕ^{ab}_{lr} of $S_i = M^i_{lr}$ to $S_j = M^j_{lr}$. Also, there is a colour-preserving isomorphism ϕ_{lr}^{ba} of $S_j = M_{lr}^j$ to $S_i = M_{lr}^i$. For $\gamma \in S$, we define

$$\phi(\gamma) = \begin{cases} \psi_{lrl'r'}^{ab}(\gamma) & \text{if } \gamma \in S_i, \\ \psi_{ba}(\gamma) & \text{if } \gamma \in S_j, \\ \gamma & \text{otherwise.} \end{cases}$$

Then $\phi \in \mathrm{Aut}_\rho(S)$ such that $\phi(\alpha) = \beta$. Therefore, the generalized Cayley graph $\mathrm{Cay}(S, \rho)$ is $\mathrm{Aut}_\rho(S)$-vertex-transitive. This completes the proof. □

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References

[1] L. Babai, *Automorphism groups, isomorphism, reconstruction*, in: Handbook of Combinatorics, pp. 1447–1540, Elsevier, Amsterdam, 1995.
[2] A. Devillers, W. Jin, C. H. Li, and C. E. Praeger, *On normal 2-geodesic transitive Cayley graphs*, J. Algebraic Combin. **39** (2014), no. 4, 903–918.
[3] S. Fan, *Vertex transitive Cayley graphs of semigroups of order a product of two primes*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **14** (2007), Bio-inspired computingtheory and applications, Part 2, suppl. S3, pp. 905–909.
[4] R. S. Gigoń, *Rectangular group congruences on a semigroup*, Semigroup Forum **87** (2013), no. 1, 120–128.
[5] C. D. Godsil, *On Cayley graph isomorphisms*, Ars Combin. **15** (1983), 231–246.
[6] D. Grynkiewicz, V. F. Lev, and O. Serra, *Connectivity of addition Cayley graphs*, J. Combin. Theory Ser. B **99** (2009), no. 1, 202–217.
[7] Y. Hao, X. Yang, and N. Jin, *On transitive Cayley graphs of strong semilattices of rectangular groups*, Ars Combin. **105** (2012), 183–192.
[8] J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.
[9] A. V. Kelarev, *On undirected Cayley graphs*, Australas. J. Combin. **25** (2002), 73–78.
[10] ———, *Graph Algebras and Automata*, Marcel Dekker, Inc., New York, 2003.
[11] A. V. Kelarev and C. E. Praeger, *On transitive Cayley graphs of groups and semigroups*, European J. Combin. **24** (2003), no. 1, 59–72.
[12] A. V. Kelarev and S. J. Quinn, *Directed graphs and combinatorial properties of semigroups*, J. Algebra **251** (2002), no. 1, 16–26.
[13] A. V. Kelarev, J. Ryan, and J. Yearwood, *Cayley graphs as classifiers for data mining: The influence of asymmetries*, Discrete Math. **309** (2009), no. 17, 5360–5369.
[14] B. Khosravi and M. Mahmoudi, *On Cayley graphs of rectangular groups*, Discrete Math. **310** (2010), no. 4, 804–811.
[15] C. H. Li, *Finite CI-groups are soluble*, Bull. London Math. Soc. **31** (1999), no. 4, 419–423.
[16] ———, *On isomorphisms of finite Cayley graphs–a survey*, Discrete Math. **256** (2002), no. 1-2, 301–334.
[17] S. Panma, *Characterization of Cayley graphs of rectangular groups*, Thai J. Math. **8** (2010), no. 3, 535–543.
[18] S. Panma, N. N. Chiangmai, U. Knauer, and Sr. Arworn, *Characterizations of Clifford semigroup digraphs*, Discrete Math. **306** (2006), no. 12, 1247–1252.
[19] S. Panma, U. Knauer, and Sr. Arworn, *On transitive Cayley graphs of right (left) groups and of Clifford semigroups*, Thai J. Math. **2** (2004), 183–195.
[20] ———, *On transitive Cayley graphs of strong semilattices of right (left) groups*, Discrete Math. **309** (2009), no. 17, 5393–5403.
[21] M. Suzuki, *Group Theory I*, Springer, New York, 1982.

[22] S. F. Wang, *A problem on generalized Cayley graphs of semigroups*, Semigroup Forum **86** (2013), no. 1, 221–223.

[23] R. J. Wilson, *Introduction to Graph Theory*, 3rd edn, Longman, New York, 1982.

[24] Y. Zhu, *Generalized Cayley graphs of semigroups I*, Semigroup Forum **84** (2012), no. 1, 131–143.

[25] ———, *Generalized Cayley graphs of semigroups II*, Semigroup Forum **84** (2012), no. 1, 144–156.

[26] ———, *Cayley-symmetric semigroups*, Bull. Korean Math. Soc. **52** (2015), no. 2, 409–419.

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