

VARIATIONAL RESULT FOR THE BIFURCATION PROBLEM OF THE HAMILTONIAN SYSTEM

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ABSTRACT. We get a theorem which shows the existence of at least four 2π -periodic weak solutions for the bifurcation problem of the Hamiltonian system with the superquadratic nonlinearity. We obtain this result by using the variational method, the critical point theory induced from the limit relative category theory.

1. Introduction

Let $H(t, z(t))$ be a C^2 function defined on $\mathbb{R}^1 \times \mathbb{R}^{2n}$ which is 2π -periodic with respect to the first variable t , and $\lambda \in \mathbb{R}$. In this paper we consider the number of the 2π -periodic weak solutions for the bifurcation problem of the following Hamiltonian system

$$(1.1) \quad \begin{aligned} \dot{p}(t) &= -\lambda q(t) - H_q(t, p(t), q(t)), \\ \dot{q}(t) &= \lambda p(t) + H_p(t, p(t), q(t)), \end{aligned}$$

where $p, q \in \mathbb{R}^n$. Let $z = (p, q)$ and J be the standard symplectic structure on \mathbb{R}^{2n} , i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Then (1.1) can be rewritten as

$$(1.2) \quad -J\dot{z} = \lambda z + H_z(t, z(t)),$$

where $\dot{z} = \frac{dz}{dt}$ and H_z is the gradient of H . We assume that $H \in C^2(\mathbb{R}^1 \times \mathbb{R}^{2n}, \mathbb{R}^1)$ satisfies the following conditions:

(H1) $H \in C^2(\mathbb{R}^1 \times \mathbb{R}^{2n}, \mathbb{R})$, $H(0, \theta) = 0$, where $\theta = (0, \dots, 0)$.

(H2) There exist $1 < p_1 \leq p_2 < 2p_1 + 1$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 \geq 0$ such that

$$\alpha_1 \|z(t)\|_{\mathbb{R}^{2n}}^{p_1+1} - \beta_1 \leq H(t, z(t)) \leq \alpha_2 \|z(t)\|_{\mathbb{R}^{2n}}^{p_2+1} \quad \text{for every } z \in \mathbb{R}^{2n}.$$

(H3) H is a 2π -periodic function with respect to t .

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Several ([1], [5], [6], [7] etc.) studied the nonlinear Hamiltonian system. Jung and Choi ([5], [6], [7]) considered (1.1) with nonsingular potential nonlinearity or jumping nonlinearity crossing one eigenvalue, or two eigenvalues, or several eigenvalues. Chang ([1]) proved that (1.1) has at least two nontrivial 2π -periodic weak solutions under some asymptotic nonlinearity. Jung and Choi ([6]) proved that (1.1) has at least m weak solutions, which are geometrically distinct and nonconstant under some jumping nonlinearity.

We are looking for the weak solutions of (1.1) under the conditions (H1)-(H3). The 2π -periodic weak solution $z = (p, q) \in E$ of (1.1) satisfies

$$\int_0^{2\pi} (\dot{z} - \lambda Jz(t) - JH_z(t, z(t))) \cdot Jw dt = 0 \text{ for all } w \in E,$$

i.e.,

$$\int_0^{2\pi} [(\dot{p} + \lambda q(t) + H_q(t, z(t))) \cdot \psi - (\dot{q} - \lambda p(t) - H_p(t, z(t))) \cdot \phi] dt = 0$$

for all $\zeta = (\phi, \psi) \in E$, where E is introduced in Section 2.

Our main result is as follows:

Theorem 1.1. *Assume that H satisfies the conditions (H1)-(H3) and that j_0 and j_1 are negative integers with $j_1 < j_0 < 0$. Then there exists a small number $\delta > 0$ such that for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$, system (1.1) has at least four nontrivial 2π -periodic weak solutions.*

The outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce the perturbed operator $A_\epsilon = \epsilon I + A$ of the operator A ($A(z(t)) = -J\dot{z}(t)$) to make the compactness of the operator $|A_\epsilon|^{-1}$ and so prove the (P.S.)* condition for the associated functional of the perturbed problem $A_\epsilon(z) = \lambda z + \epsilon z + H_z(t, z(t))$. We also recall the critical point theory induced from the limit relative category, which plays a crucial role to prove the multiplicity result. In Section 3, we prove the existence of the first weak solution and the second one of (1.1), and in Section 4 we prove the existence of the third weak solution and fourth one of (1.1) by the critical point theory induced from the limit relative category, and prove Theorem 1.1.

2. Abstract critical point theory

Let $L^2([0, 2\pi], \mathbb{R}^{2n})$ denote the set of $2n$ -tuples of the square integrable 2π -periodic functions and choose $z \in L^2([0, 2\pi], \mathbb{R}^{2n})$. Then it has a Fourier expansion $z(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{ikt}$, with $a_k = \frac{1}{2\pi} \int_0^{2\pi} z(t) e^{-ikt} dt \in \mathbb{C}^{2n}$, $a_{-k} = \bar{a}_k$ and $\sum_{k \in \mathbb{Z}} |a_k|^2 < \infty$. Let

$$A : z(t) \mapsto -J\dot{z}(t)$$

with domain

$$D(A) = \{z(t) \in H^1([0, 2\pi], \mathbb{R}^{2n}) \mid z(0) = z(2\pi)\}$$

$$= \{z(t) \in L^2([0, 2\pi], \mathbb{R}^{2n}) \mid \sum_{k \in \mathbb{Z}} (\epsilon + |k|)^2 |a_k|^2 < +\infty\},$$

where ϵ is a positive small number. Then A is self-adjoint operator. Let $\{M_\lambda\}$ be the spectral resolution of A , and let τ be a positive number such that $\tau \notin \sigma(A)$ and $[-\tau, \tau]$ contains only one element 0 of $\sigma(A)$ and r be a positive number such that $r \notin \sigma(A)$, $[j_0 - r, j_0 + r]$ contains only one element j_0 of $\sigma(A)$, $[j_1 - r, j_1 + r]$ contains only one element j_1 of $\sigma(A)$ and $j_1 + r < j_0 - r$. Let

$$P_0 = \int_{-\tau}^{\tau} dM_\lambda, \quad P_+ = \int_{\tau}^{+\infty} dM_\lambda, \quad P_- = \int_{-\infty}^{-\tau} dM_\lambda,$$

$$P_{(-\infty, j_1 - 1]} = \int_{-\infty}^{j_1 - 1} dM_\lambda, \quad P_{j_0} = \int_{j_0 - r}^{j_0 + r} dM_\lambda, \quad P_{j_1} = \int_{j_1 - r}^{j_1 + r} dM_\lambda.$$

Let

$$L_0 = P_0 L^2([0, 2\pi], \mathbb{R}^{2n}), \quad L_+ = P_+ L^2([0, 2\pi], \mathbb{R}^{2n}),$$

$$L_- = P_- L^2([0, 2\pi], \mathbb{R}^{2n}), \quad L_{(-\infty, j_1 - 1]} = P_{(-\infty, j_1 - 1]} L^2([0, 2\pi], \mathbb{R}^{2n}),$$

$$L_{j_0} = P_{j_0} L^2([0, 2\pi], \mathbb{R}^{2n}), \quad L_{j_1} = P_{j_1} L^2([0, 2\pi], \mathbb{R}^{2n}).$$

For each $u \in L^2([0, 2\pi], \mathbb{R}^{2n})$, we have the composition

$$u = u_0 + u_+ + u_-,$$

where $u_0 \in L_0$, $u_+ \in L_+$, $u_- \in L_-$. According to A , there exists a small number $\epsilon > 0$ such that $-\epsilon \notin \sigma(A)$. Let us define the space E as follows:

$$E = D(|A|^{\frac{1}{2}}) = \{z \in L^2([0, 2\pi], \mathbb{R}^{2n}) \mid \sum_{k \in \mathbb{Z}} (\epsilon + |k|) |a_k|^2 < \infty\}$$

with the scalar product

$$(z, w)_E = \epsilon(z, w)_{L^2} + (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_{L^2}$$

and the norm

$$\|z\| = (z, z)_E^{\frac{1}{2}} = \left(\sum_{k \in \mathbb{Z}} (\epsilon + |k|) |a_k|^2 \right)^{\frac{1}{2}}.$$

The space E endowed with this norm is a real Hilbert space continuously embedded in $L^2([0, 2\pi], \mathbb{R}^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-\frac{1}{2}, 2}([0, 2\pi], \mathbb{R}^{2n})$. We note that the operator $(\epsilon I + |A|)^{-1}$ is a compact linear operator from $L^2([0, 2\pi], \mathbb{R}^{2n})$ to E such that

$$((\epsilon I + |A|)^{-1}w, z)_E = \int_0^{2\pi} (w(t), z(t)) dt.$$

Let

$$A_\epsilon = \epsilon I + A.$$

Let

$$E_0 = |A_\epsilon|^{-\frac{1}{2}} L_0, \quad E_+ = |A_\epsilon|^{-\frac{1}{2}} L_+, \quad E_- = |A_\epsilon|^{-\frac{1}{2}} L_-,$$

$$E_{(-\infty, j_1 - 1]} = |A_\epsilon|^{-\frac{1}{2}} L_{(-\infty, j_1 - 1]}, \quad E_{j_0} = |A_\epsilon|^{-\frac{1}{2}} L_{j_0}, \quad E_{j_1} = |A_\epsilon|^{-\frac{1}{2}} L_{j_1}.$$

Then $E = E_0 \oplus E_+ \oplus E_-$, and for $z \in E$, z has the decomposition $z = z_0 + z_+ + z_- \in E$, where

$$(2.1) \quad z_0 = |A_\epsilon|^{-\frac{1}{2}}u_0, \quad z_+ = |A_\epsilon|^{-\frac{1}{2}}u_+, \quad z_- = |A_\epsilon|^{-\frac{1}{2}}u_-.$$

Thus we have

$$\|z_0\|_{E_0} = \|u_0\|_{L_0}, \quad \|z_+\|_{E_+} = \|u_+\|_{L_+}, \quad \|z_-\|_{E_-} = \|u_-\|_{L_-}$$

and that $E_0, E_+, E_-, E_{(-\infty, j_1-1]}, E_{j_1}, E_{j_0}$ are isomorphic to $L_0, L_+, L_-, L_{(-\infty, j_1-1]}, L_{j_1}, L_{j_0}$ respectively. Moreover $E = E_{(-\infty, j_1-1]} \oplus E_{j_1} \oplus E_{j_0} \oplus E_+$. Let us define a functional

$$f(u) = \frac{1}{2}(\|u_+\|_{L^2}^2 + \|M_+u_0\|_{L^2}^2 - \|M_-u_0\|_{L^2}^2 - \|u_-\|_{L^2}^2) - \psi_\epsilon(z),$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2}\|z\|_{L^2}^2$, $\psi(z) = \int_0^{2\pi} [\frac{\lambda}{2}z(t)^2 + H(t, z(t))]dt$, $M_+ = \int_0^\infty dM_\lambda$, $M_- = \int_{-\infty}^0 dM_\lambda$. By $H \in C^2$, $\psi(z) = \int_0^{2\pi} [\frac{\lambda}{2}z(t)^2 + H(t, z(t))] \in C^2$. Let

$$F(z) = \lambda z(t) + H_z(t, z(t)), \quad F_\epsilon(z) = \epsilon z + F(z) = \epsilon z + \lambda z(t) + H_z(t, z(t)).$$

Then (1.2) can be rewritten as

$$(2.2) \quad A_\epsilon(z) = F_\epsilon(z).$$

The Euler equation of the functional $f(u)$ is the system

$$(2.3) \quad u_+ = |A_\epsilon|^{-\frac{1}{2}}P_+F_\epsilon(z),$$

$$(2.4) \quad u_- = -|A_\epsilon|^{-\frac{1}{2}}P_-F_\epsilon(z),$$

$$(2.5) \quad M_+u_0 = |A_\epsilon|^{-\frac{1}{2}}M_+P_0F_\epsilon(z), \quad M_-u_0 = -|A_\epsilon|^{-\frac{1}{2}}M_-P_0F_\epsilon(z).$$

The system (2.3)-(2.5) is reduced to

$$(2.6) \quad A_\epsilon z_+ = P_+F_\epsilon(z_0 + z_+ + z_-) \quad \text{or} \quad z_+ = (A_\epsilon)^{-1}P_+F_\epsilon(z_0 + z_+ + z_-),$$

$$(2.7) \quad A_\epsilon z_- = P_-F_\epsilon(z_0 + z_+ + z_-) \quad \text{or} \quad z_- = (A_\epsilon)^{-1}P_-F_\epsilon(z_0 + z_+ + z_-),$$

$$(2.8) \quad A_\epsilon M_+z_0 = M_+P_0F_\epsilon(z_0 + z_+ + z_-), \quad A_\epsilon M_-z_0 = M_-P_0F_\epsilon(z_0 + z_+ + z_-).$$

It follows from (2.3)-(2.8) that $z = z_0 + z_+ + z_-$ is a solution of (1.2) if and only if $u = u_0 + u_+ + u_-$ is a critical point of f . By (2.1), we define a functional

$$I(z) = f(u(z)).$$

The functional $I(z)$ is of the form

$$I(z) = \frac{1}{2}(\| |A_\epsilon|^{\frac{1}{2}}z_+\|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}}M_+z_0\|_{L^2}^2 - \| (-|A_\epsilon|)^{\frac{1}{2}}z_-\|_{L^2}^2 - \| (-|A_\epsilon|)^{\frac{1}{2}}M_-z_0\|_{L^2}^2) - \psi_\epsilon(z).$$

Thus it suffices to find the critical points of the functional I to find the critical points of the functional f . By the following Lemma 2.1, the weak solutions of (2.2) coincide with the critical points of the functional $I(z)$.

Lemma 2.1. *Assume that H satisfies the conditions (H1)-(H3) and $\lambda \notin \mathbb{Z}$. Then $I(z)$ is continuous and Fréchet differentiable in E with Fréchet derivative (2.9)*

$$DI(z)w = \int_0^{2\pi} (JA_\epsilon z(t) - (\lambda + \epsilon)Jz(t) - JH_z(t, z(t))) \cdot Jw dt \quad \text{for all } w \in E.$$

Moreover $DI \in C$. That is, $I \in C^1$.

Proof. First we prove that $I(z) = \int_0^{2\pi} [\frac{1}{2}A_\epsilon z - \frac{\epsilon + \lambda}{2}z^2 - H(t, z(t))]dt$ is continuous in E . For $z, w \in E$,

$$\begin{aligned} & |I(z+w) - I(z)| \\ &= \left| \int_0^{2\pi} \frac{1}{2}A_\epsilon(z+w) \cdot (z+w) - \int_0^{2\pi} [H(t, z+w) + \frac{\epsilon + \lambda}{2}(z+w)^2] \right. \\ &\quad \left. - \int_0^{2\pi} \frac{1}{2}A_\epsilon(z) \cdot z + \int_0^{2\pi} [H(t, z) + \frac{\epsilon + \lambda}{2}z^2] \right| \\ &= \left| \int_0^{2\pi} \frac{1}{2}[A_\epsilon(z) \cdot w + A_\epsilon(w) \cdot z + A_\epsilon(w) \cdot w] \right. \\ &\quad \left. - \int_0^{2\pi} [H(t, z+w) - H(t, z) + \frac{\epsilon + \lambda}{2}(2z \cdot w + w^2)] \right|. \end{aligned}$$

We note that

$$\left| \int_0^{2\pi} \frac{1}{2}[A_\epsilon(z) \cdot w + A_\epsilon(w) \cdot z + A_\epsilon(w) \cdot w] \right| = O(\|w\|_{\mathbb{R}^{2n}})$$

and

$$(2.10) \quad \left| \int_0^{2\pi} [H(t, z+w) - H(t, z)]dt \right| \leq \left| \int_0^{2\pi} [H_z(t, z(t)) \cdot w + O(\|w\|_{\mathbb{R}^{2n}})]dt \right| = O(\|w\|_{\mathbb{R}^{2n}}).$$

Thus we have

$$|I(z+w) - I(z)| = O(\|w\|_{\mathbb{R}^{2n}}).$$

Next we shall prove that $I(z)$ is Fréchet differentiable in E . For $z, w \in E$,

$$\begin{aligned} & |I(z+w) - I(z) - DI(z)w| \\ &= \left| \int_0^{2\pi} \frac{1}{2}A_\epsilon(z+w) \cdot (z+w) - \int_0^{2\pi} [H(t, z+w) + \frac{\epsilon + \lambda}{2}(z+w)^2] \right. \\ &\quad \left. - \int_0^{2\pi} \frac{1}{2}A_\epsilon(z) \cdot z + \int_0^{2\pi} [H(t, z) + \frac{\epsilon + \lambda}{2}z^2] \right. \\ &\quad \left. - \int_0^{2\pi} JA_\epsilon(z) \cdot Jw + \int_0^{2\pi} [(JH_z(t, z) + (\epsilon + \lambda)Jz) \cdot Jw] \right| \\ &= \left| \int_0^{2\pi} \frac{1}{2}[A_\epsilon(w) \cdot z + A_\epsilon(w) \cdot w] \right| \end{aligned}$$

$$- \int_0^{2\pi} \left[H(t, z + w) - H(t, z) - H_z(t, z) \cdot w + \frac{\epsilon + \lambda}{2} w^2 \right] dt.$$

By (2.10), we have

$$\int_0^{2\pi} [H(t, z + w) - H(t, z) - H_z(t, z)] dt = O(\|w\|_{\mathbb{R}^{2n}}).$$

Thus

$$|I(z + w) - I(z) - DI(z)w| = O(\|w\|_{\mathbb{R}^{2n}}). \quad \square$$

Now, we recall the critical point theory on the manifold with boundary. Let E be a Hilbert space and M be the closure of an open subset of E such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \rightarrow \mathbb{R}$ be a $C^{1,1}$ functional, where W is an open set containing M . For applying the usual topological methods of the critical points theory we need a suitable notion of critical point for f on M . We recall the following notions: lower gradient of f on M , $(P.S.)_c^*$ condition and the limit relative category (see [4]).

Definition 2.1. If $z \in M$, the lower gradient of f on M at z is defined by

$$(2.11) \quad \text{grad}_M^- f(z) = \begin{cases} Df(z) & \text{if } z \in \text{int}(M), \\ Df(z) + \langle Df(z), \nu(z) \rangle^- \nu(z) & \text{if } z \in \partial M, \end{cases}$$

where we denote by $\nu(z)$ the unit normal vector to ∂M at the point z , pointing outwards. We say that z is a lower critical for f on M , if $\text{grad}_M^- f(z) = 0$.

Since the functional $I(z)$ is strongly indefinite, the notion of the $(P.S.)_c^*$ condition and the limit relative category is a very useful tool for the proof of the main theorems.

Let $(E_n)_n$ be a sequence of closed subspaces of E with the conditions:

$$(2.12) \quad E_n = E_{n-} \oplus E_0 \oplus E_{n+}, \text{ where } E_{n+} \subset E_+, E_{n-} \subset E_- \text{ for all } n, \\ (E_{n+} \text{ and } E_{n-} \text{ are subspaces of } E), \dim E_n < +\infty, E_n \subset E_{n+1}, \cup_{n \in \mathbb{N}} E_n \text{ is} \\ \text{dense in } E. \text{ Let } P_{E_n} \text{ be the orthogonal projections from } E \text{ onto } E_n. M_n = \\ M \cap E_n, \text{ for any } n, \text{ be the closure of an open subset of } E_n \text{ and has the structure} \\ \text{of a } C^2 \text{ manifold with boundary in } E_n. \text{ We assume that for any } n \text{ there exists} \\ \text{a retraction } r_n : M \rightarrow M_n. \text{ For given } B \subset E, \text{ we will write } B_n = B \cap E_n.$$

Definition 2.2. Let $c \in \mathbb{R}$. We say that f satisfies the $(P.S.)_c^*$ condition with respect to $(M_n)_n$, on the manifold M with boundary, if for any sequence $(k_n)_n$ in \mathbb{N} and any sequence $(u_n)_n$ in M such that $k_n \rightarrow \infty, u_n \in M_{k_n}, \forall n, f(u_n) \rightarrow c, \text{grad}_{M_{k_n}}^- f(u_n) \rightarrow 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $\text{grad}_M^- f(u) = 0$.

Let Y be a closed subspace of M .

Definition 2.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $\text{cat}_{M,Y}(B)$ of B in (M, Y) , as the least integer h such that there exist $h + 1$ closed subsets U_0, U_1, \dots, U_h with the following properties:

$B \subset U_0 \cup U_1 \cup \dots \cup U_h$;
 U_1, \dots, U_h are contractible in M ;
 $Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0, 1] \rightarrow M$ such that

$$\begin{aligned} F(x, 0) &= x \quad \forall x \in U_0, \\ F(x, t) &\in Y \quad \forall x \in Y, \forall t \in [0, 1], \\ F(x, 1) &\in Y \quad \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Definition 2.4. Let (X, Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X . For any subset B of X we define the limit relative category of B in (X, Y) , with respect to $(X_n)_n$, by

$$(2.13) \quad cat_{(X,Y)}^*(B) = \limsup_{n \rightarrow \infty} cat_{(X_n, Y_n)}(B_n).$$

Now we consider a theorem which gives an estimate of the number of critical points of a functional, in terms of the limit relative category of its sublevels. The theorem is proved repeating the classical arguments, using the nonsmooth version of the classical Deformation Lemma for functions on manifolds with boundary.

Let Y be a fixed subset of M . We set

$$(2.14) \quad \mathcal{B}_i = \{B \subset M \mid cat_{(M,Y)}^*(B) \geq i\},$$

$$(2.15) \quad c_i = \inf_{B \in \mathcal{B}_i} \sup_{x \in B} f(x).$$

We have the following multiplicity theorem.

Theorem 2.1. *Let $i \in \mathbb{N}$ and assume that*

- (1) $c_i < +\infty$,
- (2) $\sup_{x \in Y} f(x) < c_i$,
- (3) *the $(P.S.)_{c_i}^*$ condition with respect to $(M_n)_n$ holds.*

Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \dots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M \mid f(x) = c, grad_M^- f(x) = 0\}) \geq k.$$

Proof. Let $c = c_i$; using the $(P.S.)_c^*$ condition, with respect to $(M_n)_n$, one can prove that, for any neighborhood N of

$$K_c = \{x \mid f(x) = c, grad_M^- f(x) = 0\},$$

there exist n_0 in \mathbb{N} and $\delta > 0$ such that $\|grad_M^- f\| \geq \delta$ for all $n \geq n_0$ and all $x \in E_n \setminus N$ with $c - \delta \leq f(x) \leq c + \delta$. Moreover it is not difficult to see that, for all n , the function $\tilde{f}_n : E_n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by $\tilde{f}_n = f(x)$, if $x \in M_n$, $\tilde{f}_n(x) = +\infty$, otherwise, is ϕ -convex of order two, according to the definitions of [5]. Then the conclusion follows using the same arguments of [1, 8] and the nonsmooth version of the classical Deformation Lemma. □

Now we state the following multiplicity result (for the proof see Theorem 4.6 of [8]) which will be used in the proof of our main theorem.

Theorem 2.2. *Let E be a Hilbert space and let $E = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of E with X_1, X_2 of finite dimension. For a given subspace X of E , let P_X be the orthogonal projection from E onto X . Set*

$$C = \{x \in E \mid \|P_{X_2}x\| \geq 1\}$$

and let $f : W \rightarrow \mathbb{R}$ be a $C^{1,1}$ function defined on a neighborhood W of C . Let $1 < \rho < R, R_1 > 0$. We define

$$\begin{aligned} \Delta &= \{x_1 + x_2 \mid x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, 1 \leq \|x_2\| \leq R\}, \\ \Sigma &= \{x_1 + x_2 \mid x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = 1\} \\ &\quad \cup \{x_1 + x_2 \mid x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = R\} \\ &\quad \cup \{x_1 + x_2 \mid x_1 \in X_1, x_2 \in X_2, \|x_1\| = R_1, 1 \leq \|x_2\| \leq R\}, \\ S &= \{x \in X_2 \oplus X_3 \mid \|x\| = \rho\}, \\ B &= \{x \in X_2 \oplus X_3 \mid \|x\| \leq \rho\}. \end{aligned}$$

(a) Assume that

$$\sup f(\Sigma) < \inf f(S)$$

and

(b) Assume that the $(P.S.)_c$ condition holds for f on C , with respect to the sequence $(C_n)_n, \forall c \in [\alpha, \beta]$, where

$$\alpha = \inf f(S), \quad \beta = \sup f(\Delta).$$

(c) Moreover we assume $\beta < +\infty$ and $f|_{X_1 \oplus X_3}$ has no critical points z in $X_1 \oplus X_3$ with $\alpha \leq f(z) \leq \beta$.

Then there exist two lower critical points z_1, z_2 for f on C such that $\alpha \leq f(z_i) \leq \beta, i = 1, 2$.

3. Existence of two solutions on E_{j_0}

Throughout this section we assume that H satisfies the conditions (H1)-(H3), $\lambda \notin \mathbb{Z}$ and $\lambda < 0$. Let $I(z)$ be the functional defined in Section 2, i.e.,

$$\begin{aligned} I(z) &= \frac{1}{2}(\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|_{L^2}^2 - \| (-|A_\epsilon|)^{\frac{1}{2}} z_- \|_{L^2}^2 \\ &\quad - \| (-|A_\epsilon|)^{\frac{1}{2}} M_- z_0 \|_{L^2}^2) - \psi_\epsilon(z), \end{aligned}$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2} \|z\|_{L^2}^2, \psi(z) = \int_0^{2\pi} [\frac{\lambda}{2} z(t)^2 + H(t, z(t))] dt$.

We shall show that the functional $I(z)$ satisfies the geometric assumptions of Theorem 2.2.

Lemma 3.1 ($(P.S.)^*$ condition). *Assume that H satisfies the conditions (H1)-(H3) and $\lambda \notin \mathbb{Z}$. Then $I(z)$ satisfies the $(P.S.)_\gamma^*$ condition with respect to $(E_n)_n$ for any $\gamma \in \mathbb{R}$.*

Proof. Let $(k_n)_n$ and $(z_n)_n$ be two sequences such that $k_n \rightarrow +\infty$, and for any sequence $(z_n)_n$ with $z_n \in E_{k_n}$,

$$I(z_n) \rightarrow \gamma$$

and

$$DI_{k_n}(z_n) \rightarrow \theta,$$

where I_{k_n} is a restriction of I on E_{k_n} and $\theta = (0, \dots, 0)$. It follows from $DI_{k_n}(z_n) \rightarrow \theta$ that

$$P_{E_{k_n}} z_n - P_{E_{k_n}} A_\epsilon^{-1}((\lambda + \epsilon)z_n + H_z(t, z_n(t))) \rightarrow \theta,$$

where A_ϵ^{-1} is a compact operator. We claim that $(z_n)_n$ is bounded. By contradiction, we suppose that $\|z_n\| \rightarrow \infty$. If $w_n = \frac{z_n}{\|z_n\|}$, we can suppose that $w_n \rightharpoonup w_0$ weakly for some $w_0 \in E$. We have

$$\begin{aligned} 0 &\longleftarrow \langle P_{E_{k_n}} w_n - A_\epsilon^{-1} P_{E_{k_n}} ((\lambda + \epsilon)w_n + \frac{H_z(t, z_n(t))}{\|z_n\|}), w_n \rangle \\ &= P_{E_{k_n}} \langle w_n, w_n \rangle - \langle A_\epsilon^{-1} P_{E_{k_n}} ((\lambda + \epsilon)w_n + \frac{H_z(t, z_n(t))}{\|z_n\|}), w_n \rangle. \end{aligned}$$

Since A_ϵ^{-1} is a compact operator, $(\lambda + \epsilon)w_n$ is bounded and $\frac{H_z(t, z_n(t))}{\|z_n\|} \rightarrow 0$, $A_\epsilon^{-1}(P_{E_{k_n}}((\lambda + \epsilon)w_n + \frac{H_z(t, z_n(t))}{\|z_n\|}))$ converges to $A_\epsilon^{-1}((\lambda + \epsilon)w_0)$ and we have

$$0 = \langle w_0, w_0 \rangle - \langle A_\epsilon^{-1}((\lambda + \epsilon)w_0), w_0 \rangle = \|w_0\|^2 - \langle A_\epsilon^{-1}(\lambda + \epsilon)w_0, w_0 \rangle,$$

from which w_0 is a solution of the equation

$$A_\epsilon w = (\lambda + \epsilon)w.$$

Since $\lambda \notin \sigma(A)$, $w_0 = 0$, which is a contradiction to the fact that $\|w_0\| = 1$. Thus $(z_n)_n$ is bounded. We can suppose that $z_n \rightharpoonup z_0$ weakly in E for some z_0 in E . We have

$$\begin{aligned} &\langle P_{E_{k_n}} DI(z_n), P_+ z_n + P_- z_n \rangle \\ &= \|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2 \\ &\quad - P_{E_{k_n}} \int_0^{2\pi} (\lambda z_n(t) + \epsilon z_n(t) + H_z(t, z_n))(P_+ z_n + P_- z_n) \rightarrow 0. \end{aligned}$$

By (H1) and the boundedness of $H_z(t, z_n) \cdot (P_+ z_n + P_- z_n)$,

$$\lim_{n \rightarrow \infty} \|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2 = \int_0^{2\pi} (\lambda z(t) + \epsilon z(t) + H_z(t, z))z,$$

i.e., $\|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2$ converges strongly, which implies that, up to a subsequence, $P_{E_{k_n}} z_n$ converges strongly to z , and we prove the lemma and have

$$DI(z) = \lim_{n \rightarrow \infty} P_{E_{k_n}} DI(z_n) = 0,$$

so z is the critical point of I . □

Let us set

$$X_1 = E_{(-\infty, j_1]}, \quad X_2 = E_{j_0}, \quad X_3 = E_+.$$

Then E is the topological direct sum of the subspaces X_1, X_2 and X_3 . Let P_X be the orthogonal projection from E onto X . Let us set

$$(3.1) \quad C = \{z \in E \mid \|P_{X_2}z\| \geq 1\}.$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap E_n$. Let us define a functional $\Psi : E \setminus \{X_1 \oplus X_3\} \rightarrow E$ by

$$(3.2) \quad \Psi(z) = z - \frac{P_{X_2}z}{\|P_{X_2}z\|} = P_{X_1 \oplus X_3}z + \left(1 - \frac{1}{\|P_{X_2}z\|}\right)P_{X_2}z.$$

We have

$$(3.3) \quad \nabla\Psi(z) \cdot w = w - \frac{1}{\|P_{X_2}z\|} (P_{X_2}w - \langle \frac{P_{X_2}z}{\|P_{X_2}z\|}, w \rangle \frac{P_{X_2}z}{\|P_{X_2}z\|}).$$

Let us define the constrained functional $\tilde{I} : C \rightarrow \mathbb{R}$ by

$$(3.4) \quad \tilde{I} = I \circ \Psi.$$

Then $\tilde{I} \in C_{loc}^{1,1}$. It turns out that

$$(3.5) \quad \text{grad}_C^- \tilde{I}(\tilde{z}) = \begin{cases} P_{X_1 \oplus X_3}DI(z) + \left(1 - \frac{1}{\|P_{X_2}\tilde{z}\|_E}\right)P_{X_2}DI(z) & \text{if } z \in \text{int}(C), \\ P_{X_1 \oplus X_3}DI(z) - \langle DI(z), \frac{P_{X_2}\tilde{z}}{\|P_{X_2}\tilde{z}\|_E} \rangle + \frac{P_{X_2}\tilde{z}}{\|P_{X_2}\tilde{z}\|_E} & \text{if } z \in \partial C. \end{cases}$$

We note that if \tilde{z} is the critical point of \tilde{I} and lies in the interior of C , then $z = \Psi(\tilde{z})$ is the critical point of I . Thus it suffices to find the critical points, which lies in the interior of C , for \tilde{I} . We also note that

$$(3.6) \quad \|\text{grad}_C^- \tilde{I}(\tilde{z})\|_E \geq \|P_{X_1 \oplus X_3}DI(\Psi(\tilde{z}))\|_E \quad \forall \tilde{z} \in \partial C.$$

Let us set

$$S_{23}(\rho) = \{z \in X_2 \oplus X_3 \mid \|z\|_E = \rho\}, \quad \rho > 0,$$

$$S_{23}^{\tilde{}}(\rho) = \Psi^{-1}(S_{23}(\rho)),$$

$$\Delta_{12}(R, R_1) = \{z_1 + z_2 \mid z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, 1 \leq \|z_2\|_E \leq R\},$$

$$\Delta_{12}(\tilde{R}, R_1) = \Psi^{-1}(\Delta_{12}(R, R_1))$$

$$\begin{aligned} \Sigma_{12}(R, R_1) &= \{z_1 + z_2 \mid z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, \|z_2\|_E = 1\} \\ &\quad \cup \{z_1 + z_2 \mid z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, \|z_2\|_E = R\} \\ &\quad \cup \{z_1 + z_2 \mid z_1 \in X_1, z_2 \in X_2, \|z_1\|_E = R_1, 1 \leq \|z_2\|_E \leq R\}, \end{aligned}$$

$$\Sigma_{12}(\tilde{R}, R_1) = \Psi^{-1}(\Sigma_{12}(R, R_1)).$$

We will prove the multiplicity result by using Theorem 2.2 for $\tilde{I}, C, S_{23}^{\tilde{}}(\rho), \Delta_{12}(\tilde{R}, R_1)$ and $\Sigma_{12}(\tilde{R}, R_1)$. Now we have the following linking geometry for \tilde{I} .

Lemma 3.2. *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then there exist small numbers $\bar{\delta} > 0, R > \rho > 0, R_1 > 0, R > 1$ and $\rho > 0$ with $R > \rho$ such that for any λ with $j_1 - \bar{\delta} \leq \lambda < j_0 < 0$,*

$$(3.7) \quad \sup_{\tilde{z} \in \Sigma_{12}(\bar{R}, R_1)} \tilde{I}(\tilde{z}) < 0 < \inf_{\tilde{w} \in S_{23}(\rho)} \tilde{I}(\tilde{w}).$$

Moreover

$$-\infty < \inf_{\tilde{w} \in B_{23}(\rho)} \tilde{I}(\tilde{w}), \quad \sup_{\tilde{z} \in \Delta_{12}(\bar{R}, R_1)} \tilde{I}(\tilde{z}) < \infty.$$

Proof. It suffices to show that there exist $\bar{\delta} > 0, R > \rho > 0, R_1 > 0$ and $R > 1$ such that for any λ with $j_1 - \bar{\delta} \leq \lambda < j_0 < 0, z = \psi(\tilde{z}), w = \psi(\tilde{w}),$

$$\sup_{z \in \Sigma_{12}(R, R_1)} I(z) < \inf_{w \in S_{23}(\rho)} I(w)$$

because

$$(3.8) \quad \sup_{\tilde{z} \in \Sigma_{12}(\bar{R}, R_1)} \tilde{I}(\tilde{z}) = \sup_{z \in \Sigma_{12}(R, R_1)} I(z), \quad \inf_{\tilde{w} \in S_{23}(\rho)} \tilde{I}(\tilde{w}) = \inf_{w \in S_{23}(\rho)} I(w).$$

Let $z = z_1 + z_2 \in X_1 \oplus X_2$. By (H2), we have

$$\begin{aligned} I(z) &= \frac{1}{2} (\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|^2 - \| (-|A_\epsilon|)^{\frac{1}{2}} z_- \|_{L^2}^2 \\ &\quad - \| (-|A_\epsilon|)^{\frac{1}{2}} M_- z_0 \|^2) - \int_0^{2\pi} [\frac{1}{2}(\lambda + \epsilon)z^2 + H(t, z(t))] dt \\ &\leq \frac{j_1 + \epsilon - \lambda - \epsilon}{2} \|z_1\|_{L^2}^2 + \frac{j_0 + \epsilon - \lambda - \epsilon}{2} \|z_2\|_{L^2}^2 - \int_0^{2\pi} [\alpha_1 |z|^{p_1+1} - \beta_1] dt \\ &\leq \frac{\bar{\delta}}{2} \|z_1\|_{L^2}^2 + \frac{j_0 + \epsilon - \lambda - \epsilon}{2} \|z_2\|_{L^2}^2 - \int_0^{2\pi} [\alpha_1 |z|^{p_1+1} - \beta_1] dt. \end{aligned}$$

Since $j_0 - \lambda > 0$ and $p_1 + 1 > 2$, there exist small numbers $\bar{\delta} > 0, R > 0, R_1 > 0$ and $R > 1$ such that for any λ with $j_1 - \bar{\delta} \leq \lambda < j_0 < 0, z \in X_1 \oplus X_2, \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0$. If $z \in \Delta_{12}(R, R_1)$, then $\sup_{z \in \Delta_{12}(R, R_1)} I(z) < \frac{\bar{\delta}}{2} \|z_1\|_{L^2}^2 + \frac{j_0 - \lambda}{2} \|z_2\|^2 + 2\pi\beta_1 < \infty$. On the other hand, if $z \in X_2 \oplus X_3$, then

$$\begin{aligned} I(z) &= \frac{1}{2} (\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|^2 - \| (-|A_\epsilon|)^{\frac{1}{2}} z_- \|_{L^2}^2 \\ &\quad - \| (-|A_\epsilon|)^{\frac{1}{2}} M_- z_0 \|^2) - \int_0^{2\pi} [\frac{1}{2}(\lambda + \epsilon)z^2 + H(t, z(t))] dt \\ &\geq \frac{j_0 + \epsilon - \lambda - \epsilon}{2} \|z\|_{L^2}^2 - \int_0^{2\pi} \alpha_2 |z|^{p_2+1} dt. \end{aligned}$$

Since $j_0 - \lambda > 0$ and $p_2 + 1 > 2$, there exists a small number $\rho > 0$ with $R > \rho > 0$ such that for $z \in X_2 \oplus X_3, \inf_{z \in S_{23}(\rho)} I(z) > 0$. If $z \in B_{23}(\rho)$, then $\inf_{z \in B_{23}(\rho)} I(z) > -\infty$. Thus we prove the lemma. \square

Lemma 3.3. *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then for any λ with $j_1 \leq \lambda < 0$, \tilde{I} has no critical point \tilde{z} such that $\tilde{I}(\tilde{z}) = c$ and $\tilde{z} \in \partial C$, where $\inf_{\tilde{z} \in B_{23}(\rho)} \tilde{I}(\tilde{z}) \leq c \leq \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < 0$.*

Proof. It suffices to prove that I has no critical point $z = \psi(\tilde{z})$ such that $I(z) = c$ and $z \in X_1 \oplus X_3$. We notice that from Lemma 3.2, for fixed $z_1 \in X_1$, the functional $z_3 \mapsto I(z_1 + z_3)$ is weakly convex in X_3 , while, for fixed $z_3 \in X_3$, the functional $z_1 \mapsto I(z_1 + z_3)$ is strictly concave in X_1 . Moreover $(0, 0)$ is a critical point in $X_1 \oplus X_3$ with $I(0, 0) = 0$. So if $z = z_1 + z_3$ is another critical point for $I|_{X_1 \oplus X_3}$, then we have

$$0 = I(0, 0) \leq I(z_3) \leq I(z_1 + z_3) \leq I(z_1) \leq I(0, 0) = 0.$$

So $I(z_1 + z_3) = I(0, 0) = 0$. □

Lemma 3.4. *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then there exists a small number $\hat{\delta} > 0$ such that for any λ with $j_1 - \hat{\delta} \leq \lambda < j_0 < 0$, \tilde{I} has no critical point \tilde{z} such that $\inf_{\tilde{z} \in B_{23}(\rho)} \tilde{I}(\tilde{z}) \leq \tilde{I}(\tilde{z}) \leq \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < 0$ and $\tilde{z} \in \partial C$.*

Proof. It suffices to show that $I(z)$ has no critical point z such that

$$\inf_{z \in B_{23}(\rho)} I(z) \leq I(z) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0 \text{ and } z \in X_1 \oplus X_3.$$

By contradiction we suppose that we can find two sequences $(\lambda_n)_n$ in \mathbb{R} with $j_1 - \hat{\delta} \leq \lambda_n < j_0 < 0$ and $(z_n)_n$ in $X_1 \oplus X_3$ such that $\lambda_n \rightarrow \lambda \in [j_1, j_0)$, $\inf_{z \in B_{23}(\rho)} I(z) \leq I(z) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0$ and $DI|_{X_1 \oplus X_3}(z_n) = 0$. We claim that $(z_n)_n$ is bounded. If not we can suppose that $\|z_n\| \rightarrow +\infty$ and set $w_n = \frac{z_n}{\|z_n\|}$. Since w_n is bounded, up to a subsequence w_n converges weakly to w_0 , for some $w_0 \in X_1 \oplus X_3$. Furthermore since $P_{X_1} z_n \in E_-$, $\|P_+ P_{X_1} z_n\| = 0$ and we have

$$\begin{aligned} (3.9) \quad & \langle DI(z_n), P_{X_1} z_n \rangle \\ &= \|P_+ P_{X_1} z_n\|^2 - \|P_- P_{X_1} z_n\|^2 - \langle (\lambda + \epsilon) z_n + H_z(t, z_n), P_{X_1} z_n \rangle \\ &= -\|P_- P_{X_1} z_n\|^2 - \langle (\lambda + \epsilon) z_n + H_z(t, z_n), P_{X_1} z_n \rangle \longrightarrow 0. \end{aligned}$$

Moreover since $P_{X_3} z_n \in E_+$, $\|P_- P_{X_3} z_n\| = 0$ and we have

$$(3.10) \quad \langle DI(z_n), P_{X_3} z_n \rangle = \|P_+ P_{X_3} z_n\|^2 - \langle (\lambda + \epsilon) z_n + H_z(t, z_n), P_{X_3} z_n \rangle \longrightarrow 0.$$

Adding (3.9) to (3.10), we have

$$(3.11) \quad \lim_{n \rightarrow \infty} (\|P_+ P_{X_3} z_n\|^2 - \|P_- P_{X_1} z_n\|^2) = \lim_{n \rightarrow \infty} \langle (\lambda + \epsilon) z_n + H_z(t, z_n), P_{X_1 \oplus X_3} z_n \rangle.$$

Dividing (3.11) by $\|z_n\|^2$ and going to the limit, we get

$$(3.12) \quad \|P_+ P_{X_3} w_0\|^2 - \|P_- P_{X_1} w_0\|^2 = \langle (\lambda + \epsilon) w_0, P_{X_1 \oplus X_3} w_0 \rangle,$$

from which w_0 is the unique solution of the linear equation

$$A_\epsilon z = (\lambda + \epsilon)z.$$

Since $\lambda \notin \sigma(A)$, $w_0 = 0$, which is a contradiction to the fact $\|w_0\| = 1$. Thus $(z_n)_n$ is bounded. By the same arguments used for $(w_n)_n$, we get, up to a subsequence, $(z_n)_n$ converges strongly to a point $z \in X_1 \oplus X_3$ with $\inf_{z \in B_{23}(\rho)} I(z) \leq I(z) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0$ and $DI|_{X_1 \oplus X_3}(z) = 0$, which contradicts Lemma 3.3. Thus we prove the lemma. \square

Lemma 3.5. *The functional $-\tilde{I}$ satisfies the $(P.S.)_{-\tilde{c}}^*$ condition with respect to $(C_n)_n$ for any $-\tilde{c}$ such that*

$$0 < \inf_{\tilde{z} \in \Sigma_{12}(R, R_1)} (-\tilde{I})(\tilde{z}) \leq -\tilde{c} \leq \sup_{\tilde{z} \in B_{23}(\rho)} (-\tilde{I})(\tilde{z}).$$

Proof. Let $(h_n)_n$ be a sequence in \mathbb{N} with $h_n \rightarrow +\infty$ and $(\tilde{z}_n)_n$ be a sequence in C with $\tilde{z}_n \in C_{h_n}$ for all n , $-\tilde{I}(\tilde{z}_n) \rightarrow -\tilde{c}$ and $\text{grad}_{C_{h_n}}^- (-\tilde{I}|_{E_{h_n}})(\tilde{z}_n) \rightarrow 0$. Set $z_n = \Psi(\tilde{z}_n)$. Then $I(z_n) \rightarrow c$. We first consider the case $\tilde{z}_n \notin \partial C_{h_n}$ for large n . Since for large n $P_{E_n} \circ P_{X_2} = P_{X_2} \circ P_{E_n} = P_{X_2}$, we have

$$\begin{aligned} \text{grad}_{C_{h_n}}^- (-\tilde{I})(\tilde{z}_n) &= P_{E_{h_n}} \Psi'(\tilde{z}_n) D(-I)(z_n) = \Psi'(\tilde{z}_n) (P_{E_{h_n}} D(-I)(z_n)) \\ &= P_{E_{h_n}} P_{X_1 \oplus X_3} D(-I)(z_n) + P_{E_{h_n}} \left(1 - \frac{1}{\|P_{X_2} \tilde{z}_n\|_E}\right) P_{X_2} D(-I)(z_n) \rightarrow 0. \end{aligned}$$

Thus

$$\begin{aligned} P_{X_1 \oplus X_3} P_{E_{h_n}} D(-I)(z_n) &\rightarrow 0 \quad \text{and} \\ \left(1 - \frac{1}{\|P_{X_2} \tilde{z}_n\|_E}\right) P_{X_2} D(-I)(z_n) &\rightarrow 0. \end{aligned}$$

It is impossible that $\|P_{X_2} \tilde{z}_n\|_E \rightarrow 1$ because $\text{dist}(z_n, X_2) \rightarrow 0$. Thus

$$P_{E_{h_n}} D(-I)(z_n) \rightarrow 0.$$

Using $(P.S.)_c^*$ for I of Lemma 3.1 it follows that $(z_n)_n$ has a subsequence $(z_{k_n})_n$ such that $z_{k_n} \rightarrow z$ for some z in X_2 . Since Ψ is invertible in $\text{int}(C)$, $\tilde{z}_{k_n} \rightarrow \Psi^{-1}(z)$. Next we consider the case $\tilde{z}_n \in \partial C_{h_n}$ for infinitely many n . We claim that this case cannot occur. If $\tilde{z}_n \in \partial C_{h_n}$, then $\|P_{X_2} \tilde{z}_n\|_E = 1$. Thus we have

$$\begin{aligned} \text{grad}_{C_{h_n}}^- (-\tilde{I})(\tilde{z}_n) &= P_{E_{h_n}} (P_{X_1 \oplus X_3} D(-I)(z_n) - \langle D(-I)(z_n), P_{X_2} \tilde{z}_n \rangle^+ P_{X_2} \tilde{z}_n) \\ &\rightarrow 0. \end{aligned}$$

Using the properties of the projections, we get

$$P_{E_{h_n}} P_{X_1 \oplus X_3} D(-I)(z_n) \rightarrow 0,$$

which contradicts to Lemma 3.3. In fact, let \tilde{z} be the limit point of the subsequence z_{k_n} of \tilde{z}_n , then $\tilde{z} \in \partial C$ and

$$\text{grad}_C^- (-\tilde{I})(\tilde{z}) = P_{X_1 \oplus X_3} \text{grad}(-I)(z) - \langle \text{grad}(-I)(z), P_{X_2} \tilde{z} \rangle P_{X_2} \tilde{z}. \quad \square$$

Theorem 3.1 (Existence of two solutions on E_{j_0}). *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Let $\lambda \notin \mathbb{Z}$ and $\lambda < 0$. Then there exists a small number $\delta_1 > 0$ such that for any λ with $j_1 - \delta_1 < \lambda < j_0 < 0$, system (1.1) has at least two nontrivial 2π -periodic weak solutions on E_{j_0} .*

Proof. We note that the critical points of the functional \tilde{I} coincide with the critical points of the functional $-\tilde{I}$. Thus it suffices to find the number of the critical points of $-\tilde{I}$, which is appropriate functional for applying Theorem 2.2, to find the number of the critical points of I . Let us set

$$\delta_1 = \min\{\bar{\delta}, \hat{\delta}\},$$

where $\bar{\delta}$ is a small number introduced in Lemma 3.2 and $\hat{\delta}$ is a small number introduced in Lemma 3.4. By Lemma 3.2, there exist $R > \rho > 0, R_1 > 0, R > 1$ and $\rho > 0$ with $R > \rho$ such that for any λ with $j_1 - \delta_1 < \lambda < j_0 < 0$,

$$\begin{aligned} \sup_{\tilde{z} \in S_{23}(\rho)} (-\tilde{I})(\tilde{z}) &= \sup_{z \in S_{23}(\rho)} (-I)(z) < 0 < \inf_{z \in \Sigma_{12}(R, R_1)} (-I)(z) \\ &= \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} (-\tilde{I})(z). \end{aligned}$$

and

$$\inf_{\tilde{z} \in \Delta_{12}(\tilde{R}, R_1)} (-\tilde{I})(z) = - \sup_{z \in \Delta_{12}(\tilde{R}, R_1)} \tilde{I}(z) > -\infty,$$

$$\sup_{\tilde{z} \in B_{23}(\rho)} (-\tilde{I})(\tilde{z}) = - \inf_{\tilde{z} \in B_{23}(\rho)} \tilde{I}(\tilde{z}) < \infty,$$

so the condition (a) of Theorem 2.2 for the functional $-\tilde{I}$ is satisfied. By Lemma 3.5, the functional $-\tilde{I}$ satisfies the $(P.S.)_{-\tilde{c}}^*$ condition with respect to $(C_n)_n$ for any $-\tilde{c} \in [\alpha, \beta]$, where $\alpha = \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} (-\tilde{I})(\tilde{z})$ and $\beta = \sup_{\tilde{z} \in B_{23}(\rho)} (-\tilde{I})(\tilde{z})$, so the condition (b) of Theorem 2.2 is satisfied. By Lemma 3.4, for any λ with $j_1 - \delta_1 \leq \lambda < j_0 < 0, \tilde{I}$ has no critical point \tilde{z} such that $\inf_{\tilde{z} \in B_{23}(\rho)} \tilde{I}(\tilde{z}) \leq \tilde{I}(\tilde{z}) \leq \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < 0$ and $\tilde{z} \in \partial C$, so the condition (c) of Theorem 2.2 is satisfied. Thus by Theorem 2.2, for any λ with $j_1 - \delta_1 < \lambda < j_0 < 0$ there exists two lower critical points \tilde{z}_1, \tilde{z}_2 for $-\tilde{I}$ on C such that

$$0 < \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} (-\tilde{I})(\tilde{z}) \leq (-\tilde{I})(\tilde{z}_i) \leq \sup_{\tilde{z} \in B_{23}(\rho)} (-\tilde{I})(\tilde{z}), \quad i = 1, 2.$$

Thus the functional I has at least two lower critical points z_1, z_2 on X_2 with

$$\inf_{z \in B_{23}(\rho)} I(z) \leq I(z_i) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0, \quad i = 1, 2.$$

Thus system (1.2) has at least two nontrivial solutions on $X_2 = E_{j_0}$. Thus Theorem 3.1 is proved. □

4. Existence of two solutions on E_{j_1} and proof of Theorem 1.1

Let us set

$$X_1 = E_{(-\infty, j_1-1]}, \quad X_2 = E_{j_1}, \quad X_3 = E_{[j_0, \infty)}.$$

Then E is the topological direct sum of the subspaces X_1, X_2 and X_3 . Let P_X be the orthogonal projection from E onto X .

Lemma 4.1. *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then there exist small numbers $\delta_2 > 0, R' > \rho' > 0, R'_1 > 0, R' > 1$ and $\rho' > 0$ with $R' > \rho'$ such that for any λ with $j_1 - \delta_2 < \lambda < j_1 < j_0 < 0$,*

$$(4.1) \quad \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} \tilde{I}(\tilde{z}) < 0 < \inf_{\tilde{w} \in S_{23}(\rho')} \tilde{I}(\tilde{w}).$$

Moreover

$$-\infty < \inf_{\tilde{w} \in B_{23}(\rho')} \tilde{I}(\tilde{w}), \quad \sup_{\tilde{z} \in \Delta_{12}(\tilde{R}', R'_1)} \tilde{I}(\tilde{z}) < \infty.$$

Proof. It suffices to show that there exist $\delta_2 > 0, R' > \rho' > 0, R'_1 > 0$ and $R' > 1$ such that for any λ with $j_1 - \delta_2 < \lambda < j_1 < j_0 < 0, z = \psi(\tilde{z}), w = \psi(\tilde{w})$,

$$\sup_{z \in \Sigma_{12}(R', R'_1)} I(z) < \inf_{w \in S_{23}(\rho')} I(w)$$

because

$$(4.2) \quad \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} \tilde{I}(\tilde{z}) = \sup_{z \in \Sigma_{12}(R', R'_1)} I(z), \quad \inf_{\tilde{w} \in S_{23}(\rho')} \tilde{I}(\tilde{w}) = \inf_{w \in S_{23}(\rho')} I(w).$$

Let $z = z_1 + z_2 \in X_1 \oplus X_2$. Then $\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 = 0, \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|_{L^2}^2 = 0$. By (H2), we have

$$\begin{aligned} I(z) &= \frac{1}{2} (\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|^2 - \| (-|A_\epsilon|)^{\frac{1}{2}} z_- \|_{L^2}^2 \\ &\quad - \| (-|A_\epsilon|)^{\frac{1}{2}} M_- z_0 \|^2) - \int_0^{2\pi} [\frac{1}{2}(\lambda + \epsilon)z^2 + H(t, z(t))] dt \\ &\leq \frac{j_1 + \epsilon - \lambda - \epsilon}{2} \|z_2\|_{L^2}^2 - \int_0^{2\pi} [\alpha_1 |z|^{p_1+1} - \beta_1] dt \\ &\leq \frac{\delta_2}{2} \|z_1\|_{L^2}^2 - \int_0^{2\pi} [\alpha_1 |z|^{p_1+1} - \beta_1] dt. \end{aligned}$$

Since $j_1 - \lambda > 0$ and $p_1 + 1 > 2$, there exist small numbers $\delta_2 > 0, R' > 0, R'_1 > 0$ and $R' > 1$ such that for any λ with $j_1 - \delta_2 < \lambda < j_1 < j_0 < 0, z \in X_1 \oplus X_2, \sup_{z \in \Sigma_{12}(R', R'_1)} I(z) < 0$. If $z \in \Delta_{12}(R', R'_1)$, then $\sup_{z \in \Delta_{12}(R', R'_1)} I(z) < \frac{\delta_2}{2} \|z_1\|_{L^2}^2 + 2\pi\beta_1 < \infty$. On the other hand, if $z \in X_2 \oplus X_3$, then

$$I(z) = \frac{1}{2} (\| |A_\epsilon|^{\frac{1}{2}} z_+ \|_{L^2}^2 + \| |A_\epsilon|^{\frac{1}{2}} M_+ z_0 \|^2 - \| (-|A_\epsilon|)^{\frac{1}{2}} z_- \|_{L^2}^2$$

$$\begin{aligned}
 & - \|(-|A_\epsilon|)^{\frac{1}{2}} M_- z_0\|^2 - \int_0^{2\pi} \left[\frac{1}{2}(\lambda + \epsilon)z^2 + H(t, z(t)) \right] dt \\
 & \geq \frac{j_1 + \epsilon - \lambda - \epsilon}{2} \|z\|_{L^2}^2 - \int_0^{2\pi} \alpha_2 |z|^{p_2+1} dt.
 \end{aligned}$$

Since $j_1 - \lambda > 0$ and $p_2 + 1 > 2$, there exists a small number $\rho' > 0$ with $R' > \rho' > 0$ such that for $z \in X_2 \oplus X_3$, $\inf_{z \in S_{23}(\rho')} I(z) > 0$. If $z \in B_{23}(\rho')$, then $\inf_{z \in B_{23}(\rho')} I(z) > -\infty$. Thus we prove the lemma. \square

Lemma 4.2. *For any Λ with $j_1 - 1 < \Lambda < j_1 < 0$, there exists $\Gamma < 0$ such that for any λ with $j_1 - 1 < \Lambda < \lambda < j_1 < j_0 < 0$, if \tilde{z} is a critical pint for $\tilde{I}|_{X_1 \oplus X_3}$ with $\Gamma \leq \tilde{I}(\tilde{z}) \leq 0$, then $\tilde{z} = \theta$.*

Proof. By contradiction, we can suppose that there exist $\Lambda < 0$, a sequence $(\lambda_n)_n$ such that $\lambda_n \rightarrow \lambda \in (\Lambda, j_1)$, and a sequence $(z_n)_n$ in $X_1 \oplus X_3$ such that $I(z_n) \rightarrow 0$ and $P_{X_1 \oplus X_3} DI(z_n) = \theta$. We claim that $(z_n)_n$ is bounded. If not we suppose that $\|z_n\| \rightarrow +\infty$. Let us set $w_n = \frac{z_n}{\|z_n\|}$. We have up to a subsequence that $w_n \rightharpoonup w_0$ weakly for some $w_0 \in X_1 \oplus X_3$. Furthermore

$$\begin{aligned}
 (4.3) \quad & 0 \\
 & = \langle DI(z_n), P_{X_1 \oplus X_3} z_n \rangle \\
 & = \|P_+ P_{X_1 \oplus X_3} z_n\|^2 - \|P_- P_{X_1 \oplus X_3} z_n\|^2 - \langle (\lambda_n + \epsilon)z_n + H_z(t, z_n), P_{X_1 \oplus X_3} z_n \rangle.
 \end{aligned}$$

Dividing (4.3) by $\|z_n\|^2$ and going to the limit as $n \rightarrow \infty$, we have

$$\begin{aligned}
 \|P_+ w_0\|^2 - \|P_- w_0\|^2 & = \langle (\lambda + \epsilon)w_0 + \frac{H_z(t, z_n)}{\|z_n\|^2}, P_{X_1 \oplus X_3} w_0 \rangle \\
 & = \langle ((\lambda + \epsilon)w_0, w_0) \rangle,
 \end{aligned}$$

from which w_0 is a solution of the equation

$$A_\epsilon w = (\lambda + \epsilon)w.$$

Since $\lambda \notin \sigma(A)$, $w_0 = 0$, which is a contradiction to the fact that $\|w_0\| = 1$. Thus $(z_n)_n$ is bounded. We can suppose that $z_n \rightharpoonup z_0$ weakly in E , for some z_0 in E . We claim that, up to a subsequence, $(z_n)_n$ converges strongly to a point $z \in X_1 \oplus X_3$. We have

$$(4.4) \quad 0 = \langle P_{X_1 \oplus X_3} DI(z_n), z_n \rangle = 2I(z_n) + \int_0^{2\pi} [2H(t, z_n) - H_z(t, z_n) \cdot z_n] dt.$$

It follows from (4.4) that

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} H_z(t, z_n) \cdot z_n dt = 2 \lim_{n \rightarrow \infty} \int_0^{2\pi} H(t, z_n) dt.$$

It follows from (4.3) that

$$\lim_{n \rightarrow \infty} [\|P_+ P_{X_1 \oplus X_3} z_n\|^2 - \|P_- P_{X_1 \oplus X_3} z_n\|^2]$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} [\langle (\lambda_n + \epsilon)z_n + H_z(t, z_n), P_{X_1 \oplus X_3} z_n \rangle] \\
 &= \lim_{n \rightarrow \infty} \int_0^{2\pi} [(\lambda_n + \epsilon)z_n^2 + 2H(t, z_n)] dt.
 \end{aligned}$$

Thus $\|P_+ P_{X_1 \oplus X_3} z_n\|^2 - \|P_- P_{X_1 \oplus X_3} z_n\|^2$ converges strongly, so $(P_{X_1 \oplus X_3} z_n)_n$ converges strongly to a point $z \in X_1 \oplus X_3$. We claim that $z = \theta$. We assume that $z \neq \theta$. By (H2),

$$2\alpha_1 \|z\|_{L^2}^{p_1+1} - 4\pi\beta_1 \leq 2 \int_0^{2\pi} H(t, z(t)) dt \leq 2\alpha_2 \|z\|_{L^2}^{p_2+1}$$

for some $1 < p_1 \leq p_2 < 2p_1 + 1$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 \geq 0$. We also have

$$\begin{aligned}
 0 &= \langle DI(z_n), P_{X_1} z_n \rangle \\
 &= \|P_+ P_{X_1} z_n\|^2 - \|P_- P_{X_1} z_n\|^2 - \langle (\lambda_n + \epsilon)z_n + H_z(t, z_n), P_{X_1} z_n \rangle.
 \end{aligned}$$

Since $\|P_+ P_{X_1} z_n\|^2 = 0$,

$$(4.5) \quad -\|P_- P_{X_1} z_n\|^2 = \langle (\lambda_n + \epsilon)z_n + H_z(t, z_n), P_{X_1} z_n \rangle.$$

If $z \in X_1$, then

$$-\|P_{X_1} z\|^2 \leq (j_1 - 1 + \epsilon) \|P_{X_1} z\|_{L^2}^2 \quad \text{for } j_1 - 1 < 0$$

and

$$(4.6) \quad (j_1 - 1 + \epsilon) \|P_{X_1} z\|_{L^2}^2 \geq -\|P_{X_1} z\|^2 \geq (\lambda + \epsilon) \|P_{X_1} z\|_{L^2}^2 + 2\alpha_1 \|P_{X_1} z\|_{L^2}^{p_1+1} - 4\pi\beta_1.$$

If $z \in X_3$, then

$$\|P_{X_3} z\|^2 \leq \|P_{X_3} z\|_{L^2}^2.$$

and

$$(4.7) \quad 0 < \|P_{X_3} z\|_{L^2}^2 \leq \|P_{X_3} z\|^2 \leq (\lambda + \epsilon) \|P_{X_3} z\|_{L^2}^2 + 2\alpha_2 \|P_{X_3} z\|_{L^2}^{p_2+1}.$$

Adding (4.6) $\times (-1)$ to (4.7), we have

$$\begin{aligned}
 &(\lambda - j_1 + 1) \|P_{X_1} z\|_{L^2}^2 - (\lambda + \epsilon) \|P_{X_3} z\|_{L^2}^2 \\
 &\leq -2\alpha_1 \|P_{X_1} z\|_{L^2}^{p_1+1} + 4\pi\beta_1 + 2\alpha_2 \|P_{X_3} z\|_{L^2}^{p_2+1}.
 \end{aligned}$$

The left hand side of the above inequality is positive but the right hand side is not always positive, which is a contradiction. Thus $z = \theta$. □

Lemma 4.3. *The functional $-\tilde{I}$ satisfies the $(P.S.)_{-\tilde{c}}^*$ condition with respect to $(C_n)_n$ for any $-\tilde{c}$ such that*

$$0 < \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} (-\tilde{I})(\tilde{z}) \leq -\tilde{c} \leq \sup_{\tilde{z} \in B_{23}(\rho')} (-\tilde{I})(\tilde{z}).$$

Proof. The proof of Lemma 4.3 has the same process as that of Lemma 3.5. □

Theorem 4.1 (Existence of two solutions on E_{j_1}). *Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Let $\lambda \notin \mathbb{Z}$ and $\lambda < 0$. Then there exists a small number $\delta_2 > 0$ such that for any λ with $j_1 - \delta_2 < \lambda < j_1 < 0$, system (1.1) has at least two nontrivial 2π -periodic weak solutions on E_{j_1} .*

Proof. We note that the critical points of the functional \tilde{I} coincide with the critical points of the functional $-\tilde{I}$. Thus it suffices to find the number of the critical points of $-\tilde{I}$, which is appropriate functional for applying Theorem 2.2, to find the number of the critical points of I . By Lemma 4.1, there exist small numbers $\delta_2 > 0, R' > \rho' > 0, R'_1 > 0, R' > 1$ and $\rho' > 0$ with $R' > \rho'$ such that for any λ with $j_1 - \delta_2 < \lambda < j_1 < j_0 < 0$,

$$\sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} \tilde{I}(\tilde{z}) < 0 < \inf_{\tilde{w} \in S_{23}(\rho')} \tilde{I}(\tilde{w}),$$

so the condition (a) of Theorem 2.2 for the functional $-\tilde{I}$ is satisfied. By Lemma 4.3, the functional $-\tilde{I}$ satisfies the $(P.S.)^*_{-\tilde{c}}$ condition with respect to $(C_n)_n$ for any $-\tilde{c}$ such that

$$0 < \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} (-\tilde{I})(\tilde{z}) \leq -\tilde{c} \leq \sup_{\tilde{z} \in B_{23}(\rho')} (-\tilde{I})(\tilde{z}),$$

so the condition (b) of Theorem 2.2 is satisfied. By Lemma 4.2, for any Λ with $j_1 - 1 < \Lambda < j_1$, there exists $\Gamma < 0$ such that for any λ with $j_1 - 1 < \Lambda < \lambda < j_1 < j_0 < 0$, if \tilde{z} is a critical pint for $\tilde{I}|_{X_1 \oplus X_3}$ with $\Gamma \leq \tilde{I}(\tilde{z}) \leq 0$, then $\tilde{z} = 0$. Thus \tilde{I} has no critical point \tilde{z} such that $\inf_{\tilde{z} \in B_{23}(\rho')} \tilde{I}(\tilde{z}) \leq \tilde{I}(\tilde{z}) \leq \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}', R'_1)} \tilde{I}(\tilde{z}) < 0$ and $\tilde{z} \in \partial C$, so the condition (c) of Theorem 2.2 is satisfied. Thus by Theorem 2.2, for any λ with $j_1 - \delta_2 < \lambda < j_1 < j_0 < 0$ there exists two lower critical points \tilde{z}_1, \tilde{z}_2 for $-\tilde{I}$ on C such that

$$0 < \inf_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} (-\tilde{I})(\tilde{z}) \leq (-\tilde{I})(\tilde{z}_i) \leq \sup_{\tilde{z} \in B_{23}(\rho)} (-\tilde{I})(\tilde{z}), \quad i = 1, 2.$$

Thus the functional I has at least two lower critical points z_1, z_2 on X_2 with

$$\inf_{z \in B_{23}(\rho)} I(z) \leq I(z_i) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0, \quad i = 1, 2.$$

Thus system (1.2) has at least two nontrivial solutions on $X_2 = E_{j_1}$. Thus Theorem 4.1 is proved. \square

Proof of Theorem 1.1. Assume that H satisfies the conditions (H1)-(H3), and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Let $\lambda \notin \mathbb{Z}$ and $\lambda < 0$. Let δ_1 be the small number in Theorem 3.1 and δ_2 be the small number in Theorem 4.1. Let us set

$$\delta = \min\{\delta_1, \delta_2\}.$$

The common part of $(j_1 - \delta_1, j_0)$ and $(j_1 - \delta_2, j_1)$ is

$$(j_1 - \delta, j_1).$$

Thus for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$, Theorem 3.1 and Theorem 4.1 hold simultaneously. Thus by Theorem 3.1, system (1.2) has at least two nontrivial solutions on E_{j_0} and by Theorem 4.1, system (1.2) has at least two nontrivial solutions on E_{j_1} . Thus for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$, system (1.2) has at least four nontrivial solutions, two of which are on E_{j_0} and two of which are on E_{j_1} . Thus we prove Theorem 1.1. \square

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