Bull. Korean Math. Soc. **52** (2015), No. 4, pp. 1047–1057
http://dx.doi.org/10.4134/BKMS.2015.52.4.1047

ON THE PERIOD OF β-EXPANSION OF PISOT OR SALEM SERIES OVER $\mathbb{F}_q((x^{-1}))$

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Abstract. In [6], it is proved that the lengths of periods occurring in the β-expansion of a rational series r noted by $Per_\beta(r)$ depend only on the denominator of the reduced form of r for quadratic Pisot unit series. In this paper, we will show first that every rational r in the unit disk has strictly periodic β-expansion for Pisot or Salem unit basis under some condition. Second, for this basis, if $r = \frac{P}{Q}$ is written in reduced form with $|P| < |Q|$, we will generalize the curious property "$Per_\beta(\frac{P}{Q}) = Per_\beta(\frac{1}{Q})$".

1. Introduction

The notion of β-expansions of real numbers was introduced by A. Rényi [11]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several researchers. In this paper, we consider an analogue of this concept in algebraic function fields over finite fields. There are striking analogies between these digit systems and the classical β-expansions of real numbers. In order to pursue this analogy, we recall the definition of real β-expansions and survey the problems corresponding to our results.

Let β be a fixed real number greater than 1 and let x be a positive real number. A convergent series $\sum_{k\leq n} x_k\beta^k$ is called a β-representation of x if

$$x = \sum_{k\leq n} x_k\beta^k$$

and for all k the coefficient x_k is a non-negative integer. If moreover for every $-\infty < N < n$ we have

$$\sum_{k\leq N} x_k\beta^k < \beta^{N+1}$$

the series $\sum_{k\leq n} x_k\beta^k$ is called the β-expansion of x. The β-expansion is an analogue of the decimal or binary expansion of reals and we sometimes use the natural notation $d_\beta(x) = x_n x_{n-1}\cdots x_0.x_{-1}\cdots$. Every $x \geq 0$ has

Received March 15, 2013; Revised January 10, 2014.
2010 *Mathematics Subject Classification.* 11R06, 37B50.
Key words and phrases. formal power series, β-expansion, Pisot series, Salem series.

a unique β-expansion which is found by the greedy algorithm [11]. We can introduce lexicographic ordering on β-representations in the following way. The β-representation $x_n\beta^n + x_{n-1}\beta^{n-1} + \cdots$ is lexicographically greater than $x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$, if $k < n$ and for the corresponding infinite words we have $x_nx_{n-1}\cdots \succ \underbrace{0\cdots 00}_{(n-k)\text{times}} x_kx_{k-1}\cdots$, where the symbol $\prec$ means the common lexicographic ordering on words in an ordered alphabet.

A β-expansion is finite if there exists $m \geq 1$ such that $x_k = 0$ holds for all $k \geq m$. We denote by $Fin(\beta)$ the sets of real numbers in [0,1) with finite β-expansions. A β-expansion is periodic if there exist $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We denote by $Per(\beta)$ the sets of real numbers in [0,1) with periodic β-expansions.

An easy argument shows that $Per(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$ for every real number $\beta > 1$. In the statement [13], K. Schmidt showed that if β is a Pisot number (an algebraic integer whose conjugates have modulus < 1), then $Per(\beta) = \mathbb{Q}(\beta) \cap [0,1)$.

In the statement [8], S. Ito and H. Rao discussed the purely periodic β-expansions and they characterized all reals in $[0,1)$ having purely periodic β-expansions with Pisot unit base. Also S. Akiyama proved in the statement [2] that if β verifies finiteness conditions ($Fin(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$), then a positive constant c exists in a way that every rational in $[0,c)$ has a purely periodic β-expansion. The cubic case produces unexpected results, as stated by S. Akiyama in the statement [2], and more recently by B. Adamczewski, C. Frougny, A. Siegel and W. Steiner in the statement [1].

In the real case and with a quadratic base β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geq 1$, K. Schmidt [13] has given this theorem.

Theorem 1.1. *Suppose that β satisfies $\beta^2 = n\beta + 1$ for some $n \geq 1$. Then every $r \in \mathbb{Q} \cap [0,1)$ has a strictly periodic β-expansion. If $r = \frac{p}{q}$ is written in reduced form with $0 < p < q$, then $Per_\beta(\frac{p}{q}) = Per_\beta(\frac{1}{q})$.*

In [4] and [5], D. W. Boyd investigated the length of the period for some Salem numbers of degree 4 and 6.

In the case of formal power series over finite fields, we have proved in [6] and especially in the quadratic Pisot unit case that every rational r in the unit disk has a strictly periodic β-expansion. If $r = \frac{P}{Q}$ is written in reduced form with $|P| < |Q|$, the curious property "$Per_\beta(\frac{P}{Q}) = Per_\beta(\frac{1}{Q})$" holds.

This paper is a continuation of this work which is organized as follows: In Section 2, we define the field of formal power series over finite field as well as the analogues to Pisot and Salem numbers. We will also define the β-expansion algorithm for formal power series. In Section 3, we will prove that every rational series have purely periodic β-expansions in Pisot or Salem unit basis under some condition. We will also investigate some properties for the length of the period of the β-expansion of a rational series, avoiding difficult computations of power series.

2. β-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficient in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ the field of rational functions, $\mathbb{F}_q(x, \beta)$ the minimal extension of $\mathbb{F}_q$ containing x and β and $\mathbb{F}_q[x, \beta]$ the minimal ring containing x and β. Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q$$

whereby

$$l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{for } f \neq 0; \\ -\infty & \text{for } f = 0. \end{cases}$$

Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Then $|\cdot|$ is not archimedean. It fulfills the strict triangular inequality

$$|f+g| \leq \max\,(|f|, |g|) \quad \text{and}$$
$$|f+g| = \max\,(|f|, |g|) \quad \text{if} \quad |f| \neq |g|.$$

For $f \in \mathbb{F}_q((x^{-1}))$ define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$, where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[x]$ and $(f - [f]) \in M_0$, where $M_0 = \{f \in \mathbb{F}_q((x^{-1})) : |f| < 1\}$.

We know that $\mathbb{F}_q((x^{-1}))$ is complete with respect to the metric defined by this absolute value. We denote by $\overline{\mathbb{F}_q((x^{-1}))}$ an algebraic closure of $\mathbb{F}_q((x^{-1}))$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}_q((x^{-1}))}$. Abusing a little the notations, we will use the same symbol $|\cdot|$ for the two absolute values.

A Pisot (resp. Salem) element $w \in \mathbb{F}_q((x^{-1}))$ is an algebraic integer over $\mathbb{F}_q[x]$ such that $|w| > 1$ whose remaining conjugates in $\overline{\mathbb{F}_q((x^{-1}))}$ have an absolute value strictly smaller than 1 (resp., $|w| > 1$ whose remaining conjugates in $\overline{\mathbb{F}_q((x^{-1}))}$ have an absolute value strictly smaller than 1 and there exists at least one conjugate w_k such that $|w_k| = 1$). In 1962 Bateman and Duquette [3] introduced and characterized Pisot elements and Salem elements in the field of formal power series. They obtained the following result.

Theorem 2.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_{n-1}y^{n-1} - \cdots - A_1, \quad A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then

(i) *β is a Pisot element if and only if* $|A_{n-1}| > \max_{1 \leq i \leq n-2} |A_i|$.

(ii) *β is a Salem element if and only if* $|A_{n-1}| = \max_{1 \leq i \leq n-2} |A_i|$.

Let $\beta,\ f \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of f is an infinite sequence $(a_i)_{i\geq 1}$, $a_i \in \mathbb{F}_q[x]$ with

$$f \quad = \quad \sum_{i\geq 1} \frac{a_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, denoted $d_\beta(f)$. This is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i\geq 1}$, where $a_i = [\beta T_\beta^{i-1}(f)]$.

An equivalent definition of the β-expansion can be obtained by a greedy algorithm. This algorithm works as follows:

$$r_0 = f,\ a_i = [\beta r_{i-1}] \ \text{ and } \ r_i = \beta r_{i-1} - a_i \ \text{ for all } \ i \geq 1. \tag{1}$$

The β-expansion of f will be noted as $d_\beta(f) = (a_i)_{i\geq 1}$.

Notice that $d_\beta(f)$ is finite if and only if there is $k \geq 0$ with $T^k(f) = 0$, and that $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) such that $T_\beta^{p+s}(f) = T_\beta^p(f)$, namely the period length will be denoted by $Per_\beta(f)$.

Now, let $f \in \mathbb{F}_q((x^{-1}))$ be an element, with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}$ having $|\beta|^k \leq |f| < |\beta|^{k+1}$. Hence, $|\frac{f}{\beta^{k+1}}| < 1$. We can represent f by shifting $d_\beta(\frac{f}{\beta^{k+1}})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.d_1d_2d_3\cdots$, then $d_\beta(\beta f) = d_1.d_2d_3\cdots$.

Afterwards, we will use the following notation:

$$Per(\beta) = \{f \in \mathbb{F}_q((x^{-1})) :\ d_\beta(f) \text{ is eventually periodic}\}.$$

Remark 2.2. In contrast to the real case, there is no carry occurring, when we add two digits. So, if $z,\, w \in \mathbb{F}_q((x^{-1}))$, then we have $d_\beta(z+w) = d_\beta(z)+d_\beta(w)$ digitwise and if $c \in \mathbb{F}_q^*$, then $d_\beta(cz) = cd_\beta(z)$.

In the case of the field of formal series, the following theorems were proved independently by Hbaib-Mkaouar and Scheicher in the statements [12] and [7].

Theorem 2.3 ([7]). *A β-representation $(d_j)_{j\geq 1}$ of f in the unit disk is its β-expansion if and only if $|d_j| < |\beta|$ for $j \geq 1$.*

Theorem 2.4. *β is a Pisot or Salem element if and only if $Per(\beta) = \mathbb{F}_q(x, \beta)$.*

In the papers [9] and [10], metric results are established and the relation to continued fractions is studied.

3. Results

We study specifically the length of the period by giving a lower bound of the length of the period for every rational in Pisot or Salem unit base.

Theorem 3.1. *Let β be a Pisot or Salem unit series of algebraical degree d. Then every rational f in the unit disk satisfies*

$$Per_\beta(f) \geq \frac{-d\deg(f)}{\deg(\beta)}.$$

Proof. According to Theorem 2.4, we have $f \in Per(\beta)$, so

$$d_\beta(f) = 0.a_1 \cdots a_p\overline{a_{p+1} \cdots a_{p+s}} \quad \text{and} \quad a_p \neq a_{p+s}.$$

Hence

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s}(f - \frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p}),$$

we get

$$f(1 - \frac{1}{\beta^s}) = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s}(-\frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p}),$$

therefore

$$f(\beta^{s+p} - \beta^p) = a_1\beta^{s+p-1} + \cdots + \beta a_{p+s-1} + a_{p+s} - a_1\beta^{p-1} - \cdots - \beta a_{p-1} - a_p,$$

otherwise

$$f(\beta^{s+p} - \beta^p) = a_1\beta^{s+p-1} + \cdots + \beta a_{p+s-1} - a_1\beta^{p-1} - \cdots - \beta a_{p-1} + (a_{p+s} - a_p).$$

Since $a_1, \ldots, a_{n+s} \in \mathbb{F}_q[x]$ and $f \in \mathbb{F}_q(x)$,

$$\begin{aligned}
f((\beta^{(2)})^{s+p} - (\beta^{(2)})^p) &= a_1(\beta^{(2)})^{s+p-1} - \cdots - (\beta^{(2)})a_{p-1} + (a_{p+s} - a_p), \\
&\vdots \\
f((\beta^{(d)})^{s+p} - (\beta^{(d)})^p) &= a_1(\beta^{(d)})^{s+p-1} - \cdots - (\beta^{(d)})a_{p-1} + (a_{p+s} - a_p).
\end{aligned}$$

Multiplying each of the above, we obviously obtain

$$\begin{aligned}
&|\beta^{s+p}(\beta^{(2)})^p \ldots (\beta^{(d)})^p||f|^d \\
&= |\prod_{1 \leq i \leq d} (a_1(\beta^{(i)})^{s+p-1} - \cdots - (\beta^{(i)})a_{p-1} + (a_{p+s} - a_p))|.
\end{aligned}$$

We can note that

$$\begin{aligned}
H = \prod_{1 \leq i \leq d} &(a_1(\beta^{(i)})^{s+p-1} + \cdots + (\beta^{(i)})a_{p+s-1} - a_1(\beta^{(i)})^{p-1} - \\
&\cdots - (\beta^{(i)})a_{p-1} + (a_{p+s} - a_p)).
\end{aligned}$$

To finish this proof we need the following fundamental lemma over symmetrical polynomials:

Lemma 3.2. *Let $Q \in \mathbb{F}_q[x][y]$ (i.e., with coefficient in $\mathbb{F}_q[x]$) and $F(Y_1, Y_2, \ldots, Y_n) = Q(Y_1)Q(Y_2) \cdots Q(Y_n)$. Then there exists a polynomial T to n variables with coefficients in $\mathbb{F}_q[x]$ such that*

$$F(Y_1, Y_2, \ldots, Y_n) = T(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

where

$$\begin{cases} \sigma_1 = \sum_{i=1}^{n} Y_i, \\ \sigma_2 = \sum_{1\le i<j\le n} Y_iY_j, \\ \sigma_3 = \sum_{1\le i<j<k\le n} Y_iY_jY_k, \\ \vdots \\ \sigma_n = Y_1Y_2\cdots Y_n. \end{cases}$$

Note moreover that the total degree of T is lower or equal to the degree of Q.

Now return to the proof of Theorem 3.1. Applying Lemma 3.2, we deduce that

$$|\beta^{s+p}(\beta^{(2)})^p\cdots(\beta^{(d)})^p||f|^d \ge 1.$$

Consequently, $|\beta^s||f|^d \ge 1$, so $s\deg|\beta| + d\ \deg|f| \ge 0$. Finally, we get the result. □

The main problem involved in the rest of this section is the study of $Per_\beta(f)$ when β is a Pisot or Salem unit series and $f = \frac{P}{Q}$ is written in his reduced form with $|P| < |Q|$. Thus, β is an algebraic integer, and we write $M_\beta(y) = y^d + A_{d-1}y^{d-1} + A_{d-2}y^{d-2} + \cdots + A_0$ for the minimal polynomial of β. We have seen in the second section that, $r_0 = f = \frac{1}{Q}(P + 0\beta + 0\beta^2 + \cdots + 0\beta^{d-1}) = \frac{1}{Q}(B_{0,0} + B_{1,0}\beta + \cdots + B_{d-1,0}\beta^{d-1})$ with $B_{0,0} = P$ and $B_{1,0} = B_{2,0} = \cdots = B_{d-1,0} = 0$. We know that if $d_\beta(f) = (a_i)_{i\ge1}$, then $r_n = f\beta^n - a_1\beta^{n-1} - \cdots - a_n = \frac{1}{Q}(B_{0,n} + B_{1,n}\beta + \cdots + B_{d-1,n}\beta^{d-1})$ with $B_{0,n}, B_{1,n}, \ldots, B_{d-1,n} \in \mathbb{F}_q[x]$. So we have

$$\begin{aligned} r_{n+1} &= \beta r_n - a_{n+1} \\ &= \beta(\frac{1}{Q}(B_{0,n} + B_{1,n}\beta + \cdots + B_{d-1,n}\beta^{d-1})) - a_{n+1} \\ &= \frac{1}{Q}(B_{0,n}\beta + B_{1,n}\beta^2 + \cdots + B_{d-1,n}\beta^d - a_{n+1}Q) \\ &= \frac{1}{Q}(B_{0,n}\beta + B_{1,n}\beta^2 + \cdots + B_{d-1,n}(-A_{d-1}\beta^{d-1} - \cdots - A_0) - a_{n+1}Q). \end{aligned}$$

Hence

$$\begin{aligned} r_{n+1} = \frac{1}{Q}[&(-B_{d-1,n}A_0 - a_{n+1}Q) + (B_{0,n} - B_{d-1,n}A_1)\beta + \cdots \\ &+ (B_{d-3,n} - B_{d-1,n}A_{d-2})\beta^{d-2} + (B_{d-2,n} - B_{d-1,n}A_{d-1})\beta^{d-1}] \\ = \frac{1}{Q}&(B_{0,n+1} + B_{1,n+1}\beta + \cdots + B_{d-2,n+1}\beta^{d-2} + B_{d-1,n+1}\beta^{d-1}). \end{aligned}$$

This implies that $\mathbb{B}_{n+1} = M\mathbb{B}_n + V$, where

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -A_0 \\ 1 & 0 & \dots & 0 & 0 & -A_1 \\ 0 & 1 & \ddots & 0 & 0 & -A_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 & -A_{d-2} \\ 0 & 0 & \dots & 0 & 1 & -A_{d-1} \end{pmatrix}, \ \mathbb{B}_n = \begin{pmatrix} B_{0,n} \\ B_{1,n} \\ B_{2,n} \\ \vdots \\ B_{d-2,n} \\ B_{d-1,n} \end{pmatrix}, \text{ and}$$

$$V = \begin{pmatrix} -a_{n+1}Q \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Now, we define the application H by

$$\begin{array}{rcl} H : \mathbb{F}_q[x]^d & \longrightarrow & \mathbb{F}_q[x]^d \\ \mathbb{P} = (P_1, \dots, P_d) & \longmapsto & (Q_1, \dots, Q_d) = M\mathbb{P} + V, \end{array}$$

and $\widetilde{H}$ is induced by:

$$\begin{array}{rcl} \widetilde{H} : (\mathbb{F}_q[x]/Q\mathbb{F}_q[x])^d & \longrightarrow & (\mathbb{F}_q[x]/Q\mathbb{F}_q[x])^d \\ (\widetilde{P_1}, \dots, \widetilde{P_d}) & \longmapsto & (\widetilde{Q_1}, \dots, \widetilde{Q_d}), \end{array}$$

with $\mathbb{F}_q[x]/Q\mathbb{F}_q[x]$ is the group of polynomials (mod Q) and $(\mathbb{F}_q[x]/Q\mathbb{F}_q[x])^d$ the d-fold Cartesian product of copies of $\mathbb{F}_q[x]/Q\mathbb{F}_q[x]$. Since $\widetilde{H}$ is a natural automorphism over a finite group, so there exists n with

$$\widetilde{H}^n(\widetilde{P}, \widetilde{0}, \dots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \dots, \widetilde{0}).$$

Let

$$s = \min\{\ n > 0\ ;\ \widetilde{H}^n(\widetilde{P}, \widetilde{0}, \dots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \dots, \widetilde{0})\}.$$

Remark 3.3. For each rational f in the unit disk, we have $Per_\beta(f)$ is a multiple of s.

Indeed, if $l = Per_\beta(f)$, then $H^{k+l}(P, 0, \dots, 0) = H^k(P, 0, \dots, 0)$. Thus, $\widetilde{H}^{k+l}(\widetilde{P}, \widetilde{0}, \dots, \widetilde{0}) = \widetilde{H}^k(\widetilde{P}, \widetilde{0}, \dots, \widetilde{0})$. Since $\widetilde{H}$ is a one-to-one mapping, so $\widetilde{H}^l(\widetilde{P}, \widetilde{0}, \dots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \dots, \widetilde{0})$. Consequently l is a multiple of s.

In the real case and in a very special form, i.e., β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geq 1$, K. Schmidt [13] has given an explicit formula for the length of the periods occurring in the expansions of rational numbers. In order to obtain this result in the formal power series to every Pisot or Salem unit series, we can now state the main result of this section.

Theorem 3.4. *Let β be a Pisot (resp. Salem) unit series of minimal polynomial $M_\beta(y) = y^d + A_{d-1}y^{d-1} + A_{d-2}y^{d-2} + \cdots + A_0$ such that $\sum_{0\le i\le d-2} A_i = -1$ (resp. $\sum_{0\le i\le d-1} A_i = -1$). Then every rational in the unit disk has a purely periodic β-expansion.*

Proof. We have $M_\beta(y) = y^d + A_{d-1}y^{d-1} + A_{d-2}y^{d-2} + \cdots + A_0$ the minimal polynomial of β. We define inductively the sequences: $(C_{1,k}), \ldots, (C_{d-1,k})$, $(D_{1,k}), \ldots, (D_{d-1,k})$ and $(U_{0,k}), \ldots, (U_{d-1,k})$ of polynomials as follows. Let us have

$$U_{d-1,0} = P, \quad U_{d-2,0} = \cdots = U_{0,0} = 0, \quad D_{1,0} = \cdots = D_{d-1,0} = 0 \text{ and }$$
$$C_{1,0} = \cdots = C_{d-1,0} = 0.$$

We put $U_{0,k+1} = -A_0 U_{d-1,k}$ and

$$\begin{aligned}
D_{d-1,k+1} &= U_{d-2,k} - A_{d-1}U_{d-1,k} = U_{d-1,k+1} + QC_{d-1,k+1},\\
D_{d-2,k+1} &= U_{d-3,k} - A_{d-2}U_{d-1,k} = U_{d-2,k+1} + QC_{d-2,k+1},\\
D_{d-3,k+1} &= U_{d-4,k} - A_{d-3}U_{d-1,k} = U_{d-3,k+1} + QC_{d-3,k+1},\\
&\vdots\\
D_{1,k+1} &= U_{0,k} - A_1U_{d-1,k} = U_{1,k+1} + QC_{1,k+1},
\end{aligned}$$

such that $U_{d-1,k+1}, \ldots, U_{1,k+1}$ are respectively the rest of the Euclidean division of $D_{d-1,k+1}, \ldots, D_{1,k+1}$ by Q. This induction process yields sequences $((C_{1,k}), \ldots, (C_{d-1,k}) : k \ge 0)$, $((D_{1,k}), \ldots, (D_{d-1,k}) : k \ge 0)$ and $((U_{0,k}), \ldots, (U_{d-1,k}) : k \ge 0)$ with the following properties:

$$\begin{aligned}
&\text{For all } 0 \le i \le d-1, \qquad |U_{i,k}| < |Q|.\\
&\text{For all } 1 \le i \le d-1, \qquad |D_{i,k}| < |A_i||Q|.
\end{aligned}$$

Thus,

$$\text{for all } 1 \le i \le d-1, \quad |C_{i,k}| < |A_i| = |\beta|.$$

Consequently, we have

$$(U_{0,0}, \ldots, U_{d-2,0}, U_{d-1,0}) = (0, \ldots, 0, P) \text{ and}$$
$$\begin{pmatrix} \widetilde{U_{0,k+1}} \\ \widetilde{U_{1,k+1}} \\ \widetilde{U_{2,k+1}} \\ \vdots \\ \widetilde{U_{d-2,k+1}} \\ \widetilde{U_{d-1,k+1}} \end{pmatrix} = \widetilde{H} \begin{pmatrix} \widetilde{U_{0,k}} \\ \widetilde{U_{1,k}} \\ \widetilde{U_{2,k}} \\ \vdots \\ \widetilde{U_{d-2,k}} \\ \widetilde{U_{d-1,k}} \end{pmatrix}.$$

From our construction, we have $\widetilde{H}^{d-1}(\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{0}, \ldots, \widetilde{0}, \widetilde{P})$ which implies $\widetilde{H}^s(\widetilde{0}, \ldots, \widetilde{0}, \widetilde{P}) = (\widetilde{0}, \ldots, \widetilde{0}, \widetilde{P})$. Therefore we have for all $0 \le i \le d-1$,

$\widetilde{U_{i,k+s}} = \widetilde{U_{i,k}}$ for every $k \geq 0$. So for all $0 \leq i \leq d-1$, $U_{i,k+s} = U_{i,k}$, hence for all $1 \leq i \leq d-1$, $C_{i,k+s} = C_{i,k}$ for every $k \geq 0$. To finish this proof we need only to show the following lemma:

Lemma 3.5. *For $n \geq 1$, in the Pisot case, we have*

$$\frac{P}{Q} = \sum_{k=1}^{n} \frac{C_{d-2,k} + \cdots + C_{1,k}}{\beta^{k-1}} + \frac{\beta^{-n+1}}{Q}(U_{d-2,n} + \cdots + U_{0,n}). \tag{2}$$

(*resp. in the Salem case, we have*

$$\frac{P}{Q} = \sum_{k=1}^{n} \frac{C_{d-1,k} + \cdots + C_{1,k}}{\beta^{k-1}} + \frac{\beta^{-n+1}}{Q}(U_{d-1,n} + \cdots + U_{0,n}).) \tag{3}$$

Proof. We start showing this lemma by induction for the Pisot case: We have

$$\begin{aligned} & C_{d-2,1} + \cdots + C_{1,1} + \frac{1}{Q}(U_{d-2,1} + \cdots + U_{0,1}) \\ = & \frac{(C_{d-2,1}Q + U_{d-2,1}) + \cdots + (C_{1,1}Q + U_{1,1}) + U_{0,1}}{Q}. \end{aligned}$$

In addition, we have

$$\begin{aligned} C_{d-2,1}Q + U_{d-2,1} &= -A_{d-2}P, \\ &\vdots \\ C_{1,1}Q + U_{1,1} &= -A_1 P, \\ U_{0,1} &= -A_0 P. \end{aligned}$$

So

$$C_{d-2,1} + \cdots + C_{1,1} + \frac{1}{Q}(U_{d-2,1} + \cdots + U_{0,1}) = \frac{P}{Q}(-A_{d-2} - \cdots - A_0) = \frac{P}{Q}.$$

Consequently, (2) holds for $n = 1$. Suppose (2) holds for n, then it is easily confirmed that (2) holds for $n+1$. Thus, (2) holds for all $n \geq 1$. In the same manner with this proof, we show (3) for the Salem case. $\square$

Let us return to the proof of Theorem 3.4: As we apply the limit $(n \to \infty)$ for the expression of Lemma 3.5, we get

$$\frac{P}{Q} = \sum_{k=1}^{n} C_{d-2,k} + \cdots + C_{1,k}\beta^{k-1}$$

because $\left|\frac{\beta^{-n+1}}{Q}(U_{d-2,n} + \cdots + U_{0,n})\right| < 1$. Thus, obviously we obtain

$$\frac{P}{Q} = \sum_{k \geq 1} \frac{C_{d-2,k} + \cdots + C_{1,k}}{\beta^{k-1}}.$$

Since $|C_{i,k}| < |\beta|$ and $C_{i,k+s} = C_{i,k}$ for all $k \geq 1$ and $1 \leq i \leq d-1$, according to Theorem 2.3, $\frac{P}{Q}$ has a purely periodic β-expansions of period s.

With the same manner as the Pisot case, we prove that $\frac{P}{Q}$ has a purely periodic β-expansions of period s in the Salem case. □

Corollary 3.6. *Let β be a Pisot (resp. Salem) unit series of minimal polynomial $M_\beta(y) = y^d + A_{d-1}y^{d-1} + A_{d-2}y^{d-2} + \cdots + A_0$ such that $\sum_{0\le i\le d-2} A_i = -1$ (resp. $\sum_{0\le i\le d-1} A_i = -1$) and $(P,Q) \in \mathbb{F}_q[x]^2$ with $|P| < |Q|$. Then*

$$Per_\beta(\frac{P}{Q}) = s = \min\{\ n > 0\ ; \widetilde{H}^n(\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0})\}.$$

Proof. As we have the β-expansion of $\frac{P}{Q}$ is purely periodic of period s according to Theorem 3.4 and $Per_\beta(\frac{P}{Q})$ is a multiple of s, we conclude that

$$Per_\beta(\frac{P}{Q}) = s = \min\{\ n > 0\ ;\ \widetilde{H}^n(\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0})\}.$$ □

Corollary 3.7. *Let β be a Pisot (resp. Salem) unit series of minimal polynomial $M_\beta(y) = y^d + A_{d-1}y^{d-1} + A_{d-2}y^{d-2} + \cdots + A_0$ such that $\sum_{0\le i\le d-2} A_i = -1$ (resp. $\sum_{0\le i\le d-1} A_i = -1$) and $(P,Q) \in \mathbb{F}_q[x]^2$ with $|P| < |Q|$ and $P \wedge Q = 1$. Then*

$$Per_\beta(\frac{P}{Q}) = Per_\beta(\frac{1}{Q}).$$

Proof. Let $Per_\beta(\frac{P}{Q}) = \min\{\ n > 0\ ;\ \widetilde{H}^n(\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0})\}$ and since $P \wedge Q = 1$ then $\widetilde{P}$ is invertible in $\mathbb{F}_q[x]/Q\mathbb{F}_q[x]$. Hence,

$$\begin{aligned} &\min\{\ n > 0\ ;\ \widetilde{H}^n(\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{P}, \widetilde{0}, \ldots, \widetilde{0})\} \\ = &\min\{\ n > 0\ ;\ \widetilde{H}^n(\widetilde{1}, \widetilde{0}, \ldots, \widetilde{0}) = (\widetilde{1}, \widetilde{0}, \ldots, \widetilde{0})\}. \end{aligned}$$

From the well known properties of automorphisms of finite abelian groups we can now conclude the following statement

$$Per_\beta(\frac{P}{Q}) = Per_\beta(\frac{1}{Q}).$$ □

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