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# DERIVATIONS ON CONVOLUTION ALGEBRAS

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Abstract. In this paper, we investigate derivations on the noncommutative Banach algebra $L_0^\infty(\omega)^*$ equipped with an Arens product. As a main result, we prove the Singer-Wermer conjecture for the noncommutative Banach algebra $L_0^\infty(\omega)^*$. We then show that a derivation on $L_0^\infty(\omega)^*$ is continuous if and only if its restriction to $\mathrm{rad}(L_0^\infty(\omega)^*)$ is continuous. We also prove that there is no nonzero centralizing derivation on $L_0^\infty(\omega)^*$. Finally, we prove that the space of all inner derivations of $L_0^\infty(\omega)^*$ is continuously homomorphic to the space $L_0^\infty(\omega)^*/L^1(\omega)$.

## 1. Introduction

Let $\omega$ be a weight function on $\mathbb{R}^+ = [0,\infty)$, i.e., a continuous function $\omega : \mathbb{R}^+ \to [1,\infty)$ so that $\omega(0) = 1$ and

$$w(x+y) \le w(x)\, w(y)$$

for all $x, y \in \mathbb{R}^+$. Let also $L^1(\omega)$ be the Banach space of all Lebesgue measurable functions $\phi$ on $\mathbb{R}^+$ such that

$$\|\phi\|_{1,\,\omega} = \int_0^\infty |\phi(x)|\, w(x)\, dx < \infty.$$

Then $L^1(\omega)$ with the norm $\|\cdot\|_{1,\,\omega}$ and the convolution product "$*$" defined by

$$\phi * \psi(x) = \int_0^x \phi(y)\psi(x-y)\, dy \quad (x \in \mathbb{R}^+)$$

is a Banach algebra. We also assume that $L_0^\infty(\omega)$ is the Banach space of all Lebesgue measurable functions $f$ on $\mathbb{R}^+$ such that

$$\|f\chi_{(x,\infty)}\|_{\infty,\omega} = \text{ess sup}\{f(y)\chi_{(x,\infty)}(y)/\omega(y) : y \in \mathbb{R}^+\} \to 0$$

as $x \to \infty$. For every $f \in L_0^\infty(\omega)$ and $\phi \in L^1(\omega)$, the function $f \circ \phi$ defined by

$$f \circ \phi(x) = \int_0^\infty f(x+y)\phi(y)\, dy \qquad (x \in \mathbb{R}^+)$$

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is an element in $C_0(\omega)$, the Banach space of all continuous complex-valued functions $f$ on $\mathbb{R}^+$ such that $f/\omega$ vanishes at infinity; see [10]. So, we may endow the dual of $L_0^\infty(\omega)$, represented by $L_0^\infty(\omega)^*$, with the first Arens product "$\cdot$" defined as follows. For any $\phi$ in $L^1(\omega)$, $f$ in $L_0^\infty(\omega)$ and $m, n$ in $L_0^\infty(\omega)^*$, the element $m \cdot n$ is defined by

$$\langle m \cdot n, f\rangle = \langle m, nf\rangle,$$

where $\langle nf, \phi\rangle = \langle n, f \circ \phi\rangle$. Then $L_0^\infty(\omega)^*$ with this product is a Banach algebra; see [10]; see also Lau and Pym [8] for the locally compact group case. Note that since $\phi \cdot \psi = \phi * \psi$ for $\phi, \psi \in L^1(\omega)$, $L^1(\omega)$ may be regarded as a subspace of $L_0^\infty(\omega)^*$ and then $L^1(\omega)$ is a closed ideal in $L_0^\infty(\omega)^*$ [10].

Since $\omega(x) \geq 1$ for all $x \in \mathbb{R}^+$, the Banach algebra $L^1(\omega)$ has a bounded approximate identity, for example the sequence $\{i\chi_{(0,1/i)}\}_{i\in\mathbb{N}}$ is one of them; see [14]. We denote by $\Lambda(L_0^\infty(\omega)^*)$ the set of all weak$^*-$cluster points of an approximate identity in $L^1(\omega)$ bounded by one. Note that for any $n \in L_0^\infty(\omega)^*$ and $\phi \in L^1(\omega)$, the maps

$$m \mapsto m \cdot n \quad \text{and} \quad m \mapsto \phi \cdot m$$

are weak$^*$-weak$^*$ continuous on $L_0^\infty(\omega)^*$. Hence if $u \in \Lambda(L_0^\infty(\omega)^*)$, then $u$ is a mixed identity with norm one, that is, $\|u\| = 1$ and $u \cdot \phi = \phi \cdot u = \phi$ for all $\phi \in L^1(\omega)$. Goldstine's theorem implies that every mixed identity for $L_0^\infty(\omega)^*$ is a right identity for $L_0^\infty(\omega)^*$. Also, note that if $u$ is a right identity for $L_0^\infty(\omega)^*$ and $(e_\alpha)$ is a bounded net in $L^1(\omega)$ such that $e_\alpha \to u$ in the weak$^*$ topology of $L_0^\infty(\omega)^*$, then for every $\phi \in L^1(\omega)$, we have

$$\phi \cdot e_\alpha \to \phi$$

in the weak topology. Passing to convex combinations, we can suppose that $\phi \cdot e_\alpha \to \phi$ in the norm topology. This together with the fact that $L^1(\omega)$ is a commutative Banach algebra shows that $(e_\alpha)$ is a bounded approximate identity for $L^1(\omega)$. Thus $u \in \Lambda(L_0^\infty(\omega)^*)$. These facts can be summarized by saying that $u \in \Lambda(L_0^\infty(\omega)^*)$ if and only if it is a right identity for $L_0^\infty(\omega)^*$ with norm one or, equivalently, $u$ is a mixed identity with norm one.

By $M(\omega)$ we mean the Banach space of all complex regular Borel measures $\mu$ on $\mathbb{R}^+$ such that

$$\|\mu\|_\omega = \int_0^\infty \omega(x)\, d|\mu|(x) < \infty,$$

where $|\mu|$ denotes the total variation of $\mu$. By the usual way, the Banach space $M(\omega)$ can be identified with the dual of $C_0(\omega)$. So, we may define the convolution multiplication "$*$" by

$$\langle \mu * \nu, g\rangle = \int_0^\infty \int_0^\infty g(x+y)\, d\mu(x)\, d\nu(y)$$

for all $\mu, \nu \in M(\omega)$ and $g \in C_0(\omega)$. Then the Banach space $M(\omega)$ with the convolution product $*$ and the norm $\|\cdot\|_\omega$ is a commutative Banach algebra. Let us remark from [11, Theorem 3.6(i) and Example 4.13(c)] that if $u$ is an element of $\Lambda(L_0^\infty(\omega)^*)$, then $u \cdot L_0^\infty(\omega)^*$ is isometrically isomorphic to $M(\omega)$. This implies that $u \cdot L_0^\infty(\omega)^*$ is commutative and so for every $k, m, n \in L_0^\infty(\omega)^*$

$$k \cdot m \cdot n = k \cdot u \cdot m \cdot u \cdot n = k \cdot u \cdot n \cdot u \cdot m = k \cdot n \cdot m. \tag{1}$$

Let $\mathrm{rAnn}(L_0^\infty(\omega)^*)$ be the set of all $r \in L_0^\infty(\omega)^*$ such that $L_0^\infty(\omega)^* \cdot r = \{0\}$. For every $m \in L_0^\infty(\omega)^*, f \in L_0^\infty(\omega)$ and $\phi \in L^1(\omega)$, we have

$$\langle \phi \cdot m, f \rangle = \langle \phi, mf \rangle = \langle mf, \phi \rangle = \langle m, f \circ \phi \rangle.$$

Also it is well-known from [11, Proposition 2.3(b) and Example 4.13(c)] that

$$L_0^\infty(\omega) \circ L^1(\omega) = C_0(\omega).$$

These facts show that

$$\mathrm{rAnn}(L_0^\infty(\omega)^*) = C_0(\omega)^\perp := \{r \in L_0^\infty(\omega)^* : r|_{C_0(\omega)} = 0\}.$$

Let us recall from [11, Theorem 3.6(iv) and Example 4.13(c)] that if $u \in \Lambda(L_0^\infty(\omega)^*)$, then the Banach space $L_0^\infty(\omega)^*$ is the Banach space direct sum of $u \cdot L_0^\infty(\omega)^*$ and $\mathrm{rAnn}(L_0^\infty(\omega)^*)$.

Let $A$ be a Banach algebra; a linear mapping $D : A \to A$ is called a *derivation* if

$$D(ab) = D(a)b + aD(b).$$

In 1955, Singer and Wermer [17] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. They conjectured that the continuity requirement for the derivations is unnecessary and can be removed. Johnson [5] showed the automatic continuity of derivations of commutative semisimple Banach algebras and so the conjecture can be established for these types of Banach algebras. Finally in 1988, Thomas [19] gave an affirmative answer to the Singer-Wermer conjecture. Obviously, because of inner derivations, this result does not remain valid for noncommutative Banach algebras. There are, however, various noncommutative versions of the Singer-Wermer theorem for Banach algebras [11, 12, 16]. For example, Sinclair [16] proved that every continuous derivation of a Banach algebra $A$ leaves primitive ideals of $A$ invariant. Mathieu and Runde [12] gave another noncommutative extension of the Singer-Wermer theorem: Every centralizing derivation on a noncommutative Banach algebra has its image in the radical of algebra; see Posner [15] for the noncommutative version of the Singer-Wermer theorem for prime rings; see also [1, 6, 7, 20] for range inclusion results for derivations on not necessarily commutative Banach algebras.

Isik and et al. [4] gave some interesting results on the structure of the Banach algebra $L^\infty(G)^*$, for an infinite compact group $G$. Lau and Pym [8] introduced the subspace $L_0^\infty(G)$ of $L^\infty(G)$ consisting of bounded measurable functions on locally compact group $G$ that vanish at infinity. For a locally compact group, they proved most of the results obtained in [4] for $L_0^\infty(G)^*$. In fact, they introduced a sensible replacement for $L^\infty(G)$, when $G$ is compact. Maghsoudi and et al. [10] have introduced and studied a semigroup analogue of $L_0^\infty(G)^*$. They showed that some aspects of the theory of $L^\infty(G)^*$ when $G$ is a compact group hold for $L_0^\infty(\omega)^*$. Other aspects of analysis on these Banach algebras have been studied by many authors; see for example, [9, 13, 18].

Since $\mathrm{rAnn}(L_0^\infty(\omega)^*) = C_0(\omega)^\perp$, an easy application of the Hahn-Banach's theorem shows that there is a non-zero element $r \in \mathrm{rAnn}(L_0^\infty(\omega)^*)$; see for example [2, Corollary 6.8 of Chapter 3]. Hence for every $u \in \Lambda(L_0^\infty(\omega)^*)$, we have $u \cdot r = 0$, $r \cdot u = r$ and

$$r \cdot L_0^\infty(\omega)^* \cdot r = \{0\}.$$

These relations imply that $L_0^\infty(\omega)^*$ is a noncommutative Banach algebra. It also follows that $L_0^\infty(\omega)^*$ is not a prime ring. Therefore, we cannot apply the well-known results concerning derivations of commutative Banach algebra and derivation of prime rings to $L_0^\infty(\omega)^*$. It is natural to ask whether the results hold for $L_0^\infty(\omega)^*$. In this paper we investigate the truth of these results for $L_0^\infty(\omega)^*$.

This paper is organized as follows: In Section 2, we prove that the range of every derivation on the noncommutative Banach algebra $L_0^\infty(\omega)^*$ is contained in the radical of $L_0^\infty(\omega)^*$ and that every derivation on $L_0^\infty(\omega)^*$ leaves the primitive ideals of $L_0^\infty(\omega)^*$ invariant. We also show that a derivation on $L_0^\infty(\omega)^*$ is continuous if and only if its restriction to $\mathrm{rad}(L_0^\infty(\omega)^*)$ is continuous. In Section 3, we study centralizing derivations and inner derivations of $L_0^\infty(\omega)^*$. We show that there is no nonzero centralizing derivation on $L_0^\infty(\omega)^*$ and prove that the space of all inner derivations of $L_0^\infty(\omega)^*$ is continuously homomorphic to the space $L_0^\infty(\omega)^*/L^1(\omega)$.

## 2. The range and automatic continuity of derivations

Let $\mathrm{rad}(L_0^\infty(\omega)^*)$ denote the radical of Banach algebra $L_0^\infty(\omega)^*$. The main result of this paper is the following theorem.

**Theorem 2.1.** *Let $\omega$ be a weight function on $\mathbb{R}^+$ and $D$ be a derivation on $L_0^\infty(\omega)^*$. Then $D$ maps $L_0^\infty(\omega)^*$ into* $\mathrm{rad}(L_0^\infty(\omega)^*) = \mathrm{rAnn}(L_0^\infty(\omega)^*)$.

*Proof.* First, note that every element of $\mathrm{rAnn}(L_0^\infty(\omega)^*)$ is nilpotent. So

$$\mathrm{rAnn}(L_0^\infty(\omega)^*) \subseteq \mathrm{rad}(L_0^\infty(\omega)^*).$$

Since $\omega(x) \geq 1$ for all $x \in \mathbb{R}^+$, the Banach algebra $M(\omega)$ is semisimple; see for example [3]. This together with the fact that $M(\omega)$ is isometrically isomorphism to $L_0^\infty(\omega)^*/\mathrm{rAnn}(L_0^\infty(\omega))^*$ yields that

$$\mathrm{rad}(L_0^\infty(\omega)^*)/\mathrm{rAnn}(L_0^\infty(\omega)^*) = 0.$$

It follows that $\mathrm{rad}(L_0^\infty(\omega)^*)$ is contained in $\mathrm{rAnn}(L_0^\infty(\omega)^*)$. Thus

$$\mathrm{rad}(L_0^\infty(\omega)^*) = \mathrm{rAnn}(L_0^\infty(\omega)^*).$$

For every $k \in L_0^\infty(\omega)^*$ and $r \in \mathrm{rAnn}(L_0^\infty(\omega)^*)$ we have

$$k \cdot D(r) = D(k \cdot r) - D(k) \cdot r = 0.$$

Hence $D$ maps $\mathrm{rAnn}(L_0^\infty(\omega)^*)$ into $\mathrm{rAnn}(L_0^\infty(\omega)^*)$, which implies that the function

$$\overline{D} : L_0^\infty(\omega)^*/\mathrm{rAnn}(L_0^\infty(\omega)^*) \to L_0^\infty(\omega)^*/\mathrm{rAnn}(L_0^\infty(\omega)^*)$$

defined by

$$\overline{D}(m + \mathrm{rAnn}(L_0^\infty(\omega)^*)) = D(m) + \mathrm{rAnn}(L_0^\infty(\omega)^*)$$

is well defined. It is easy to see that $\overline{D}$ is a derivation on semisimple and commutative Banach algebra $L_0^\infty(\omega)^*/\mathrm{rAnn}(L_0^\infty(\omega)^*)$. Thus

$$\overline{D}(m + \mathrm{rAnn}(L_0^\infty(\omega)^*)) = 0$$

for all $m \in L_0^\infty(\omega)^*$. Hence $D(m)$ is an element in $\mathrm{rAnn}(L_0^\infty(\omega)^*)$. That is,

$$D(L_0^\infty(\omega)^*) \subseteq \mathrm{rAnn}(L_0^\infty(\omega)^*),$$

as claimed. □

As an immediate consequence of this theorem we have the following result.

**Corollary 2.2.** *Let $\omega$ be a weight function on $\mathbb{R}^+$. Then the following statements hold*:

(i) *Any derivation on $L_0^\infty(\omega)^*$ leaves primitive ideals of $L_0^\infty(\omega)^*$ invariant.*
(ii) *The composition of two derivations on $L_0^\infty(\omega)^*$ is always a derivation on $L_0^\infty(\omega)^*$.*

Let us recall that a linear mapping $T : L_0^\infty(\omega)^* \to L_0^\infty(\omega)^*$ is called *spectrally bounded* if there exists $\alpha \geq 0$ such that $r(T(m)) \leq \alpha r(m)$ for all $m \in L_0^\infty(\omega)^*$, where $r(\cdot)$ stands for the spectral radius.

**Corollary 2.3.** *Let $\omega$ be a weight function on $\mathbb{R}^+$. Then the following statements hold*:

(i) *Any derivation on $L_0^\infty(\omega)^*$ is spectrally bounded.*
(ii) *Zero is the only weak$^*$-weak$^*$ continuous derivation on $L_0^\infty(\omega)^*$.*

*Proof.* (i) Let $D$ be a derivation on $L_0^\infty(\omega)^*$. If $m \in L_0^\infty(\omega)^*$, then by Theorem 2.1, $D(m)^i = 0$ for all $i \geq 2$. Hence

$$r(D(m)) = \lim_{i\to\infty} \|D(m)^i\|^{1/i} = 0.$$

So $D$ is spectrally bounded.

(ii) Let $D$ be a weak$^*$-weak$^*$ continuous derivation on $L_0^\infty(\omega)^*$ and $\phi \in L^1(\omega)$. Invoke Cohen's factorization theorem to conclude that $\phi = \phi_1 * \phi_2$ for some $\phi_1, \phi_2 \in L^1(\omega)$. By definition of $\mathrm{rAnn}(L_0^\infty(\omega)^*)$, we have

$$L^1(\omega) \cdot \mathrm{rAnn}(L_0^\infty(\omega)^*) = \{0\}. \tag{2}$$

Proposition 2.3 of [10] together with the fact that $\mathrm{rAnn}(L_0^\infty(\omega)^*) = C_0(\omega)^\perp$ implies that

$$\mathrm{rAnn}(L_0^\infty(\omega)^*) \cdot L^1(\omega) = \{0\}. \tag{3}$$

From (2) and (3) it follows that

$$D(\phi) = D(\phi_1 * \phi_2) = D(\phi_1) \cdot \phi_2 + \phi_1 \cdot D(\phi_2) = 0.$$

This shows that $D(L^1(\omega)) = \{0\}$. From weak$^*$ density of $L^1(\omega)$ in $L_0^\infty(\omega)^*$ we infer that $D = 0$. Hence (ii) holds. □

We conclude the section by the following result.

**Theorem 2.4.** *Let $\omega$ be a weight function on $\mathbb{R}^+$ and $D$ be a derivation on $L_0^\infty(\omega)^*$. Then $D$ is continuous if and only if $D|_{\mathrm{rad}(L_0^\infty(\omega)^*)}$ is continuous.*

*Proof.* Choose $u \in \Lambda(L_0^\infty(\omega)^*)$. Let $(u \cdot m_\alpha)_{\alpha\in A}$ be a net in $L_0^\infty(\omega)^*$ such that

$$u \cdot m_\alpha \to 0 \quad \text{and} \quad D(u \cdot m_\alpha) \to m \in L_0^\infty(\omega)^*$$

in the norm topology of $L_0^\infty(\omega)^*$. Suppose that $m \neq 0$ and $i \in \mathbb{N}$. Choose $\alpha_0 \in A$ such that $\|D(u \cdot m_\alpha)\| \geq \|m\|/2$ and $\|u \cdot m_\alpha\| < \|m\|/i$ for all $\alpha \geq \alpha_0$. In view of Theorem 2.1, we have

$$\begin{aligned}\|m\|/2 &\leq \|D(u \cdot m_\alpha)\| = \|D(u) \cdot m_\alpha\| \\ &= \|D(u) \cdot u \cdot m_\alpha\| \leq \|D(u)\|\|u \cdot m_\alpha\| \\ &\leq \|D(u)\|\|m\|/i.\end{aligned}$$

Hence $\|D(u)\| \geq i/2$ for all $i \in \mathbb{N}$. This contradiction shows that $m = 0$. By closed graph theorem, $D|_{u\cdot L_0^\infty(\omega)^*}$ is continuous. Thus there exists $M_1 > 0$ such that for every $m \in L_0^\infty(\omega)^*$

$$\|D(u \cdot m)\| \leq M_1\|u \cdot m\|.$$

Now, let $D|_{\mathrm{rad}(L_0^\infty(\omega)^*)}$ be continuous. Then there exists $M_2 > 0$ such that $\|D(r)\| \leq M_2\,\|r\|$ for all $r \in \mathrm{rad}(L_0^\infty(\omega)^*)$. For any $m \in L_0^\infty(\omega)^*$ set

$$r_m = m - u \cdot m \in \mathrm{rad}(L_0^\infty(\omega)^*).$$

Then

$$\begin{aligned}\|D(m)\| &= \|D(u \cdot m) + D(r_m)\| \\ &\leq \|D(u \cdot m)\| + \|D(r_m)\| \\ &\leq M_1\,\|u \cdot m\| + M_2\,\|m - u \cdot m\| \\ &\leq M_1\,\|m\| + 2M_2\,\|m\| \\ &= (M_1 + 2M_2)\,\|m\|.\end{aligned}$$

Therefore, $D$ is bounded. The converse is clear. □

## 3. Centralizing derivations and inner derivations

In the sequel, $\mathrm{Z}(L_0^\infty(\omega)^*)$ denotes the center of $L_0^\infty(\omega)^*$ and we write $[m, n] = m \cdot n - n \cdot m$ for all $m, n \in L_0^\infty(\omega)^*$.

**Theorem 3.1.** *Let $\omega$ be a weight function on $\mathbb{R}^+$ and $D$ be a derivation on $L_0^\infty(\omega)^*$. Then the following assertions are equivalent.*

(a) $D = 0$.
(b) *There exists* $i \in \mathbb{N}$ *such that* $D(m^i) = 0$ *for all* $m \in L_0^\infty(\omega)^*$.
(c) *There exists* $i \in \mathbb{N}$ *such that* $D(m^i) \in Z(L_0^\infty(\omega)^*)$ *for all* $m \in L_0^\infty(\omega)^*$.
(d) *There exists* $i \in \mathbb{N}$ *such that* $[D(m), m^i] \in Z(L_0^\infty(\omega)^*)$ *for all* $m \in L_0^\infty(\omega)^*$.
(e) *There exists* $i \in \mathbb{N}$ *such that* $[D(m), m^i] = 0$ *for all* $m \in L_0^\infty(\omega)^*$.

*Proof.* The implications (a)$\Rightarrow$ (b)$\Rightarrow$ (c) are clear. It follows from Theorem 2.1 that

$$[D(m), m^{i-1}] = D(m) \cdot m^{i-1} - m^{i-1} \cdot D(m) = D(m^i).$$

Hence (c)$\Rightarrow$ (d). Let us show that (d)$\Rightarrow$ (e) and (e)$\Rightarrow$ (a). Assume that (d) holds. Choose $u \in \Lambda(L_0^\infty(\omega)^*)$. Then

$$\begin{aligned}[D(m), m^i] &= [D(m), m^i] \cdot u = u \cdot [D(m), m^i] \\ &= u \cdot D(m) \cdot m^i = 0.\end{aligned}$$

So, we obtain (e). To complete the proof, suppose that there is $i \in \mathbb{N}$ such that $[D(m), m^i] = 0$ for all $m \in L_0^\infty(\omega)^*$. Fix $u \in \Lambda(L_0^\infty(\omega)^*)$. Then

$$D(u) = [D(u), u^i] = 0.$$

For every $r \in \mathrm{rAnn}(L_0^\infty(\omega)^*)$, we have

$$\begin{aligned}0 &= [D(r+u), (r+u)^i] = [D(r), (r+u)^i] \\ &= D(r)(r^i + c_1 r^{i-1} u + c_2 r^{i-2} u^2 + \cdots + c_{i-1} r u^{i-1} + u^i) = D(r)\end{aligned}$$

for some $c_1, \ldots, c_{i-1} \geq 0$. Now, let $m \in L_0^\infty(\omega)^*$. Then

$$r_m = m - u \cdot m \in \mathrm{rAnn}(L_0^\infty(\omega)^*).$$

Thus

$$D(m) = D(u \cdot m) + D(r_m) = D(u) \cdot m + D(r_m) = 0.$$

Therefore, $D = 0$. So (e) implies (a). □

A mapping $T : L_0^\infty(\omega)^* \to L_0^\infty(\omega)^*$ is called *centralizing* if $[T(m), m] \in \mathrm{Z}(L_0^\infty(\omega)^*)$ for all $m \in L_0^\infty(\omega)^*$; in a special case when $[T(m), m] = 0$ for all $m \in L_0^\infty(\omega)^*$, $T$ is called *commuting*.

**Corollary 3.2.** *Let* $\omega$ *be a weight function on* $\mathbb{R}^+$. *Then the following statements hold*:
(i) *Zero is the only centralizing derivation on* $L_0^\infty(\omega)^*$.
(ii) *Zero is the only commuting derivation on* $L_0^\infty(\omega)^*$.

A derivation $D$ on $L_0^\infty(\omega)^*$ is said to be *inner* if there exists $n_0 \in L_0^\infty(\omega)^*$ such that $D(m) = [m, n_0]$ for all $m \in L_0^\infty(\omega)^*$.

**Proposition 3.3.** *Let* $\omega$ *be a weight function on* $\mathbb{R}^+$ *and* $D$ *be a derivation on* $L_0^\infty(\omega)^*$. *Then the following assertions are equivalent.*

(a) *$D$ is inner.*

(b) *There exists* $n_0 \in L_0^\infty(\omega)^*$ *such that* $D(m^2) = [m^2, n_0]$ *for all* $m \in L_0^\infty(\omega)^*$.
(c) *There exists* $n_0 \in L_0^\infty(\omega)^*$ *such that the mapping* $m \mapsto D(m) + n_0 \cdot m$ *is commuting.*
(d) *There exists* $n_0 \in L_0^\infty(\omega)^*$ *such that the mapping* $m \mapsto D(m) + n_0 \cdot m$ *is centralizing.*

*Proof.* It is obvious that (a) implies (b) and that (c) implies (d). Suppose that there is $n_0 \in L_0^\infty(\omega)^*$ such that $D(m^2) = [m^2, n_0]$ for all $m \in L_0^\infty(\omega)^*$, then by (1) we have

$$\begin{aligned}[D(m) + n_0 \cdot m, m] &= D(m) \cdot m + n_0 \cdot m^2 - m \cdot n_0 \cdot m \\ &= D(m^2) + n_0 \cdot m^2 - m \cdot n_0 \cdot m \\ &= (m^2 \cdot n_0 - n_0 \cdot m^2) + n_0 \cdot m^2 - m^2 \cdot n_0 \\ &= 0.\end{aligned}$$

This shows that the mapping $m \mapsto D(m) + n_0 \cdot m$ is commuting. So (b) implies (c). Finally, let $n_0 \in L_0^\infty(\omega)^*$. We define $\tilde{D} : L_0^\infty(\omega)^* \to L_0^\infty(\omega)^*$ by

$$\tilde{D}(m) = D(m) - [m, n_0].$$

It is plain that $\tilde{D}$ is a derivation on $L_0^\infty(\omega)^*$. It follows from Theorem 2.1 that

$$[\tilde{D}(m), m] = \tilde{D}(m) \cdot m = [D(m) + n_0 \cdot m, m].$$

If (d) holds, then $\tilde{D}$ is a centralizing derivation. So $\tilde{D}(m) = 0$ by Corollary 3.2. This shows that

$$D(m) = [m, n_0]$$

for all $m \in L_0^\infty(\omega)^*$. That is, $D$ is inner. Hence (d) implies (a). □

In the following, let $\mathrm{InnD}(L_0^\infty(\omega)^*)$ be the space of all inner derivations on $L_0^\infty(\omega)^*$.

**Theorem 3.4.** *Let* $\omega$ *be a weight function on* $\mathbb{R}^+$. *Then the mapping* $\mathcal{I} : m + L^1(\omega) \mapsto \mathcal{I}_m$ *is a continuous homomorphism from* $L_0^\infty(\omega)^*/L^1(\omega)$ *onto* $\mathrm{Inn\,D}(L_0^\infty(\omega)^*)$, *where* $\mathcal{I}_m(n) = [n, m]$ *for all* $m \in L_0^\infty(\omega)^*$. *Furthermore,* $\mathcal{I}$ *is an isomorphism if and only if* $Z(L_0^\infty(\omega)^*) = L^1(\omega)$.

*Proof.* First note that if $u \in \Lambda(L_0^\infty(\omega)^*)$, then $u$ is a mixed identity and $u \cdot L_0^\infty(\omega)^*$ is commutative. Let $m \in L_0^\infty(\omega)^*$ and $\phi \in L^1(\omega)$. Then

$$\begin{aligned}\phi \cdot m &= u \cdot \phi \cdot u \cdot m = u \cdot m \cdot u \cdot \phi \\ &= u \cdot m \cdot \phi. \end{aligned} \tag{4}$$

Since $L^1(\omega)$ is an ideal in $L_0^\infty(\omega)^*$ and $u$ is a mixed identity, we get

$$u \cdot m \cdot \phi = m \cdot \phi. \tag{5}$$

From (4) and (5) we see that

$$\phi \cdot m = m \cdot \phi. \tag{6}$$

So the mapping $\mathcal{I}$ is well defined. Obviously, $\mathcal{I}$ is a homomorphism. To see that $\mathcal{I}$ is continuous, let $n \in L_0^\infty(\omega)^*$ and $\phi \in L^1(\omega)$. Then

$$\begin{aligned}\|\mathcal{I}_m(n)\| &= \|n \cdot m - m \cdot n\| \\ &\le \|n \cdot m - \phi \cdot n\| + \|\phi \cdot n - m \cdot n\| \\ &\le \|n\| \, \|m - \phi\| + \|\phi - m\| \, \|n\| \\ &= 2\|n\| \, \|m - \phi\|\end{aligned}$$

for all $m \in L_0^\infty(\omega)^*$. This implies that

$$\|\mathcal{I}(m + L^1(\omega))\| = \|\mathcal{I}_m\| \le 2\|m - \phi\|$$

for all $m \in L_0^\infty(\omega)^*$ and $\phi \in L^1(\omega)$. Hence

$$\begin{aligned}\|\mathcal{I}(m + L^1(\omega))\| &\le 2\inf\{\|m - \phi\| : \phi \in L^1(\omega)\} \\ &= 2\inf\{\|m + \phi\| : \phi \in L^1(\omega)\} = 2\,\|m + L^1(\omega)\|.\end{aligned}$$

Therefore, $\mathcal{I}$ is continuous.

Now, let $m \in \mathrm{Z}(L_0^\infty(\omega)^*)$. Then $\mathcal{I}_m = 0$ on $L_0^\infty(\omega)^*$. Thus $\mathcal{I}(m + L^1(\omega)) = 0$. It follows that $m \in L^1(\omega)$ if $\mathcal{I}$ is an isomorphism. This together with (6) shows that

$$\mathrm{Z}(L_0^\infty(\omega)^*) = L^1(\omega).$$

To complete the proof, let $m \in L_0^\infty(\omega)^*$ and $\mathcal{I}(m + L^1(\omega)) = 0$. Then

$$\mathcal{I}_m(n) = n \cdot m - m \cdot n = 0$$

for all $n \in L_0^\infty(\omega)^*$. This shows that $m \in \mathrm{Z}(L_0^\infty(\omega)^*)$. So, if $\mathrm{Z}(L_0^\infty(\omega)^*) = L^1(\omega)$, then $m \in L^1(\omega)$ and so $\mathcal{I}$ is an isomorphism. □

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