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THE EXTENDED REAL LINE AS A JULIA SET

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Abstract. A recursive family $\{F_n\}$ of holomorphic functions on the Riemann sphere is defined and some elementary properties of this family is described. Then the Julia set of F_n is computed. Finally this family as a real recursive family is studied and it is shown that F_n is chaotic on a specific subset of $\mathbb{R}$.

1. Introduction and preliminaries

Suppose $\hat{\mathbb{C}}$ is the Riemann sphere and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a non-constant holomorphic map. By f^n we mean $\underbrace{f \circ \cdots \circ f}_{n \text{ times}}$. The point $p \in \hat{\mathbb{C}}$ is an *attracting* (*repelling*) *fixed point* of f, if $f(p) = p$ and $|f'(p)| < 1$ ($|f'(p)| > 1$). The *basin of attraction* of p is the set of all points z such that $\lim_{n\to\infty} f^n(z) = p$. Generally, if $f^k(p) = p$ for some $k \geq 1$, then p is called a *periodic point* of f.

The *Fatou set* of f consists of all points $z \in \hat{\mathbb{C}}$ that have an open neighborhood on which $\{f^n\}$ forms a normal family. The *Julia set* of f, denoted by $J(f)$, is the complement of the Fatou set. It is clear from the definition that the Fatou set is an open subset of $\hat{\mathbb{C}}$ and therefore the Julia set is a closed subset of $\hat{\mathbb{C}}$. The attracting fixed points belong to the Fatou set and the repelling fixed points belong to the Julia set and also the boundary of the basin of attraction of a fixed point is a subset of the Julia set.

Holomorphic functions on the Riemann sphere have various Julia sets with very different shapes. Among these different sets, circles, the extended real line and closed intervals are well-known smooth Julia sets. More specifically the following results are known.

- The Julia set of a polynomial of degree $d \geq 2$ is homeomorphic to a closed interval if and only if it is linearly conjugate to $\pm\phi_d$, where ϕ_d is the Chebyshev polynomial of degree d (see [3, Lemma B.3]).

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- If the Julia set of a rational function $f(z)$ of degree $d \geq 2$ is the unit circle, then
$$f(z) = \alpha \frac{z - a_1}{1 - \overline{a_1} z} \frac{z - a_2}{1 - \overline{a_2} z} \cdots \frac{z - a_d}{1 - \overline{a_d} z}$$
where $a_j \in \hat{\mathbb{C}} \setminus \partial\mathbb{D}$, $\alpha \in \partial\mathbb{D}$, and $\mathbb{D}$ is the unit disc (see [4, Lemma 15.5]).
- The Julia set of
$$f(z) = \alpha z \frac{z - a_1}{1 - \overline{a_1} z} \frac{z - a_2}{1 - \overline{a_2} z} \cdots \frac{z - a_d}{1 - \overline{a_d} z}$$
where $a_j \in \mathbb{D}$, $\alpha \in \partial\mathbb{D}$, and $d \geq 1$, is the unit circle (see [4, page 73]).
- If f is a polynomial of degree $d \geq 2$, then its Julia set is the unit circle if and only if $f(z) = az^d$, where $a \in \partial\mathbb{D}$ (see [1, Theorem 1.3.1]).

It is also known that if f and g are two commuting ($f \circ g = g \circ f$) rational functions of degree at least two, then $J(f) = J(g)$ (see [1, Theorem 4.2.9]). Another basic result states that if f and g are two rational functions that are conjugate under a Möbius function φ (i.e., $f \circ \varphi = \varphi \circ g$), then the image of the Julia set of g under φ is the Julia set of f (see [1, Theorem 3.1.4]). Therefore under suitable conjugacy, some of the above statements about the unit circle can be modified to $\mathbb{R} \cup \{\infty\}$.

Recall that a continuous function $f : X \to X$, where X is a metric space with metric d, is called *chaotic* if the following three conditions hold:

1. The set of the *periodic points* of f is dense in X.
2. f is topologically *transitive* (i.e., for every pair of open subsets U, V of X there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$).
3. f has *sensitive dependence on initial conditions* (i.e., there exists $\delta > 0$ such that for any $x \in X$ and any open subset U containing x, there exist $y \in U$ and $n \geq 0$ such that $d(f^n(x),\ f^n(y)) > \delta$).

See [2, page 50] for more details.

In this paper we introduce a family $\{F_n\}$ of rational functions recursively and show that the Julia set of F_n is $\mathbb{R} \cup \{\infty\}$. We use two different methods to compute the Julia set of F_n, firstly by exploiting the commutativity of this family and secondly by establishing the conjugacy of this family with another well known family of functions. Finally, we study F_n as a real function and show that it is chaotic on a specific subset of $\mathbb{R}$.

2. The recursive family and its properties

In this section we will define a family of rational functions $\{F_n\}$ on $\hat{\mathbb{C}}$ recursively and determine their Julia sets.

Recall that the Newton iteration function associated to $f(z) = z^2 + 1$, that is $N(z) = z - \frac{f(z)}{f'(z)} = \frac{z^2-1}{2z}$, satisfies the relation
$$F_2(z) = \frac{zF_1(z) - 1}{z + F_1(z)}$$

for $F_1(z) = z$ and $F_2(z) = N(z)$. Motivated by this observation we define the following recursive family.

Let $F_1(z) = z$ and, for $n \geq 2$,

$$F_n(z) = \frac{zF_{n-1}(z) - 1}{z + F_{n-1}(z)}. \tag{2.1}$$

Note that if $F_{n-1}(z_0) = \infty$, then

$$F_n(z_0) = \lim_{z \to z_0} \frac{zF_{n-1}(z) - 1}{z + F_{n-1}(z)}.$$

The following lemmas describe some properties of this family.

Lemma 2.1. *For $k \geq 1$ and $n > k$,*

$$F_n(z) = \frac{F_k(z)F_{n-k}(z) - 1}{F_k(z) + F_{n-k}(z)}.$$

Proof. We proceed by induction on k. The case $k = 1$ is (2.1). Assume for $k > 1$ and $n > k$ we have

$$F_n(z) = \frac{F_k(z)F_{n-k}(z) - 1}{F_k(z) + F_{n-k}(z)}. \tag{2.2}$$

Now suppose $n > k + 1$. Then F_n satisfies (2.2). From (2.1) we obtain

$$F_k(z) = \frac{zF_{k+1}(z) + 1}{z - F_{k+1}(z)}.$$

By employing the latter for F_k and the definition of F_{n-k} in (2.2), the result follows by straightforward computations. □

Lemma 2.2. *For $n \geq 1$ and $k \geq 1$, $F_n \circ F_k = F_{nk}$.*

Proof. We proceed by induction on n. The case $n = 1$ and $k \geq 1$ follows from the definition of F_1. By employing (2.1), the induction hypothesis, and Lemma 2.1, we have

$$\begin{aligned} F_{n+1}(F_k(z)) &= \frac{F_k(z)F_n(F_k(z)) - 1}{F_k(z) + F_n(F_k(z))} \\ &= \frac{F_k(z)F_{nk}(z) - 1}{F_k(z) + F_{nk}(z)} \\ &= F_{nk+k}(z). \end{aligned}$$

□

Corollary 2.3. *For $n \geq 1$ and $k \geq 1$, $F_n \circ F_k = F_k \circ F_n$.*

Lemma 2.4. *For $n \geq 2$, degree of F_n is at least 2.*

Proof. Suppose $F_n(z) = \frac{p_n(z)}{q_n(z)}$ where p_n and q_n are two polynomials. It can be shown by induction that $p_n(z) = zp_{n-1}(z) - q_{n-1}(z)$ and $q_n(z) = zq_{n-1}(z) + p_{n-1}(z)$ where $p_1(z) = z$ and $q_1(z) = 1$. Since $\deg p_1 > \deg q_1$ and $\deg p_n = \deg p_{n-1} + 1$, therefore $\deg p_n > \deg q_n$ and consequently $F_n(\infty) = \infty$. By induction on n and employing (2.1) one can easily see that $\pm i$ are the fixed points of F_n. Thus $\deg F_n \geq 2$ unless $F_n(z) = z$ that is impossible. □

The following theorem will determine the Julia set of F_n.

Theorem 2.5. *For $n \geq 2$, the Julia set of F_n is $\mathbb{R} \cup \{\infty\}$.*

Proof. We first show that the Julia set of $F_2(z) = \frac{z^2-1}{2z}$ is $\mathbb{R} \cup \{\infty\}$. The points $z = \pm i$ are attracting fixed points and ∞ is a repelling fixed point of F_2. Therefore ∞ belongs to $J(F_2)$ and the set of all iterated preimages of ∞, $\{z \in \hat{\mathbb{C}} :\ F_2^n(z) = \infty \text{ for some } n \geq 0\}$, is everywhere dense in $J(F_2)$ (see [4, Corollary 4.13]). Since all of the solutions of $\frac{z^2-1}{2z} = c$ for real c are real, we conclude that $J(F_2) \subseteq \mathbb{R} \cup \{\infty\}$. Since the boundary of the basins of attraction of the fixed points i and $-i$ are subsets of $J(F_2)$, the upper half-plane is the basin of attraction of i and the lower half-plane is the basin of attraction of $-i$. Therefore $J(F_2) = \mathbb{R} \cup \{\infty\}$.

On the other hand, by Corollary 2.3, $F_2 \circ F_n = F_n \circ F_2$. Thus $J(F_n) = J(F_2) = \mathbb{R} \cup \{\infty\}$ (see [1, Theorem 4.2.9]). □

By differentiating of (2.1) one can see that the fixed points $\pm i$ are attracting. By applying these fixed points we are going to prove the following theorem.

Theorem 2.6. *F_n and z^n are conjugate.*

Proof. Suppose φ is a Möbius function such that $\varphi(0) = -i$ and $\varphi(\infty) = i$. Thus $\varphi(z) = i\frac{z-a}{z+a}$, for some $a \in \mathbb{C}$. By induction on n and employing (2.1), we conclude that $F_n \circ \varphi(z) = i\frac{z^n+a^2}{z^n-a^2}$. It can be shown that $F_n \circ \varphi(z) = \varphi(z^n)$ if and only if $a = -1$. Thus F_n and z^n are conjugate under $\varphi(z) = i\frac{z+1}{z-1}$. □

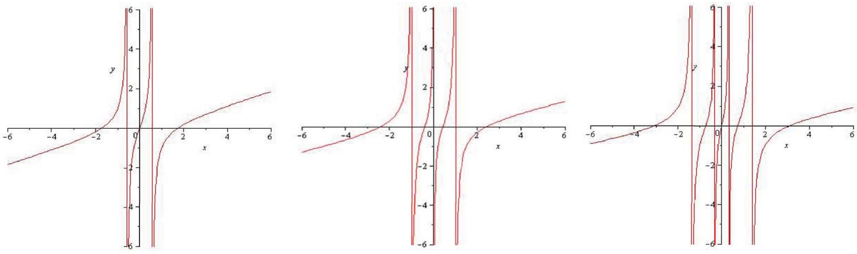
Remark 1.

- In the proof of Theorem 2.6, it is possible to suppose $\varphi(0) = i$ and $\varphi(\infty) = -i$.
- Since the image of the unit circle under φ in Theorem 2.6 is the extended real line, it is possible to conclude Theorem 2.5 from Theorem 2.6.

Now we are going to describe $F_n(z) = \frac{p_n(z)}{q_n(z)}$ more specifically. By Theorem 2.6, F_n is a rational function of degree n and p_n and q_n are two relatively prime polynomials of degree n and $n-1$, respectively. The following proposition shows that F_n has n distinct real roots and $n-1$ real poles.

Proposition 2.7. *All the roots of q_n and p_n are real and distinct.*

Proof. According to the above discussion, $F_n(z) = \infty$ if only if $q_n(z) = 0$ or $z = \infty$. On the other hand $F_n(z) = \infty$ if and only if $\varphi((\varphi^{-1}(z))^n) = \infty$. Since $\varphi(1) = \infty$, therefore $(\varphi^{-1}(z))^n = 1$. Thus $\varphi^{-1}(z)$ must be an n-th root of unity (i.e., 1, $e^{\frac{2\pi i}{n}}, \dots, e^{\frac{2(n-1)\pi i}{n}}$) and consequently $z = \infty, \varphi(e^{\frac{2\pi i}{n}}), \dots, \varphi(e^{\frac{2(n-1)\pi i}{n}})$. Therefore $q_n(z) = 0$ has $n-1$ distinct roots that are real.



$F_3(x) = \frac{x^3-3x}{3x^2-1}$ $F_4(x) = \frac{x^4-6x^2+1}{4x^3-4x}$ $F_5(x) = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}$

FIGURE 1. The graph of F_n, for $n = 3, 4, 5$

In addition, $p_n(z) = 0$ if and only if $F_n(z) = 0$. Since $\varphi(-1) = 0$, the similar argument shows that $F_n(z) = 0$ if and only if $(\varphi^{-1}(z))^n = -1$. Thus p_n has n real and distinct roots. □

Figure 1 illustrates the roots and poles of $F_n(z)$ for $n = 3, 4, 5$.

3. F_n as a real rational function

In this section we consider F_n as a real function and show that F_n is chaotic on some subset of $\mathbb{R}$. Figure 1 shows the graph of F_n, for $n = 3, 4, 5$.
Since F_n is not defined on the roots of q_n as a real function, we are only able to study the dynamics of F_n on the set $\mathbb{R} \setminus B_n$, where

$$B_n = \bigcup_{k=0}^{\infty} F_n^{-k}\Big(\Big\{x \in \mathbb{R} : \quad x \text{ is a pole of } F_n\Big\}\Big) = \bigcup_{k=1}^{\infty} F_n^{-k}(\infty).$$

Definition of B_n shows that $F_n(\mathbb{R} \setminus B_n) \subseteq \mathbb{R} \setminus B_n$.

On the other hand $\varphi(e^{i\theta}) = i\frac{e^{i\theta}+1}{e^{i\theta}-1} = \cot\frac{\theta}{2}$. Thus if $C(\theta) = \cot\frac{\theta}{2}$ and $D_n : S^1 \to S^1$ is given by

$$D_n(\theta) = n\theta \pmod{2\pi},$$

where S^1 is the unit circle, then the relation $F_n \circ \varphi(z) = \varphi(z^n)$ can be written as $F_n \circ C(\theta) = C \circ D_n(\theta)$ when $z = e^{i\theta}$. Therefore if we reduce the domain of D_n to $S^1 \setminus A_n$, where $A_n = \cup_{k\geq 1} D_n^{-k}(0)$, then the diagram

$$\begin{array}{ccc} S^1 \setminus A_n & \xrightarrow{D_n} & S^1 \setminus A_n \\ {\scriptstyle C(\theta)}\downarrow & & \downarrow{\scriptstyle C(\theta)} \\ \mathbb{R} \setminus B_n & \xrightarrow[F_n]{} & \mathbb{R} \setminus B_n \end{array} \tag{3.1}$$

is commutative. Since A_n is dense in S^1, $C(A_n \setminus \{0\}) = B_n$, and $C(\theta)$ is a homeomorphism, if $0 < \theta < 2\pi$, then we have the following result.

Proposition 3.1. *B_n is dense in $\mathbb{R}$.*

It is known that $D_n : S^1 \to S^1$ is chaotic (see [2, page 50]). Now we will show that

Theorem 3.2. *F_n is chaotic on $\mathbb{R} \setminus B_n$.*

Proof. First we will show that $D_n : S^1 \setminus A_n \to S^1 \setminus A_n$ is chaotic. Note that

(1) The periodic points of D_n are dense in S^1. The points of A_n are eventually maps to 0, so $x \neq 0$ and $x \in A_n$ is not a periodic point of D_n. Thus the periodic points of D_n are dense in $S^1 \setminus A_n$.
(2) For the nonempty open subset U of S^1, there is $k \in \mathbb{N}$ such that $D_n^k(U) = S^1$. Therefore $D_n^k(U \setminus A_n) = S^1 \setminus A_n$ since $D_n(A_n) = A_n$. If V is a nonempty open subset of S^1, then $D_n^k(U \setminus A_n) \cap (V \setminus A_n) \neq \emptyset$. This implies that D_n is transitive on $S^1 \setminus A_n$.
(3) Since $D_n^k(U \setminus A_n) = S^1 \setminus A_n$ for some $k \in \mathbb{N}$, if $x \in U \setminus A_n$, then there is $y \in U \setminus A_n$ such that $d(D_n^k(x), D_n^k(y)) > 1/2$. Thus D_n has sensitive dependence on initial conditions on $S^1 \setminus A_n$.

Therefore D_n is chaotic on $S^1 \setminus A_n$.

Since the diagram (3.1) is commutative, the density of the periodic points of F_n and topological transitivity of F_n are concluded easily from the above discussion. Also for each open set U in $\mathbb{R} \setminus B_n$ there is $k \in \mathbb{N}$ such that $F_n^k(U) = \mathbb{R} \setminus B_n$. Thus F_n has sensitive dependence on initial conditions on $\mathbb{R} \setminus B_n$. □

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References

[1] A. F. Beardon, *Iteration of Rational Functions*, Cambridge, 1991.
[2] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd. ed., Addison-Wesley, 1989.
[3] J. Milnor, *Pasting together Julia sets: A worked out example of mating*, Experiment. Math. **13** (2004), no. 1, 55–92.
[4] ———, *Dynamics in One Complex Variable*, 3rd. ed., Princeton University Press, 2006.

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