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ON THE DIOPHANTINE EQUATION $(an)^x + (bn)^y = (cn)^z$

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Abstract. In 1956, Jeśmanowicz conjectured that, for any positive integer n and any primitive Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$, the equation $(an)^x + (bn)^y = (cn)^z$ has the unique solution $(x, y, z) = (2, 2, 2)$. In this paper, under some conditions, we prove the conjecture for the primitive Pythagorean triples $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$.

1. Introduction

Let n be a positive integer and let (a, b, c) be a primitive Pythagorean triple such that $a^2+b^2 = c^2$, $(a, b, c) = 1$ and $2 \mid b$. Clearly, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \tag{1.1}$$

has the solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [10] showed there is no other solutions when $n = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [4] further proved that when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)$, then (1.1) has only the solution $(x, y, z) = (2, 2, 2)$. Moreover, he conjectured that for any positive integer n, (1.1) has the unique solution $(x, y, z) = (2, 2, 2)$. Since then, many special cases of Jeśmanowicz' conjecture have been solved for $n = 1$. In 1959, Lu [8] proved the conjecture when $n = 1$ and $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$. Later, Deḿjanenko [1] verified the conjecture if $n = 1$ and $c = b + 1$. Recently, Miyazaki [9] showed that Jeśmanowicz' conjecture is true if $n = 1$ and $a \equiv \pm 1 \pmod b$ or $c \equiv 1 \pmod b$. This result generalized the results of Lu and Deḿjanenko. For more results, see [5, 6, 11, 12, 13, 15].

When $n > 1$, only a few results on this conjecture are known. For any positive integer t with $t > 1$, let $P(t)$ denote the product of distinct prime factors of t. In 1998, Deng and Cohen [3] proved that if $n > 1$, $c = b + 1$, a is a prime power and either $P(b) \mid n$ or $P(n) \nmid b$, then (1.1) has only the solution

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$(x, y, z) = (2, 2, 2)$. In 1999, Le [7] gave certain necessary conditions for (1.2) to have positive integer solutions (x, y, z) with $(x, y, z) \neq (2, 2, 2)$.

Recently, some special cases of the Pythagorean triple $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ have been considered. Then (1.1) becomes

$$(n(4k^2 - 1))^x + (4nk)^y = (n(4k^2 + 1))^z. \tag{1.2}$$

For instance, Yang and Tang [16] proved that Jeśmanowicz' conjecture is true when $k = 2$. Tang and Weng [14] proved that Jeśmanowicz' conjecture is true for $(a, b, c) = (F_k - 2, 2^{2^{k-1}+1}, F_k)$, where $F_k = 2^{2^k} + 1$ is k-th Fermat number. Deng [2] proved the following results: (1) For $k = 2^s (1 \leq s \leq 4)$ and any positive integer n, the only solution of (1.2) is $(x, y, z) = (2, 2, 2)$; (2) For $k = 2^s (s \geq 0)$ and any positive integer n with $P(a) \mid n$ or $P(n) \nmid a$, the only solution of (1.2) is $(x, y, z) = (2, 2, 2)$.

In this paper, we always assume that $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$. The following results are proved.

Theorem 1.1. *Suppose that the positive integer n is such that $P(a) \mid n$. Then the only solution of* (1.2) *is $(x, y, z) = (2, 2, 2)$.*

Theorem 1.2. *Let $k = p^\alpha$, where $\alpha \geq 0$ and p is a prime with $p \equiv -1 \pmod 4$. Suppose that n is a positive integer with $P(n) \nmid a$. Then the only solution of* (1.2) *is $(x, y, z) = (2, 2, 2)$.*

2. Lemmas

Lemma 2.1 ([8, Theorem])**.** *Let $n = 1$. Then* (1.2) *has the only positive integer solution $(x, y, z) = (2, 2, 2)$.*

Lemma 2.2 ([2, Corollary 2.4])**.** *Let (a, b, c) be any primitive Pythagorean triple such that the Diophantine equation $a^x + b^y = c^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$. If (x, y, z) is a solution of* (1.2) *with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied*:

(1) *$x > z > y$ and $P(n) \mid b$;*
(2) *$y > z > x$ and $P(n) \mid a$.*

Lemma 2.3. *Suppose that the positive integer n is such that $P(n) \mid b$. If (x, y, z) is a solution of* (1.2) *with $(x, y, z) \neq (2, 2, 2)$ and $2 \mid z$, then $\frac{b^y}{n^{z-y}}$ is not a square number.*

Proof. By Lemma 2.2, we have $x > z > y$. Assume that $\frac{b^y}{n^{z-y}} = d^2$ is a square number. By (1.2), we have

$$\frac{b^y}{n^{z-y}} = c^z - a^x \cdot n^{x-z}.$$

Write $z = 2z_1$. Then

$$a^x \cdot n^{x-z} = c^z - d^2 = (c^{z_1} - d)(c^{z_1} + d).$$

Noting that a is odd and

$$(c^{z_1} - d, c^{z_1} + d) \le (2c^{z_1}, 2d) \le (2c^{z_1}, 2b^y) = 2,$$

we have $a = a_1 a_2$ with $\gcd(a_1, a_2) = 1$, $a_1^x \mid c^{z_1} + d$ and $a_2^x \mid c^{z_1} - d$. Let $a_i = \max\{a_1, a_2\}$. By $a_1 a_2 = a = (2k-1)(2k+1)$, we have $a_i \ge 2k+1$. Thus

$$\begin{aligned} a_i^x &> (a_i^2)^{z_1} \ge (4k^2 + 1 + 4k)^{z_1} = (c+b)^{z_1} \\ &\ge c^{z_1} + b^{z_1} > c^{z_1} + b^{\frac{y}{2}} \ge c^{z_1} + d > c^{z_1} - d, \end{aligned}$$

a contradiction. □

3. Proof of Theorem 1.1

Since $P(a) \mid n$ and $(a,b) = 1$, it follows that $P(n) \nmid b$. By Lemma 2.2, we have $y > z > x$ and $P(n) \mid a$. Thus $P(n) = P(a)$. By (1.2), we have

$$a^x = n^{z-x}(c^z - b^y \cdot n^{y-z}). \tag{3.1}$$

By $P(n) = P(a)$, we have $n^{z-x} = a^x$. It follows that

$$b^y \cdot n^{y-z} = c^z - 1. \tag{3.2}$$

Let $k = 2^l \cdot k_1$, $2 \nmid k_1$. Noting that $a = 4k^2 - 1$, $b = 4k$, $c = 4k^2 + 1$, we have

$$(2^{l+2} \cdot k_1)^y \cdot n^{y-z} = (2^{2l+2} \cdot k_1^2 + 1)^z - 1 = z \cdot 2^{2l+2} \cdot k_1^2 + \frac{z(z-1)}{2}(2^{2l+2} \cdot k_1^2)^2 + \cdots. \tag{3.3}$$

By $x < z < y$, we have $y \ge 2$. The power of 2 in $(2^{l+2} \cdot k_1)^y \cdot n^{y-z}$ is $(l+2)y \ge 2l+4$. By (3.3), we have $2 \mid z$. Write $z = 2z_1$. It follows that

$$b^y \cdot n^{y-z} = \big(c^{z_1} - 1\big)\big(c^{z_1} + 1\big).$$

Note that $(c^{z_1} + 1, b^y) = 2$, we have $\frac{b^y}{2} \mid c^{z_1} - 1$. However,

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \frac{(c-a)^{z_1}(c+a)^{z_1}}{2} \ge c^{z_1} + a^{z_1} > c^{z_1} - 1,$$

a contradiction.

4. Proof of Theorem 1.2

We suppose that (1.2) has a solution $(x, y, z) \ne (2, 2, 2)$, and will observe that this leads to a contradiction. By Lemma 2.1, we may assume that $n \ge 2$.

By $P(n) \nmid a$ and Lemma 2.2, we have $y < z < x$ and $P(n) \mid b$. Write $n = 2^r \cdot p^s$, where $r + s \ge 1$. By (1.2), we have

$$b^y = n^{z-y}(c^z - a^x \cdot n^{x-z}), \tag{4.1}$$

or equally

$$2^{2y} \cdot p^{\alpha y} = 2^{r(z-y)} \cdot p^{s(z-y)}\big((4 \cdot p^{2\alpha} + 1)^z - (4 \cdot p^{2\alpha} - 1)^x \cdot 2^{r(x-z)} \cdot p^{s(x-z)}\big). \tag{4.2}$$

Case 1. $r \ge 1$, $s = 0$. Then $n = 2^r$. By (4.2),

$$2^{2y} \cdot p^{\alpha y} = 2^{r(z-y)}\big((4 \cdot p^{2\alpha} + 1)^z - (4 \cdot p^{2\alpha} - 1)^x \cdot 2^{r(x-z)}\big).$$

So $2y = r(z-y)$ and

$$p^{\alpha y} = (4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot 2^{r(x-z)}. \tag{4.3}$$

If $r(x-z) \geq 3$, by (4.3), we have $(-1)^{\alpha y} \equiv 1 \pmod 4$. Thus $2 \mid \alpha y$. Also by (4.3), we have $5^z \equiv 1 \pmod 8$. Hence $2 \mid z$. By Lemma 2.3, this is impossible. Then $r(x-z) = 1$ or 2. If $\alpha y \geq 2$, then by (4.3), we have $(-1)^x \cdot 2^{r(x-z)} \equiv 1 \pmod{p^2}$. Note that $p \geq 3$, this is impossible. Then $\alpha y = 1$. We have $\alpha = 1,\ y = 1$. It follows that

$$(4\cdot p^2-1)^x \cdot 2^{r(x-z)} = (4\cdot p^2+1)^z - p. \tag{4.4}$$

We have $(-1)^x \cdot 2^{r(x-z)} \equiv 1 \pmod p$. Note that $r(x-z) = 1, 2$ and $p \equiv -1 \pmod 4$, we have $p = 3$. By (4.4), we get

$$35^x \cdot 2^{r(x-z)} = 37^z - 3. \tag{4.5}$$

If $r(x-z) = 1$, then $r = 1,\ x-z = 1$. Note that $2y = r(z-y)$ and $y = 1$, we have $z = 3y = 3$ and $x = 4$. Then $3001250 = 35^4 \cdot 2 = 37^3 - 3 = 50650$, this is impossible.

If $r(x-z) = 2$, then $r = 2,\ x-z = 1$ or $r = 1,\ x-z = 2$. If $r = 2,\ x-z = 1$, then we have $z = 2y = 2$ and $x = 3$. By (4.5), we have $171500 = 35^3 \cdot 2^2 = 37^2 - 3 = 1366$, this is impossible. If $r = 1,\ x-z = 2$, we have $z = 3y = 3$ and $x = 5$. It follows that $35^5 \cdot 2^2 = 37^3 - 3$, this is impossible.

Case 2. $r = 0,\ s \geq 1$. Then $n = p^s$. By (4.2),

$$2^{2y} \cdot p^{\alpha y} = p^{s(z-y)}\big((4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot p^{s(x-z)}\big).$$

So $\alpha y = s(z-y)$ and

$$2^{2y} = (4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot p^{s(x-z)}. \tag{4.6}$$

By (4.6), we have $(-1)^{x+s(x-z)} \equiv 1 \pmod 4$. Then $2 \mid x + s(x-z)$. If $y = 1$, then

$$4 = (4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot p^{s(x-z)}. \tag{4.7}$$

By (4.7), we have $4 \equiv 1 \pmod p$, so $p = 3$. We have $5^z - 3^{x+s(x-z)} \equiv 5^z - 1 \equiv 4 \pmod 8$, so $2 \nmid z$. Note that $\alpha y = s(z-y)$, we have $\alpha = s(z-1)$, so $2 \mid \alpha$. By (4.7), we have $2^z \equiv 4 \pmod{2\cdot 3^\alpha + 1}$. Let $(\frac{*}{*})$ denote the Jacobi symbol. Note that $2\cdot 3^\alpha + 1 \equiv 3 \pmod 8$, we have

$$-1 = \Big(\frac{2}{2\cdot 3^\alpha+1}\Big) = \Big(\frac{2}{2\cdot 3^\alpha+1}\Big)^z = \Big(\frac{4}{2\cdot 3^\alpha+1}\Big) = 1,$$

a contradiction. So $y \geq 2$. If $p = 3$, then, by (4.6), we have $5^z \equiv 3^{x+s(x-z)} \equiv 1 \pmod 8$. Thus $2 \mid z$. By Lemma 2.3, this is impossible. If $p > 3$, by (4.6), we have $(-1)^z \equiv 1 \pmod 3$. So $2 \mid z$. By Lemma 2.3, this is impossible.

Case 3. $r \geq 1,\ s \geq 1$. It is obviously that $2p \nmid (4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot 2^{r(x-z)} \cdot p^{s(x-z)}$. By (4.2), we have $2y = r(z-y), \alpha y = s(z-y)$ and

$$1 = (4\cdot p^{2\alpha}+1)^z - (4\cdot p^{2\alpha}-1)^x \cdot 2^{r(x-z)} \cdot p^{s(x-z)}, \tag{4.8}$$

or equally

$$(4 \cdot p^{2\alpha} - 1)^x \cdot 2^{r(x-z)} \cdot p^{s(x-z)} = (4 \cdot p^{2\alpha} + 1)^z - 1.$$

So $\alpha r = 2s$. Note that $p^{2\alpha} \mid (4 \cdot p^{2\alpha} + 1)^z - 1$, we have $s(x - z) \geq 2\alpha$. It turns out that $rs(x - z) \geq 2\alpha r = 4s$, i.e., $r(x - z) \geq 4$. By (4.8), we have $5^z \equiv 1 \pmod 8$, so $2 \mid z$. By Lemma 2.3, this is impossible.

This completes the proof of Theorem 1.2.

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