

EQUIVARIANT CROSSED MODULES AND COHOMOLOGY OF GROUPS WITH OPERATORS

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ABSTRACT. In this paper we study equivariant crossed modules in its link with strict graded categorical groups. The resulting Schreier theory for equivariant group extensions of the type of an equivariant crossed module generalizes both the theory of group extensions of the type of a crossed module and the one of equivariant group extensions.

1. Introduction

Crossed modules and categorical groups have been used widely and independently, and in various contexts. In 1976 Brown and Spencer [3] showed that crossed modules are defined by \mathcal{G} -groupoids, and hence crossed modules can be studied by means of category theory. The notion of \mathcal{G} -groupoid is also called *strict 2-group* by Baez and Lauda [1], or *strict categorical group* by Joyal and Street [11].

A *categorical group* is a monoidal category in which every morphism is invertible and every object has a weak inverse. (Here, a weak inverse of an object x is an object y such that $x \otimes y$ and $y \otimes x$ are both isomorphic to the unit object.) A *strict categorical group* is a strict monoidal category in which every morphism is invertible and every object has a strict inverse (so that $x \otimes y$ and $y \otimes x$ are actually equal to the unit object).

Graded categorical groups were originally introduced by Fröhlich and Wall in [9]. Cegarra et al. [6] have proved a precise theorem on the homotopy classification of graded categorical groups and their homomorphisms thanks to the 3-dimensional equivariant cohomology group in the sense of [7]. Then, these results were applied to give an appropriate treatment of the equivariant group extensions with a non-abelian kernel in [6].

Brown and Mucuk classified group extensions of the type of a crossed module in [2]. Another generalized version of group extension is the equivariant group extension stated by Cegarra et al. thanks to the graded categorical group theory

Received July 30, 2013; Revised November 21, 2014.

2010 *Mathematics Subject Classification.* 18D10, 18D30, 20E22, 20J06.

Key words and phrases. Γ -crossed module, strict graded categorical group, regular graded monoidal functor, equivariant extension, equivariant cohomology.

(see [6]). One can recognize a generalization of both theories by means of Γ -crossed modules and strict graded categorical groups, which we deal with in this work.

The plan of this paper, briefly, is as follows. After this introductory Section 1, Section 2 is devoted to recalling some fundamental results and notions of reduced graded categorical groups, the obstruction theory of a monoidal Γ -functor and a result on factor sets that will be used in the next section. In Section 3 we introduce the notion of *strict graded categorical group*, and show that any Γ -equivariant crossed module is defined by a strict graded categorical group. Then, we prove that the category $\Gamma\mathbf{Grstr}$ of strict Γ -graded categorical groups and regular Γ -graded monoidal functors is equivalent to the category $\Gamma\mathbf{Cross}$ of Γ -equivariant crossed modules (Theorem 7). A morphism in the category $\Gamma\mathbf{Cross}$ consists of a homomorphism of Γ -equivariant crossed modules, $(f_1, f_0) : \mathcal{M} \rightarrow \mathcal{M}'$, and an element of the group of equivariant 2-cocycles $Z^2(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$. This result contains a classical one, Theorem 1 [3].

Last, Section 4 is dedicated to stating Schreier theory for equivariant group extensions of the type of a Γ -crossed module by means of Γ -graded monoidal functors (Theorem 10, Theorem 12), then the classification theorem of group extensions of the type of a crossed module of Brown and Mucuk (Theorem 5.2 [2]) and that of Γ -group extensions of Cegarra et al. (Theorem 4.1 [6]) are obtained as particular cases.

2. Reduced graded categorical groups

Throughout Γ is a fixed group. Let us recall that a Γ -group Π means a group Π enriched with a left Γ -action by automorphisms, and that a (*left*) Γ -equivariant Π -module is a Γ -module A , that is, an abelian Γ -group, endowed with a Π -module structure such that $\sigma(xa) = (\sigma x)(\sigma a)$ for all $\sigma \in \Gamma$, $x \in \Pi$ and $a \in A$. A Γ -homomorphism $f : \Pi \rightarrow \Pi'$ of Γ -groups is a group homomorphism satisfying $f(\sigma x) = \sigma f(x)$, $\sigma \in \Gamma$, $x \in \Pi$.

We regard the group Γ as a category with one object, say $*$, where the morphisms are elements of Γ and the composition is the group operation. A category \mathbb{P} is Γ -graded if there is a functor $gr : \mathbb{P} \rightarrow \Gamma$. The grading gr is said to be *stable* if for any object $X \in \text{Ob}\mathbb{P}$ and any $\sigma \in \Gamma$ there exists an isomorphism f in \mathbb{P} with domain X and $gr(f) = \sigma$. A Γ -graded monoidal category $\mathbb{P} = (\mathbb{P}, gr, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ consists of:

1. a stable Γ -graded category (\mathbb{P}, gr) , Γ -graded functors $\otimes : \mathbb{P} \times_{\Gamma} \mathbb{P} \rightarrow \mathbb{P}$ and $I : \Gamma \rightarrow \mathbb{P}$,
2. natural isomorphisms of grade 1 $\mathbf{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$, $\mathbf{l}_X : I \otimes X \xrightarrow{\sim} X$, $\mathbf{r}_X : X \otimes I \xrightarrow{\sim} X$ such that, for all $X, Y, Z, T \in \text{Ob}\mathbb{P}$, the following two coherence conditions hold:

$$\begin{aligned} \mathbf{a}_{X,Y,Z} \otimes \mathbf{a}_{X \otimes Y, Z, T} &= (id_X \otimes \mathbf{a}_{Y, Z, T}) \mathbf{a}_{X, Y \otimes Z, T} (\mathbf{a}_{X, Y, Z} \otimes id_T), \\ (id_X \otimes \mathbf{l}_Y) \mathbf{a}_{X, I, Y} &= \mathbf{r}_X \otimes id_Y. \end{aligned}$$

A *graded categorical group* is a graded monoidal category \mathbb{P} in which every object is invertible and every morphism is an isomorphism. In this case, the subcategory $\text{Ker}\mathbb{P}$ consisting of all objects of \mathbb{P} and all morphisms of grade 1 in \mathbb{P} is a categorical group.

If \mathbb{P}, \mathbb{P}' are Γ -monoidal categories, then a *graded monoidal functor* $(F, \tilde{F}, F_*) : \mathbb{P} \rightarrow \mathbb{P}'$ consists of a Γ -graded functor $F : \mathbb{P} \rightarrow \mathbb{P}'$, natural isomorphisms of grade 1 $\tilde{F}_{X,Y} : FX \otimes FY \rightarrow F(X \otimes Y)$, and an isomorphism of grade 1 $F_* : I' \rightarrow FI$, such that, for all $X, Y, Z \in \text{Ob}\mathbb{P}$, the following coherence conditions hold:

$$\tilde{F}_{X,Y \otimes Z}(id_{FX} \otimes \tilde{F}_{Y,Z})\mathbf{a}_{FX,FY,FZ} = F(\mathbf{a}_{X,Y,Z})\tilde{F}_{X \otimes Y,Z}(\tilde{F}_{X,Y} \otimes id_{FZ}),$$

$$F(\mathbf{r}_X)\tilde{F}_{X,I}(id_{FX} \otimes F_*) = \mathbf{r}_{FX}, \quad F(\mathbf{l}_X)\tilde{F}_{I,X}(F_* \otimes id_{FX}) = \mathbf{l}_{FX}.$$

Let (F, \tilde{F}, F_*) , (F', \tilde{F}', F'_*) be two Γ -graded monoidal functors. A *graded monoidal natural equivalence* $\theta : F \xrightarrow{\sim} F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X : FX \rightarrow F'X$ are of grade 1, and for all $X, Y \in \text{Ob}\mathbb{P}$, the following coherence conditions hold:

$$(1) \quad \tilde{F}'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X \otimes Y}\tilde{F}_{X,Y}, \quad \theta_I F_* = F'_*.$$

The authors of [6] showed that any Γ -graded categorical group $\mathbb{P} = (\mathbb{P}, gr, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ determines a triple (Π, A, h) , where

- the set $\Pi = \pi_0\mathbb{P}$ of 1-isomorphism classes of the objects in \mathbb{P} is a Γ -group,
- the set $A = \pi_1\mathbb{P}$ of 1-automorphisms of the unit object I is a Γ -equivariant Π -module,
- the third invariant is an equivariant cohomology class $\bar{h} \in H^3_\Gamma(\Pi, A)$ in the sense of [7].

Based on the data: a Γ -group Π , a Γ -equivariant Π -module A and an element $h \in Z^3(\Pi, A)$, the Γ -graded categorical group $\int_\Gamma(\Pi, A, h)$ was constructed in [6] as follows. Objects of $\int_\Gamma(\Pi, A, h)$ are elements $x \in \Pi$ and their morphisms are pairs $(a, \sigma) : x \rightarrow y$ consisting of an element $a \in A$ and $\sigma \in \Gamma$ such that $\sigma x = y$.

The composition $(x \xrightarrow{(a,\sigma)} y \xrightarrow{(b,\tau)} z)$ is given by

$$(b, \tau)(a, \sigma) = (b + \tau a + h(x, \tau, \sigma), \tau\sigma).$$

The graded tensor product of two morphisms is given by

$$(x \xrightarrow{(a,\sigma)} y) \otimes (x' \xrightarrow{(b,\sigma')} y') = (xx' \xrightarrow{(a+yb+h(x,x',\sigma),\sigma)} yy').$$

The unit constrains are identities, $\mathbf{l}_x = (0, 1) = \mathbf{r}_x : x \rightarrow x$. The associativity isomorphisms are

$$\mathbf{a}_{x,y,z} = (h(x, y, z), 1) : (xy)z \rightarrow x(yz).$$

The stable Γ -grading is $gr(a, \sigma) = \sigma$. The unit graded functor $I : \Gamma \rightarrow \int_{\Gamma}(\Pi, A, h)$ is given by

$$I(* \xrightarrow{\sigma} *) = (1 \xrightarrow{(0, \sigma)} 1).$$

In the case where Π, A, \bar{h} are three invariants of a Γ -graded categorical group \mathbb{P} , then $\int_{\Gamma}(\Pi, A, h)$ is graded monoidally equivalent to \mathbb{P} . We call $\int_{\Gamma}(\Pi, A, h)$ a *reduction* of the Γ -graded categorical group \mathbb{P} , and simply denoted by $\mathcal{S}_{\mathbb{P}}$.

A Γ -functor $F : \int_{\Gamma}(\Pi, A, h) \rightarrow \int_{\Gamma}(\Pi', A', h')$ is of *type* (φ, f) if

$$F(x) = \varphi(x), \quad F(a, \sigma) = (f(a), \sigma), \quad x \in \Pi, \quad a \in A, \quad \sigma \in \Gamma,$$

where $\varphi : \Pi \rightarrow \Pi'$ is a Γ -homomorphism (so that A' becomes a Γ -equivariant Π -module via φ) and $f : A \rightarrow A'$ is a homomorphism of Γ -equivariant Π -modules (that is, a homomorphism which is both of Γ - and Π -modules). In this case, we call (φ, f) a *pair of Γ -homomorphisms*, and call the function

$$(2) \quad \xi = \varphi^* h' - f_* h$$

an *obstruction* of the Γ -functor F , where φ^*, f_* are canonical homomorphisms

$$Z_{\Gamma}^3(\Pi, A) \xrightarrow{f_*} Z_{\Gamma}^3(\Pi, A') \xleftarrow{\varphi^*} Z_{\Gamma}^3(\Pi', A').$$

From Propositions 4 and 5 [15] on the monoidal functors of type (φ, f) , one obtains the following proposition with some appropriate modifications.

Proposition 1. *Let \mathbb{P} and \mathbb{P}' be Γ -graded categorical groups, $\mathcal{S}_{\mathbb{P}}$ and $\mathcal{S}_{\mathbb{P}'}$ be their reductions, respectively.*

- i) *Every Γ -graded monoidal functor $(F, \tilde{F}) : \mathbb{P} \rightarrow \mathbb{P}'$ induces one $S_F : \mathcal{S}_{\mathbb{P}} \rightarrow \mathcal{S}_{\mathbb{P}'}$.*
- ii) *Every Γ -graded monoidal functor $\int_{\Gamma}(\Pi, A, h) \rightarrow \int_{\Gamma}(\Pi', A', h')$ is a Γ -functor of type (φ, f) .*

The following proposition is a part of Theorem 3.2 [6].

Proposition 2. *Let \mathbb{P}, \mathbb{P}' be two Γ -graded categorical groups. Any Γ -graded functor $(F, \tilde{F}) : \mathbb{P} \rightarrow \mathbb{P}'$ induces a pair of Γ -homomorphisms (φ, f) , which realizes a Γ -graded monoidal functor, if and only if its obstruction ξ vanishes in $H_{\Gamma}^3(\pi_0 \mathbb{P}, \pi_1 \mathbb{P}')$. Then, there is a bijection*

$$\text{Hom}_{(\varphi, f)}[\mathbb{P}, \mathbb{P}'] \leftrightarrow H_{\Gamma}^2(\pi_0 \mathbb{P}, \pi_1 \mathbb{P}'),$$

where $\text{Hom}_{(\varphi, f)}[\mathbb{P}, \mathbb{P}']$ is the set of all homotopy classes of monoidal Γ -functors from \mathbb{P} to \mathbb{P}' inducing the pair of Γ -homomorphisms (φ, f) .

- *Definition of factor sets with coefficients in a categorical group.*

The notion of factor set in the Schreier-Eilenberg-Mac Lane theory for group extensions has been raised to categorical level by Grothendieck [10] and was also applied in [4], [5], [17]. In this paper we use this notion to define a *strict* graded categorical group.

Definition. A factor set \mathcal{F} on Γ with coefficients in a categorical group \mathbb{G} (or a pseudo-functor from Γ to the category of categorical groups in the sense of Grothendieck [10]) consists of a family of monoidal autoequivalences $F^\sigma : \mathbb{G} \rightarrow \mathbb{G}$, $\sigma \in \Gamma$, and isomorphisms between monoidal functors $\theta^{\sigma,\tau} : F^\sigma F^\tau \rightarrow F^{\sigma\tau}$, $\sigma, \tau \in \Gamma$, satisfying the conditions:

- i) $F^1 = id_{\mathbb{G}}$,
- ii) $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}$, $\sigma \in \Gamma$,
- iii) $\theta^{\sigma\tau,\gamma} \circ \theta^{\sigma,\tau} F^\gamma = \theta^{\sigma,\tau\gamma} \circ F^\sigma \theta^{\tau,\gamma}$ for all $\sigma, \tau, \gamma \in \Gamma$.

We write $\mathcal{F} = (\mathbb{G}, F^\sigma, \theta^{\sigma,\tau})$, or simply (F, θ) .

The following lemma comes from an analogous result on graded monoidal categories [5] or a part of Theorem 1.2 [18]. We sketch the proof since we need some of its details.

Lemma 3. Each Γ -graded categorical group (\mathbb{P}, gr) determines a factor set \mathcal{F} on Γ with coefficients in the categorical group $\text{Ker } \mathbb{P}$.

Proof. For $\sigma \in \Gamma$, we define a monoidal autoequivalence $F^\sigma = (F^\sigma, \tilde{F}^\sigma) : \text{Ker } \mathbb{P} \rightarrow \text{Ker } \mathbb{P}$ as follows: for each $X \in \text{Ker } \mathbb{P}$, since the grading gr is stable, there exists an isomorphism $\Upsilon_X^\sigma : X \xrightarrow{\sim} F^\sigma X$, where $F^\sigma X \in \text{Ker } \mathbb{P}$, and $gr(\Upsilon_X^\sigma) = \sigma$. In particular, when $\sigma = 1$ we take $F^1 X = X$ and $\Upsilon_X^1 = id_X$. A morphism $f : X \rightarrow Y$ of grade 1 in $\text{Ker } \mathbb{P}$ is carried to the unique morphism $F^\sigma(f)$ in $\text{Ker } \mathbb{P}$ determined by

$$F^\sigma(f) = \Upsilon_Y^\sigma \circ f \circ (\Upsilon_X^\sigma)^{-1}.$$

The natural isomorphism $\tilde{F}_{X,Y}^\sigma : F^\sigma X \otimes F^\sigma Y \xrightarrow{\sim} F^\sigma(X \otimes Y)$ is determined by

$$\tilde{F}_{X,Y}^\sigma = (\Upsilon_X^\sigma \otimes \Upsilon_Y^\sigma) \circ (\Upsilon_{X \otimes Y}^\sigma)^{-1}.$$

Furthermore, for each pair $\sigma, \tau \in \Gamma$ there is an isomorphism of monoidal functors $\theta^{\sigma,\tau} : F^\sigma F^\tau \xrightarrow{\sim} F^{\sigma\tau}$, with $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}$, which is defined, for any $X \in \text{Ob } \mathbb{P}$, by

$$\theta_X^{\sigma,\tau} = \Upsilon_{F^\tau X}^\sigma \circ \Upsilon_X^\tau \circ (\Upsilon_X^{\sigma\tau})^{-1}.$$

The pair (F, θ) constructed as above is a factor set. □

3. Strict graded categorical groups and Γ -crossed modules

The objective of this paper is to extend the results on crossed modules and on equivariant group extensions. The notion of Γ -crossed module is a generalization of that of crossed module of groups of Whitehead [19]. First, observe that if B is a Γ -group, then the group $\text{Aut } B$ of automorphisms of B is also a Γ -group with the action

$$(\sigma f)(b) = \sigma(f(\sigma^{-1}b)), \quad b \in B, \quad f \in \text{Aut } B.$$

Then, the homomorphism $\mu : B \rightarrow \text{Aut } B, b \mapsto \mu_b$ (μ_b is the inner automorphism of B given by conjugation with b) is a homomorphism of Γ -groups.

Indeed, for all $\sigma \in \Gamma, a, b \in B$, one has

$$\mu_{\sigma b}(a) = \sigma b + a - \sigma b = \sigma(b + \sigma^{-1}a - b) = \sigma(\mu_b(\sigma^{-1}a)) = (\sigma\mu_b)(a).$$

Definition. Let B, D be Γ -groups. A Γ -crossed module is a quadruple (B, D, d, ϑ) , where $d : B \rightarrow D, \vartheta : D \rightarrow \text{Aut} B$ are Γ -homomorphisms satisfying the following conditions:

- $C_1.$ $\vartheta d = \mu,$
- $C_2.$ $d(\vartheta_x(b)) = \mu_x(d(b)),$
- $C_3.$ $\sigma(\vartheta_x(b)) = \vartheta_{\sigma x}(\sigma b),$

where $\sigma \in \Gamma, x \in D, b \in B, \mu_x$ is the inner automorphism given by conjugation with x .

A Γ -crossed module is also called an Γ -equivariant crossed module by Noohi [12].

Examples. Standard examples of Γ -crossed modules are:

1. (B, D, i, μ) , where $i : B \rightarrow D$ is an inclusion Γ -homomorphism of a normal subgroup.
2. $(B, D, \mathbf{0}, \vartheta)$, where B is a D -module, $\mathbf{0} : B \rightarrow D$ is the zero Γ -homomorphism, and ϑ is the module action.
3. $(B, \text{Aut } B, \mu, 0)$, where $\mu : B \rightarrow \text{Aut } B$ is the Γ -homomorphism of any Γ -group B which is given by conjugation.
4. (B, D, p, ϑ) , where $p : B \rightarrow D$ is a Γ -surjective such that $\text{Ker } p \subset Z(B)$, ϑ is given by conjugation.

Note on notations. For convenience, we denote by the addition for the operation in B and by the multiplication for that in D . In this paper the Γ -crossed module (B, D, d, ϑ) is sometimes denoted by $B \xrightarrow{d} D$, or $B \rightarrow D$. In this section notations $\mathcal{M}, \mathcal{M}'$ refer to the Γ -crossed modules $(B, D, d, \vartheta), (B', D', d', \vartheta')$, respectively.

The following properties follow from the definition of Γ -crossed module.

Proposition 4. *Let \mathcal{M} be a Γ -crossed module.*

- i) $\text{Ker} d$ is a Γ -subgroup in $Z(B)$.
- ii) $\text{Im} d$ is both a normal subgroup in D and a Γ -group.
- iii) The Γ -homomorphism ϑ induces one $\varphi : D \rightarrow \text{Aut}(\text{Ker} d)$ by

$$\varphi_x = \vartheta_x|_{\text{Ker} d}.$$

- iv) $\text{Ker} d$ is a left Γ -equivariant $\text{Coker} d$ -module under the actions

$$sa = \varphi_x(a), \quad \sigma s = [\sigma x], \quad a \in \text{Ker} d, \quad x \in s \in \text{Coker} d.$$

The groups $\text{Ker} d$ and $\text{Coker} d$ are denoted by $\pi_1 \mathcal{M}$ and $\pi_0 \mathcal{M}$, respectively.

It is well known that each crossed module of groups can be seen as a strict categorical group (see [3], [11, Remark 3.1]). Crossed modules of groups can be enriched in some ways to become, for example, *crossed bimodules over rings*, or *equivariant crossed modules*. In the former case, each crossed bimodule can be seen as a strict Ann-category [13]. In the later case, we shall show that each

crossed module of Γ -groups can be identified with a *strict Γ -graded categorical group*. We now state this definition.

Recall that if (F, \tilde{F}, F_*) is a monoidal functor between categorical groups, then the isomorphism $F_* : I' \rightarrow FI$ can be deduced from F and \tilde{F} , so we can omit F_* when not necessary. A monoidal functor $(F, \tilde{F}) : \mathbb{G} \rightarrow \mathbb{G}'$ between two categorical groups is termed *regular* if

$$F(x) \otimes F(y) = F(x \otimes y), \quad F(b) \otimes F(c) = F(b \otimes c)$$

for all $x, y \in \text{Ob}\mathbb{G}$, $b, c \in \text{Mor}\mathbb{G}$. A factor set (F, θ) on Γ with coefficients in a categorical group \mathbb{G} is *regular* if $\theta^{\sigma, \tau} = id$ and F^σ is a regular monoidal functor, for all $\sigma \in \Gamma$.

Definition. A graded categorical group (\mathbb{P}, gr) is said to be *strict* if

- i) $\text{Ker } \mathbb{P}$ is a strict categorical group,
- ii) \mathbb{P} induces a regular factor set (F, θ) on Γ with coefficients in the categorical group $\text{Ker } \mathbb{P}$.

Equivalently, a graded categorical group (\mathbb{P}, gr) is *strict* if it is a Γ -graded extension of a strict categorical group by a regular factor set.

For each Γ -crossed module \mathcal{M} , we construct a strict Γ -graded categorical group $\mathbb{P}_{\mathcal{M}} := \mathbb{P}$ associated to \mathcal{M} .

Objects of \mathbb{P} are elements of the group D , a σ -morphism $x \rightarrow y$ is a pair (b, σ) , where $b \in B, \sigma \in \Gamma$, such that $\sigma x = d(b)y$. The composition of two morphisms is defined by

$$(3) \quad (x \xrightarrow{(b, \sigma)} y \xrightarrow{(c, \tau)} z) = (x \xrightarrow{(\tau b + c, \tau \sigma)} z).$$

This composition is associative and unitary since B is a Γ -group.

For any morphism (b, σ) in \mathbb{P} , one has

$$(b, \sigma)^{-1} = (-\sigma^{-1}b, \sigma^{-1}),$$

so that \mathbb{P} is a groupoid. The tensor operation on objects is given by the multiplication in the group D , and for two morphisms $(x \xrightarrow{(b, \sigma)} y), (x' \xrightarrow{(c, \sigma)} y')$, then

$$(4) \quad (x \xrightarrow{(b, \sigma)} y) \otimes (x' \xrightarrow{(c, \sigma)} y') = (xx' \xrightarrow{(b + \vartheta_y c, \sigma)} yy'),$$

which is a functor thanks to the compatibility of the action ϑ with the Γ -action and conditions of the definition of Γ -crossed module, as below.

For morphisms $(x \xrightarrow{(b, \sigma)} y \xrightarrow{(c, \tau)} x), (x' \xrightarrow{(b', \sigma)} y' \xrightarrow{(c', \tau)} z')$ in \mathbb{P} ,

$$\begin{aligned} & (x \xrightarrow{(b, \sigma)} y \xrightarrow{(c, \tau)} x) \otimes (x' \xrightarrow{(b', \sigma)} y' \xrightarrow{(c', \tau)} z') \\ \stackrel{(3)}{=} & (x \xrightarrow{(\tau b + c, \tau \sigma)} z) \otimes (x' \xrightarrow{(\tau b' + c', \tau \sigma)} z') \\ \stackrel{(4)}{=} & (xx' \xrightarrow{(\tau b + c + \vartheta_z(\tau b' + c'), \tau \sigma)} zz'), \end{aligned}$$

$$\begin{aligned}
 & [(x \xrightarrow{(b,\sigma)} y) \otimes (x' \xrightarrow{(b',\sigma)} y')] \circ [(y \xrightarrow{(c,\tau)} z) \otimes (y' \xrightarrow{(c',\tau)} z')] \\
 \stackrel{(4)}{=} & (xx' \xrightarrow{(b+\vartheta_y b',\sigma)} yy') \circ (yy' \xrightarrow{(c+\vartheta_z c',\tau)} zz') \\
 \stackrel{(3)}{=} & (xx' \xrightarrow{(\tau(b+\vartheta_y b')+c+\vartheta_z c',\tau\sigma)} zz').
 \end{aligned}$$

The fact that \otimes is a functor is equivalent to

$$\tau b + c + \vartheta_z(\tau b' + c') = \tau(b + \vartheta_y b') + c + \vartheta_z c'.$$

This follows from

$$\tau(\vartheta_y b') \stackrel{(C_3)}{=} \vartheta_{\tau y}(\tau b') = \vartheta_{(dc)z}(\tau b') \stackrel{(C_1)}{=} \mu_c(\vartheta_z(\tau b')).$$

The associativity and unit constraints with respect to tensor product are strict. The graded functor is defined by $gr(b, \sigma) = \sigma$, and the unit graded functor $I : \Gamma \rightarrow \mathbb{P}$ by

$$I(* \xrightarrow{\sigma} *) = (1 \xrightarrow{(0,\sigma)} 1).$$

Since $\text{Ob}\mathbb{P} = D$ is a group and $x \otimes y = xy$, every object of \mathbb{P} is invertible, whence $\text{Ker}\mathbb{P}$ is a strict categorical group.

We next show that \mathbb{P} induces a regular factor set (F, θ) on Γ with coefficients in $\text{Ker}\mathbb{P}$. For each $x \in D, \sigma \in \Gamma$, we set $F^\sigma(x) = \sigma x, \Upsilon_x^\sigma = (x \xrightarrow{(0,\sigma)} \sigma x)$. Then, according to the proof of Lemma 3, we have $F^\sigma(b, 1) = (\sigma b, 1)$ and $\theta^{\sigma,\tau} = id$. Now, it follows from the Γ -crossed module structure of $B \rightarrow D$ that F^σ is a regular monoidal functor.

Thus, \mathbb{P} is a strict Γ -graded categorical group.

Conversely, for each strict Γ -graded categorical group \mathbb{P} , we construct a Γ -crossed module associated to \mathbb{P} as follows.

Set

$$D = \text{Ob}\mathbb{P}, \quad B = \{x \xrightarrow{b} 1 \mid x \in D, gr(b) = 1\}.$$

The operations in D and in B are given by

$$xy = x \otimes y, \quad b + c = b \otimes c,$$

respectively. Then, D becomes a group in which the unit is 1, the inverse of x is x^{-1} ($x \otimes x^{-1} = 1$). B is a group in which the zero is the morphism $(1 \xrightarrow{id_1} 1)$ and the negative of $(x \xrightarrow{b} 1)$ is the morphism $(x^{-1} \xrightarrow{\bar{b}} 1)(b \otimes \bar{b} = id_1)$.

By the definition of \mathbb{P} , its kernel $\text{Ker}\mathbb{P}$ is a strict categorical group and \mathbb{P} has a regular factor set (F, θ) . Thus, D, B are Γ -groups in which the actions are respectively defined by

$$\sigma x = F^\sigma(x), \quad x \in D, \sigma \in \Gamma,$$

$$\sigma b = F^\sigma(b), \quad b \in B.$$

The correspondences $d : B \rightarrow D$ and $\vartheta : D \rightarrow \text{Aut } B$ are given by

$$d(x \xrightarrow{b} 1) = x,$$

$$\vartheta_y(x \xrightarrow{b} 1) = (yxy^{-1} \xrightarrow{id_y + b + id_{y^{-1}}} 1),$$

respectively. Since B, D are Γ -groups, it is easy to see that d, ϑ are Γ -homomorphisms.

Definition. A homomorphism $(f_1, f_0) : \mathcal{M} \rightarrow \mathcal{M}'$ of Γ -crossed modules consists of Γ -homomorphisms $f_1 : B \rightarrow B', f_0 : D \rightarrow D'$ satisfying

- $H_1.$ $f_0 d = d' f_1,$
- $H_2.$ $f_1(\vartheta_x b) = \vartheta'_{f_0(x)} f_1(b)$ for all $x \in D, b \in B.$

The following lemmas state the relation between homomorphisms of Γ -crossed modules and graded monoidal functors between corresponding associated graded categorical groups. Observe that a morphism $(x \xrightarrow{(b,\sigma)} y)$ in $\mathbb{P}_{B \rightarrow D}$ can be written as

$$x \xrightarrow{(0,\sigma)} \sigma x \xrightarrow{(b,1)} y,$$

and a Γ -graded monoidal functor $(F, \tilde{F}) : \mathbb{P}_{\mathcal{M}} \rightarrow \mathbb{P}_{\mathcal{M}'}$ defines a function $f : D^2 \cup D \times \Gamma \rightarrow B$ by

$$(5) \quad (f(x, y), 1) = \tilde{F}_{x,y}, \quad (f(x, \sigma), \sigma) = F(x \xrightarrow{(0,\sigma)} \sigma x)$$

Lemma 5. Let $(f_1, f_0) : \mathcal{M} \rightarrow \mathcal{M}'$ be a homomorphism of Γ -crossed modules. Then, there exists a Γ -graded monoidal functor $(F, \tilde{F}) : \mathbb{P}_{\mathcal{M}} \rightarrow \mathbb{P}_{\mathcal{M}'}$ defined by $F(x) = f_0(x), F(b, 1) = (f_1(b), 1),$ if and only if $f = p^* \varphi,$ where $\varphi \in Z^2_{\Gamma}(\text{Coker } d, \text{Ker } d'), p : D \rightarrow \text{Coker } d$ is a canonical projection.

Proof. Since f_0 is a homomorphism and $Fx = f_0(x), \tilde{F}_{x,y} : Fx Fy \rightarrow F(xy)$ is a morphism of grade 1 in \mathbb{P}' if and only $df(x, y) = 1',$ or $f(x, y) \in \text{Ker } d' \subset Z(B').$

Also, since f_0 is a Γ -homomorphism, $(Fx \xrightarrow{(f(x,\sigma),\sigma)} F\sigma x)$ is a morphism of grade σ in \mathbb{P}' if and only if $df(x, \sigma) = 1',$ or $f(x, \sigma) \in \text{Ker } d' \subset Z(B').$ In particular, when $\sigma = 1$ then $f(x, 1_{\Gamma}) = f_1(0) = 0.$

The fact that f_1 is a group homomorphism is equivalent to the condition of F preserving composition of morphisms of grade 1. The condition of F preserving the composition of morphisms of form $(0, \sigma)$ is equivalent to

$$(6) \quad \tau f(x, \sigma) + f(\sigma x, \tau) = f(x, \tau \sigma).$$

- The condition of $\tilde{F}_{x,y}$ being natural isomorphisms.
- For morphisms of grade 1, consider the diagram

$$\begin{array}{ccc} F(x)F(y) & \xrightarrow{\tilde{F}_{x,y}} & F(xy) \\ \downarrow F(b,1) \otimes F(c,1) & & \downarrow F[(b,1) \otimes (c,1)] \\ F(x')F(y') & \xrightarrow{\tilde{F}_{x',y'}} & F(x'y'). \end{array}$$

Since the homomorphisms f_0, f_1 satisfy the condition H_2 , the following equation holds:

$$F(b, 1) \otimes F(c, 1) = F[(b, 1) \otimes (c, 1)].$$

Then, since $f(x, y), f(x', y') \in Z(B')$, the above diagram commutes if and only if

$$f(x, y) = f(x', y').$$

So, \tilde{F} defines a function $\varphi : \text{Coker}^2 d \rightarrow \text{Ker } d'$,

$$\varphi(r, s) = f(x, y), \quad r = px, \quad s = py,$$

where $p : D \rightarrow \text{Coker } d$ is a canonical projection.

- For morphisms of form $(0, \sigma)$, the diagram

$$\begin{array}{ccc} F(x)F(y) & \xrightarrow{\tilde{F}_{x,y}} & F(xy) \\ \downarrow F(0,\sigma) \otimes F(0,\sigma) & & \downarrow F[(0,\sigma) \otimes (0,\sigma)] \\ F(\sigma x)F(\sigma y) & \xrightarrow{\tilde{F}_{\sigma x, \sigma y}} & F(\sigma x)(\sigma y) = F\sigma(xy) \end{array}$$

commutes if and only if

$$\sigma f(x, y) - f(\sigma x, \sigma y) = f(x, \sigma) + \vartheta'_{F(\sigma x)} f(y, \sigma) - f(xy, \sigma),$$

or

$$(7) \quad \sigma f(x, y) - f(\sigma x, \sigma y) = f(x, \sigma) + (\sigma x)f(y, \sigma) - f(xy, \sigma).$$

• The commutativity of diagram

$$\begin{array}{ccc} Fx & \xrightarrow{(f(x,\sigma),\sigma)} & F\sigma x \\ \downarrow F(b,1) & & \downarrow F(\sigma b,1) \\ Fy & \xrightarrow{(f(y,\sigma),\sigma)} & F\sigma y \end{array}$$

leads to

$$f(x, \sigma) + f_1(\sigma b) = \sigma f_1(b) + f(y, \sigma).$$

Since f_1 is a Γ -homomorphism, it follows that $f(x, \sigma) = f(y, \sigma)$. This gives a function $\varphi : \text{Coker } d \times \Gamma \rightarrow \text{Ker } d'$

$$\varphi(r, \sigma) = f(x, \sigma), \quad r = px.$$

Therefore, one obtains a function

$$\varphi : \text{Coker}^2 d \cup \text{Coker } d \times \Gamma \rightarrow \text{Ker } d'$$

which is normalized in the sense that

$$\varphi(1, s) = \varphi(r, 1) = 0 = \varphi(r, 1_\Gamma).$$

The first two equations hold since $F(1) = 1'$ and (F, \tilde{F}) is compatible with the unit constraints. The last one holds since $f(x, 1_\Gamma) = 0$.

The compatibility of (F, \tilde{F}) with the associativity constraints implies

$$(8) \quad \vartheta'_{F_x}(f(y, z)) + f(x, yz) = f(x, y) + f(xy, z).$$

It follows from the relations (6)-(8) that $\varphi \in Z^2_\Gamma(\text{Coker } d, \text{Ker } d')$. □

Note that the strict Γ -graded categorical group \mathbb{P} induces a Γ -action on the group D of objects and on the group B of morphisms of grade 1, we state the following definition.

Definition. A Γ -graded monoidal functor $(F, \tilde{F}) : \mathbb{P} \rightarrow \mathbb{P}'$ between two strict Γ -graded categorical groups is called *regular* if:

- $S_1.$ $F(x \otimes y) = F(x) \otimes F(y)$,
- $S_2.$ $F(\sigma x) = \sigma F(x)$,
- $S_3.$ $F(\sigma b) = \sigma F(b)$,
- $S_4.$ $F(b) \otimes F(c) = F(b \otimes c)$

for $x, y \in \text{Ob } \mathbb{P}$, and b, c are morphisms of grade 1 in \mathbb{P} .

The Γ -graded monoidal functor mentioned in Lemma 5 is regular.

Thanks to Lemma 5, one can define the category \mathbf{rCross} whose objects are Γ -crossed modules and whose morphisms are triples (f_1, f_0, φ) , where $(f_1, f_0) : \mathcal{M} \rightarrow \mathcal{M}'$ is a homomorphism of Γ -crossed modules and $\varphi \in Z^2_\Gamma(\text{Coker } d, \text{Ker } d')$. The composition with the morphism $(f'_1, f'_0, \varphi') : \mathcal{M}' \rightarrow \mathcal{M}''$ is given by

$$(f'_1, f'_0, \varphi') \circ (f_1, f_0, \varphi) = (f'_1 f_1, f'_0 f_0, (f'_1)_* \varphi + (f_0)^*(\varphi')).$$

Lemma 6. Let \mathbb{P} and \mathbb{P}' be corresponding strict Γ -graded categorical groups associated to Γ -crossed modules \mathcal{M} and \mathcal{M}' , and let $(F, \tilde{F}) : \mathbb{P} \rightarrow \mathbb{P}'$ be a regular Γ -graded monoidal functor. Then, the triple (f_1, f_0, φ) , where

- i) $f_0(x) = F(x)$, $(f_1(b), 1) = F(b, 1)$, $\sigma \in \Gamma$, $b \in B$, $x, y \in D$,
- ii) $p^* \varphi = f$ for f is given by (5),

is a morphism in the category \mathbf{rCross} .

Proof. By the condition S_1 , f_0 is a group homomorphism, and by the condition S_2 , f_0 is a Γ -homomorphism. Since F preserves the composition of morphisms of grade 1, f_1 is a group homomorphism. Moreover, f_1 is a Γ -homomorphism thanks to the condition S_3 . Each element $b \in B$ can be seen as a morphism $(db \xrightarrow{(b,1)} 1)$ in \mathbb{P} , and hence $(f_0(db) \xrightarrow{(f_1(b),1)} 1')$ is a morphism in \mathbb{P}' , that means H_1 holds:

$$f_0(db) = d'(f_1(b)).$$

By the condition S_4 and the fact that f_1 is a homomorphism, H_2 is satisfied:

$$f_1(\vartheta_y c) = \vartheta'_{f_0(y)} f_1(c).$$

Thus, (f_1, f_0) is a homomorphism of Γ -crossed modules. By Lemma 5, the function f determines a function $\varphi \in Z^2_\Gamma(\text{Coker } d, \text{Ker } d')$, where $f = p^* \varphi$, $p : D \rightarrow \text{Coker } d$ is the canonical projection. Therefore, (f_1, f_0, φ) is a morphism in \mathbf{rCross} . □

Denote by

$$\mathbf{rGrstr}$$

the category of strict Γ -graded categorical groups and regular Γ -graded monoidal functors, we have the following result.

Theorem 7 (Classification Theorem). *There exists an equivalence*

$$\begin{aligned} \Phi : \mathbf{rCross} &\rightarrow \mathbf{rGrstr}, \\ (B \rightarrow D) &\mapsto \mathbb{P}_{B \rightarrow D} \\ (f_1, f_0, \varphi) &\mapsto (F, \tilde{F}) \end{aligned}$$

where $F(x) = f_0(x)$, $F(b, 1) = (f_1(b), 1)$, and

$$F(x \xrightarrow{(0, \sigma)} \sigma x) = (\varphi(px, \sigma), \sigma), \quad \tilde{F}_{x,y} = (\varphi(px, py), 1)$$

for $x, y \in D$, $b \in B$, $\sigma \in \Gamma$.

Proof. Let \mathbb{P}, \mathbb{P}' be the Γ -graded categorical groups associated to Γ -crossed modules $\mathcal{M}, \mathcal{M}'$, respectively. By Lemma 5, the correspondence $(f_1, f_0, \varphi) \mapsto (F, \tilde{F})$ defines an injection on the homsets,

$$\Phi : \text{Hom}_{\mathbf{rCross}}(\mathcal{M}, \mathcal{M}') \rightarrow \text{Hom}_{\mathbf{rGrstr}}(\mathbb{P}_{\mathcal{M}}, \mathbb{P}_{\mathcal{M}'}).$$

By Lemma 6, Φ is surjective.

If \mathbb{P} is a strict Γ -graded categorical group and $\mathcal{M}_{\mathbb{P}}$ is its associated Γ -crossed module, then $\Phi(\mathcal{M}_{\mathbb{P}}) = \mathbb{P}$ (rather than an isomorphism). Thus, Φ is an equivalence. \square

When $\Gamma = \mathbf{1}$, the trivial group, then the category $\mathbf{1Cross} = \mathbf{Cross}$ has crossed modules as objects, the morphisms are triples (f_1, f_0, φ) , where $(f_1, f_0) : \mathcal{M} \rightarrow \mathcal{M}'$ is a homomorphism of crossed modules and $\varphi \in Z^2(\text{Coker}d, \text{Ker}d')$, and the category $\mathbf{1Grstr} = \mathbf{Grstr}$ has strict categorical groups as objects, the morphisms are regular monoidal functor. Thus, one obtains the following result.

Corollary 8 (Theorem 5 [14]). *There exists an equivalence*

$$\begin{aligned} \Phi : \mathbf{Cross} &\rightarrow \mathbf{Grstr}, \\ (B \rightarrow D) &\mapsto \mathbb{P}_{B \rightarrow D} \\ (f_1, f_0, \varphi) &\mapsto (F, \tilde{F}) \end{aligned}$$

where $F(x) = f_0(x)$, $F(b) = f_1(b)$, $\tilde{F}_{x,y} = \varphi(\bar{x}, \bar{y})$ for $x, y \in D$, $b \in B$.

The two categories mentioned in Corollary 8 have two corresponding subcategories:

- The category 2-Gp of \mathcal{G} -groupoids (by Brown and Spencer [3]) is a subcategory of the category \mathbf{Grstr} in which the morphisms consist of monoidal functors (F, \tilde{F}) with $\tilde{F} = id$.

- The category $\mathbf{CrossMd}$ of crossed modules (in the sense of usual) is the subcategory of the category \mathbf{Cross} in which the morphisms consist of triples (f_1, f_0, φ) with $\varphi = 0$.

The categories **2-Gp** and **CrossMd** are equivalent via a functor which is a restriction of Φ . Thus, Corollary 8 contains Theorem 1 [3].

4. Equivariant group extensions, Γ -crossed modules and equivariant group cohomology

In this section we develop a theory of equivariant group extensions of the type of a Γ -crossed module which extends both group extension theory of the type of a crossed module [2, 8, 16] and equivariant group extension theory [6].

Definition. Let $B \xrightarrow{d} D$ be a Γ -crossed module and Q be a Γ -group. An *equivariant group extension* of B by Q of type $B \xrightarrow{d} D$ is a diagram of Γ -homomorphisms

$$\mathcal{E} : \begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 1, \\ & & \parallel & & \downarrow \varepsilon & & \\ & & B & \xrightarrow{d} & D & & \end{array}$$

where the top row is exact, the family (B, E, j, ϑ^0) is a Γ -crossed module in which ϑ^0 is given by conjugation, and (id, ε) is a homomorphism of Γ -crossed modules.

Two equivariant extensions of B by Q of type $B \xrightarrow{d} D$ are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 1, & E & \xrightarrow{\varepsilon} & D \\ & & \parallel & & \downarrow \alpha & & \parallel & & & \\ 0 & \longrightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & Q \longrightarrow 1, & E' & \xrightarrow{\varepsilon'} & D \end{array}$$

such that $\varepsilon'\alpha = \varepsilon$. Obviously, α is a Γ -isomorphism.

In the diagram

$$(9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 1, \\ & & \parallel & & \downarrow \varepsilon & & \downarrow \psi \\ & & B & \xrightarrow{d} & D & \xrightarrow{q} & \text{Coker}d \end{array}$$

where q is a canonical Γ -homomorphism, since the top row is exact and $q \circ \varepsilon \circ j = q \circ d = 0$, there is a Γ -homomorphism $\psi : Q \rightarrow \text{Coker}d$ such that the right hand side square commutes. Moreover, ψ is dependent only on the equivalence class of the extension \mathcal{E} , and we say that the extension \mathcal{E} *induces* ψ . The set of equivalence classes of equivariant extensions of B by Q of type $B \rightarrow D$ inducing $\psi : Q \rightarrow \text{Coker}d$ is denoted by

$$\text{Ext}_{B \rightarrow D}^\Gamma(Q, B, \psi).$$

Now, in order to study this set we apply the obstruction theory to Γ -graded monoidal functors between strict Γ -graded categorical groups ${}_{\Gamma}\text{Dis } Q$ and $\mathbb{P}_{B \rightarrow D}$, where the *discrete* Γ -graded categorical group ${}_{\Gamma}\text{Dis } Q$ is defined by

$${}_{\Gamma}\text{Dis } Q = \int_{\Gamma} (Q, 0, 0).$$

This is just the strict Γ -graded categorical group associated to the Γ -crossed module $(0, Q, 0, 0)$ (see Section 3). Thus, the objects of ${}_{\Gamma}\text{Dis } Q$ are the elements of Q and its morphisms $\sigma : x \rightarrow y$ are the elements $\sigma \in \Gamma$ with $\sigma x = y$. Composition of morphisms is multiplication in Γ . The graded tensor product is given by

$$(x \xrightarrow{\sigma} y) \otimes (x' \xrightarrow{\sigma'} y') = (xx' \xrightarrow{\sigma\sigma'} yy').$$

We first prove the following lemma.

Lemma 9. *Let $B \rightarrow D$ be a Γ -crossed module, and let $\psi : Q \rightarrow \text{Coker } d$ be a Γ -homomorphism. For each Γ -graded monoidal functor $(F, \tilde{F}) : {}_{\Gamma}\text{Dis } Q \rightarrow \mathbb{P}_{B \rightarrow D}$ which satisfies $F(1) = 1$ and induces a pair of Γ -homomorphisms $(\psi, 0) : (Q, 0) \rightarrow (\text{Coker } d, \text{Ker } d)$, there exists an equivariant group extension \mathcal{E}_F of B by Q of type $B \rightarrow D$ inducing ψ .*

The extension \mathcal{E}_F is called an *equivariant crossed product extension associated to the Γ -graded monoidal functor F* .

Proof. Let $(F, \tilde{F}) : {}_{\Gamma}\text{Dis } Q \rightarrow \mathbb{P}$ be a Γ -graded monoidal functor. By (5), it defines a function $f : Q \times Q \cup (Q \times \Gamma) \rightarrow B$ which is normalized in the sense that

$$(10) \quad f(x, 1_{\Gamma}) = 0 = f(x, 1) = f(1, y).$$

The first equality holds since F preserves identities, the rest ones hold since $F(1) = 1$ and F is compatible with the unit constraints.

It follows from the definition of morphism in \mathbb{P} that

$$(11) \quad \sigma F(x) = df(x, \sigma)F(\sigma x),$$

$$(12) \quad F(x)F(y) = df(x, y)F(xy).$$

According to the proof of Lemma 5, the function f satisfies the equations (6)-(8), but it here takes values in B instead of $\text{Ker } d'$.

Firstly, we construct the equivariant crossed product $E_0 = B \times_f Q$. The Γ -group structure of E_0 is given by the rules

$$(b, x) + (c, y) = (b + \vartheta_{Fx}(c) + f(x, y), xy),$$

$$\sigma(b, x) = (\sigma b + f(x, \sigma), \sigma x).$$

Thanks to the conditions (6), (10) and (12), $B \times_f Q$ is actually a group. The zero is $(0, 1)$ and $-(b, x) = (b', x^{-1})$, where $\vartheta_{Fx}(b') = -b - f(x, x^{-1})$. Moreover, E_0 is a Γ -group owing to the conditions (7), (8) and (11).

Then, the following sequence is exact

$$\mathcal{E}_F : 0 \rightarrow B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \rightarrow 1,$$

where $j_0(b) = (b, 1)$, $p_0(b, x) = x$, $b \in B$, $x \in Q$. Since $j_0(B)$ is a normal subgroup in E_0 , $j_0 : B \rightarrow E_0$ is a Γ -crossed module in which the action $\vartheta^0 : E_0 \rightarrow \text{Aut } B$ is given by conjugation.

Next, we embed \mathcal{E}_F into the diagram (9) as follows. Since (F, \tilde{F}) induces a Γ -homomorphism $\psi : Q \rightarrow \text{Coker } d$ by $\psi(x) = [Fx] \in \text{Coker } d$, the elements Fx are representatives of $\text{Coker } d$ in D . Then, the map $\varepsilon : E_0 \rightarrow D$ given by

$$(13) \quad \varepsilon(b, x) = db.Fx$$

is a Γ -homomorphism thanks to the conditions (11) and (12).

It is easy to see that $\varepsilon \circ j_0 = d$. Besides, for all $(b, x) \in E_0$, $c \in B$, one has $\vartheta_{(b,x)}^0(c) = \vartheta_{\varepsilon(b,x)}(c)$, as calculated below:

$$\begin{aligned} \vartheta_{(b,x)}^0(c) &= j_0^{-1}[\mu_{(b,x)}(c, 1)] = \mu_b[\vartheta_{Fx}(c)], \\ \vartheta_{\varepsilon(b,x)}(c) &= \vartheta_{db.Fx}(c) = \mu_b[\vartheta_{Fx}(c)]. \end{aligned}$$

Finally, for all $x \in Q$,

$$q\varepsilon(b, x) = q(db.Fx) = q(Fx) = \psi(x) = \psi p_0(b, x),$$

so that the extension \mathcal{E}_F induces the Γ -homomorphism $\psi : Q \rightarrow \text{Coker } d$. \square

Under the hypothesis of Lemma 9, we state the following theorem.

Theorem 10 (Schreier theory for equivariant group extensions of the type of a Γ -crossed module). *There is a bijection*

$$\text{Ext}_{B \rightarrow D}^\Gamma(Q, B, \psi) \leftrightarrow \text{Hom}_{(\psi, 0)}[\text{Dis}_\Gamma Q, \mathbb{P}_{B \rightarrow D}].$$

Proof. First, every Γ -graded monoidal functor (F, \tilde{F}) is homotopic to one (G, \tilde{G}) in which $G(1) = 1$. Hence, we can restrict our attention to this kind of Γ -graded monoidal functor $(F, \tilde{F}) : \text{Dis}_\Gamma Q \rightarrow \mathbb{P}_{B \rightarrow D}$. The, by Lemma 9, each Γ -graded monoidal functor (F, \tilde{F}) defines an equivariant extension $\mathcal{E}_F \in \text{Ext}_{B \rightarrow D}^\Gamma(Q, B, \psi)$. The correspondence $[F, \tilde{F}] \mapsto [\mathcal{E}_F]$ defines an injection

$$\Omega : \text{Hom}_{(\psi, 0)}[\text{Dis}_\Gamma Q, \mathbb{P}_{B \rightarrow D}] \rightarrow \text{Ext}_{B \rightarrow D}^\Gamma(Q, B, \psi)$$

thanks to Step 1 below.

Step 1: The Γ -graded monoidal functors $F, F' : \Gamma \text{ Dis } Q \rightarrow \mathbb{P}$ are homotopic if and only if the corresponding associated equivariant extensions $\mathcal{E}_F, \mathcal{E}_{F'}$ are equivalent.

We first recall that every graded monoidal functor (F, \tilde{F}) is homotopic to one (G, \tilde{G}) in which $G(1) = 1$. Hence, we can restrict our attention to this kind of graded monoidal functors.

Let $F, F' : \Gamma \text{ Dis } Q \rightarrow \mathbb{P}$ be homotopic by a homotopy $\alpha : F \rightarrow F'$. Then, there exists a function $g : Q \rightarrow B$ such that $\alpha_x = (g(x), 1)$, that is,

$$(14) \quad Fx = dg(x)F'x.$$

The naturality of α gives

$$(15) \quad f(x, \sigma) + g(\sigma x) = \sigma g(x) + f'(x, \sigma).$$

The coherence condition (1) of the homotopy α implies $g(1) = 0$ and

$$(16) \quad f(x, y) + g(xy) = g(x) + \vartheta_{F'x}g(y) + f'(x, y).$$

Let $\mathcal{E}_{F'}$ be the extension associated to F' by Lemma 9. We write

$$\begin{aligned} \alpha^* : E_F &\rightarrow E_{F'}. \\ (b, x) &\mapsto (b + g(x), x) \end{aligned}$$

Then, thanks to the equations (15) and (16), α^* is a Γ -homomorphism. Further, the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j_0} & E_F & \xrightarrow{p_0} & Q \longrightarrow 1, & & E_F & \xrightarrow{\varepsilon} & D \\ & & \parallel & & \downarrow \alpha^* & & \parallel & & & & \\ 0 & \longrightarrow & B & \xrightarrow{j'_0} & E_{F'} & \xrightarrow{p'_0} & Q \longrightarrow 1, & & E_{F'} & \xrightarrow{\varepsilon'} & D \end{array}$$

and hence α^* is an isomorphism. It remains to show that $\varepsilon'\alpha^* = \varepsilon$. It follows from the equations (13) and (14) that

$$\begin{aligned} \varepsilon'\alpha^*(b, x) &= \varepsilon'(b + g(x), x) = d(b + g(x))F'x \\ &= d(b)d(g(x))F'x = d(b)Fx = \varepsilon(b, x). \end{aligned}$$

Thus, two extensions \mathcal{E}_F and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if $\alpha^* : E_F \rightarrow E_{F'}$ is an isomorphism, then

$$\alpha^*(b, x) = (b + g(x), x),$$

where $g : Q \rightarrow B$ is a function with $g(1) = 0$. Thus, $\alpha_x = (g(x), 1)$ is a homotopy of F and F' as we see by retracing our steps.

Step 2: Ω is surjective.

Assume that \mathcal{E} is an equivariant extension of B by Q of type $B \rightarrow D$ inducing $\psi : Q \rightarrow \text{Coker } d$ as in the commutative diagram (9). We prove that \mathcal{E} is equivalent to an extension \mathcal{E}_F associated to some Γ -graded monoidal functor $(F, \tilde{F}) : \Gamma\text{Dis}Q \rightarrow \mathbb{P}_{B \rightarrow D}$.

For each $x \in Q$, choose a representative $u_x \in E$ such that $p(u_x) = x$, $u_1 = 0$. An element in E can be uniquely written as $b + u_x$ for $b \in B$, $x \in Q$. The representatives $\{u_x\}$ induce a normalized function $f : Q \times Q \cup Q \times \Gamma \rightarrow B$ by

$$(17) \quad u_x + u_y = f(x, y) + u_{xy},$$

$$(18) \quad \sigma u_x = f(x, \sigma) + u_{\sigma x},$$

and the automorphisms φ_x of B by

$$\varphi_x = \mu_{u_x} : b \mapsto u_x + b - u_x.$$

It follows from the condition H_2 of the homomorphism (id, ε) of Γ -crossed modules that

$$\vartheta_{\varepsilon u_x} = \mu_{u_x} = \varphi_x.$$

Then, the Γ -group structure of E can be described by

$$(b + u_x) + (c + u_y) = b + \varphi_x(c) + f(x, y) + u_{xy},$$

$$\sigma(b + u_x) = \sigma b + f(x, \sigma) + u_{\sigma x}.$$

Since $\psi(x) = \psi p(u_x) = q\varepsilon(u_x)$, $\varepsilon(u_x)$ is a representative of $\psi(x)$ in D . Thus, we define a Γ -graded monoidal functor $(F, \tilde{F}) : \text{Dis}_\Gamma Q \rightarrow \mathbb{P}$ as follows.

$$Fx = \varepsilon(u_x), F(x \xrightarrow{\sigma} \sigma x) = (f(x, \sigma), \sigma), \tilde{F}_{x,y} = (f(x, y), 1).$$

The equations (18) and (17) show that $F(\sigma)$ and $\tilde{F}_{x,y}$ are actually morphisms in \mathbb{P} , respectively. The normality of the function $f(x, \sigma)$ gives $F(id_x) = id_{Fx}$. Clearly, $F(1) = 1$. This together with the normality of the function $f(x, y)$ imply the compatibility of (F, \tilde{F}) with the unit constraints. The associativity law and the Γ -group properties of B imply the equations (6)-(8), respectively, in which ϑ_{Fx} is replaced by φ_x . These equations show that (F, \tilde{F}) is compatible with the associativity constraints, $\tilde{F}_{x,y}$ is a natural isomorphism and F preserves the composition of morphisms, respectively.

Finally, it is easy to check that the equivariant crossed product extension \mathcal{E}_F associated to (F, \tilde{F}) is equivalent to the extension \mathcal{E} by the Γ -isomorphism $\alpha : (b, x) \mapsto b + u_x$. □

Moreover, each equivariant group extension of B by Q studied in [6] may be viewed as an equivariant group extension of type of Γ -crossed module $(B, \text{Aut } B, \mu, 0)$. Then, $\mathbb{P}_{B \rightarrow \text{Aut } B}$ is just the *holomorph* Γ -graded categorical group of a Γ -group B , $\text{Hol}_\Gamma B$.

Corollary 11 ([6, Theorem 4.2]). *For Γ -groups B and Q , there exists a bijection*

$$\text{Ext}_\Gamma(Q, B) \leftrightarrow \text{Hom}_\Gamma[\text{Dis}_\Gamma Q, \text{Hol}_\Gamma B].$$

Let $\mathbb{P} = \mathbb{P}_{B \rightarrow D}$ be the Γ -graded categorical group associated to the Γ -crossed module $B \rightarrow D$. Since $\pi_0 \mathbb{P} = \text{Coker } d$ and $\pi_1 \mathbb{P} = \text{Ker } d$, the reduced graded categorical group of \mathbb{P} is

$$\mathcal{S}_\mathbb{P} = \int_\Gamma (\text{Coker } d, \text{Ker } d, h), \quad h \in Z_\Gamma^3(\text{Coker } d, \text{Ker } d).$$

Then, by (2), the pair of Γ -homomorphisms $(\psi, 0) : (Q, 0) \rightarrow (\text{Coker } d, \text{Ker } d)$ induces an *obstruction*

$$\psi^* h \in Z_\Gamma^3(Q, \text{Ker } d).$$

Under this notion of obstruction, we state the following theorem.

Theorem 12. *Let (B, D, d, ϑ) be a Γ -crossed module, and let $\psi : Q \rightarrow \text{Coker}d$ be a Γ -homomorphism. Then, the vanishing of $\overline{\psi^*h}$ in $H_\Gamma^3(Q, \text{Ker}d)$ is necessary and sufficient for there to exist an equivariant extension of B by Q of type $B \rightarrow D$ inducing ψ . Further, if $\overline{\psi^*h}$ vanishes, then the equivalence classes of such extensions are bijective with $H_\Gamma^2(Q, \text{Ker}d)$.*

Proof. By the assumption, $\overline{\psi^*h} = 0$, thus by Proposition 2, there exists a Γ -graded monoidal functor $(\Psi, \tilde{\Psi}) : {}_\Gamma \text{Dis} Q \rightarrow \mathcal{S}_\mathbb{P}$. Then, composition of $(\Psi, \tilde{\Psi})$ and $(H, \tilde{H}) : \mathcal{S}_\mathbb{P} \rightarrow \mathbb{P}$ is a Γ -graded monoidal functor $(F, \tilde{F}) : {}_\Gamma \text{Dis} Q \rightarrow \mathbb{P}$. It is easy to see that F induces the pair of Γ -homomorphisms $(\psi, 0)$, hence by Lemma 9, we obtain an associated extension \mathcal{E}_F .

Conversely, suppose that there is an equivariant extension as in the diagram (9). Let \mathbb{P}' be the Γ -graded categorical group associated to the Γ -crossed module $B \rightarrow E$. By Lemma 5, there is a Γ -graded monoidal functor $F : \mathbb{P}' \rightarrow \mathbb{P}$. Since the reduced graded categorical group of \mathbb{P}' is ${}_\Gamma \text{Dis} Q$, by Proposition 1, F induces a Γ -graded monoidal functor of type $(\psi, 0)$ from ${}_\Gamma \text{Dis} Q$ to $\mathcal{S}_\mathbb{P} = \int_\Gamma (\text{Coker}d, \text{Ker}d, h)$. Now, by Proposition 2, the obstruction of the pair $(\psi, 0)$ vanishes in $H_\Gamma^3(Q, \text{Ker}d)$, that is, $\overline{\psi^*h} = 0$.

The final assertion of the theorem follows from Proposition 2 and Theorem 10. \square

Note that if $\Gamma = \mathbf{1}$, the trivial group, then the set $\text{Ext}_{B \rightarrow D}^1(Q, B, \psi)$ is just the set of equivalence classes of group extensions of the type of a crossed module studied in [2, 8, 16]. Thus, we obtain the following consequence.

Corollary 13 ([2, Theorem 5.2]). *Let (B, D, d, ϑ) be a crossed module, and let $\psi : Q \rightarrow \text{Coker}d$ be a group homomorphism. Then, there exists a 3-dimensional cohomology class $k(B, D, \psi) \in H^3(Q, \text{Ker}d)$, called the obstruction, whose vanishing is necessary and sufficient for there to exist an extension of B by Q of type $B \rightarrow D$ inducing ψ . Further, if $k(B, D, \psi)$ vanishes, then the equivalence classes of such extensions are bijective with $H^2(Q, \text{Ker}d)$.*

For the Γ -crossed module $(B, \text{Aut} B, \mu, 0)$, since $\text{Coker} \mu = \text{Out} B, \text{Ker} \mu = Z(B)$, Theorem 12 contains Theorem 4.1 [6].

Corollary 14 ([6, Theorem 4.1]). *Let B, D be Γ -groups and let $\psi : Q \rightarrow \text{Out} B$ be a Γ -homomorphism. Then, there exists the obstruction $\text{Obs}(\psi) \in H_\Gamma^3(Q, Z(B))$ whose vanishing is necessary and sufficient for there to exist an equivariant extension of B by Q inducing ψ . Further, if $\text{Obs}(\psi)$ vanishes, then the equivalence classes of such extensions are bijective with $H_\Gamma^2(Q, Z(B))$.*

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