經營科學
第32卷 第2號
2015年 6月, pp.69-78
http://dx.doi.org/10.7737/KMSR.2015.32.2.069

대형할인매점의 요일별 고객 방문 수 분석 및 예측 : 베이지언 포아송 모델 응용을 중심으로*

김범수 · 이준겸†
서강대학교 경영학부

Estimating Heterogeneous Customer Arrivals to a Large Retail store : A Bayesian Poisson model perspective

Bumsoo Kim · Joonkyum Lee†
Sogang Business School, Sogang University

■ Abstract ■

This paper considers a Bayesian Poisson model for multivariate count data using multiplicative rates. More specifically we compose the parameter for overall arrival rates by the product of two parameters, a common effect and an individual effect. The common effect is composed of autoregressive evolution of the parameter, which allows for analysis on seasonal effects on all multivariate time series. In addition, analysis on individual effects allows the researcher to differentiate the time series by whatevercharacterization of their choice. This type of model allows the researcher to specifically analyze two different forms of effects separately and produce a more robust result. We illustrate a simple MCMC generation combined with a Gibbs sampler step in estimating the posterior joint distribution of all parameters in the model. On the whole, the model presented in this study is an intuitive model which may handle complicated problems, and we highlight the properties and possible applications of the model with an example, analyzing real time series data involving customer arrivals to a large retail store.

Keywords : Bayesian Analysis, Multivariate Poisson Model, Large Scale Problem, MCMC Method, Customer Arrival Model

논문접수일 : 2015년 01월 30일　　논문게재확정일 : 2015년 03월 28일
논문수정일 : 2015년 03월 26일
* We thank the three anonymous reviewers for their valuable comments and insights. This paper was supported by the Sogang University Research Grant of 201310041.01 and 201410042.01.
† 교신저자, jklee@sogang.ac.kr

1. Introduction

Arguably one of the most important discrete stochastic process which can be observed in nature, is the discrete-valued count related process. Its importance arises from the fact that the application of the process can be extended to many areas including but not restricted to business, economics, system reliability and engineering. Notably, problems involving number of customer arrivals to a store on a given day, number of system failures over a month and number of fatalities due to a disease in a given year are all examples of discrete count process.

Most of the literature investigating discrete count process utilizes the Poisson model as the basis-model, and developed various extensions for different research questions. Zeger [14] introduced quasi-likelihood methods in estimating model parameters of a time series regression involving discrete count data. Harvey and Fernandes [9] develop a recursionalgorithm similar to Kalman filter methods to construct the likelihood function. In their study, they assume a Gamma process on the stochastic evolution of the latent mean and extend their model to include explanatory variables as well as handling other count related models such as multinomial and binomial models. Wedel et al. [13] utilize discrete mixing distributions and formulate a finite mixture model for analyzing heterogeneous count data. Although these works enhance the understanding of the Poisson model in general, limitations exist as parameter estimation becomes extremely difficult for more complicated models where analytic form of the distribution function is not available. Furthermore, these limitations are especially highlighted in high-dimension multivariate time series analysis due to the difficulty in estimating the joint covariance matrix.

To circumvent the shortcomings of classic statistical methods, the stochastic analysis of count data has been also studied in the Bayesian literature. The model for Poisson panel data with multiple random effects has been developed by Chib et al. [7] via Markov Chain Monte Carlo (MCMC) methods, where they assess the model fit with Bayes factors. McCabe and Martin [11] present Bayesian first order autoregressive model for producing coherent forecasts of low count time series data. In business related literature, Manchanda and Chintagunta [10] use Bayesian hierarchical Poisson regression to investigate the effect of marketing activities on consumer behavior in the pharmaceutical industry and more specifically, they examine the number of sales calls directed to individual physicians from panel count data. All in all, through Bayesian MCMC methods, researchers started to analyze more complicated problems involving count data.

Still, the task of analyzing stochastic count data remains difficult and limited when it comes to multivariate count data, due to the interdependency between multiple time series. However, when studying real world phenomena, it is common to find these interdependencies and there is a need to study multivariate data. For example, when studying an epidemic over a region there could be an overall effect stemming from a common environment (i.e. region, common source of water) which works as the source of interdependency between individual counts. Therefore when interdependencies are present, separating the common effect and individual effects helps the researcher in analyzing the problem in more detail, but due to computational difficulties, literature on multivariate count data has been relatively scarce.

In this paper, we address the above problem with a Bayesian Poisson model for multivariate count data using multiplicative rates, meaning that we compose the parameter for overall arrival rates by the product of common effect and individual effect. It is a simple idea but, this type of formulation allows the researcher to specifically analyze two different forms of effects separately and produce a more robust result. Therefore, the Bayesian multivariate Poisson model presented in this paper contribute to the current literature on multivariate count models, by introducing a simple method for analyzing a complicated problem as well as being able to provide intuitive interpretation of results. We highlight the properties and possible applications of the Bayesian multivariate Poisson model with an example, analyzing real data involving customer arrivals to a retail store.

Rest of the paper is organized as follows. First we present the basic model studied in the literature regarding Bayesian Poisson model and show its multivariate extension in section 2. In section 3, we illustrate the use of the multivariate model with an example. Here, we describe the data studied and give the setting for the example in detail. We then conclude the paper with limitations and possible future research in section 4.

2. Bayesian Model for Discrete Count Data

2.1 The Basic Model

Framework for the basic Poisson stochastic model with dynamic evolution of the rates was proposed by Smith and Miller [12]. Here, the evolution of the Poisson rates follows a Markovian structure with an exponential likelihood function. Let us define X_t as the number of events occurring over a given time period t and let λ_t be the Poisson rate in the same time period. Unconditionally the occurrences $X_t^{'}s$ are correlated but given the Poisson rates λ_t one can assume conditional independence and model the occurrences with a non-homogeneous Poisson model.

$$\mathrm{p}(X_t \mid \lambda_t) = \frac{\lambda_t^{X_t} e^{-\lambda_t}}{X_t!} \tag{1}$$

The Markovian structure for the evolution of the Poisson rates is given as follows :

$$\lambda_t = \frac{\lambda_{t-1}}{\gamma}\epsilon_t \tag{2}$$

where $(\epsilon_t \mid H^{t-1}, \gamma) \sim Beta[\gamma\alpha_{t-1}, (1-\gamma)\alpha_{t-1}]$ with $H^{t-1} = \{X_1, X_2, \cdots, X_{t-1}\}$, $\alpha_{t-1} > 0$ and $0 < \gamma < 1$. Notice that one can think of H^{t-1} as the past history of the occurrences and γ as the discounting factor for the evolution process. By assuming $0 < \gamma < 1$, it is implied that there exists a stochastic ordering between two consecutive Poisson rates, $\gamma\lambda_t < \lambda_{t-1}$ and it can be shown that the conditional distributions of the consecutive rates are scaled beta densities given by :

$$\mathrm{p}(\lambda_t \mid \lambda_{t-1}, H^{t-1}, \gamma) \tag{3}$$
$$= \frac{\Gamma(\alpha_{t-1})}{\Gamma(\gamma\alpha_{t-1})\Gamma((1-\gamma)\alpha_{t-1})}\left(\frac{\gamma}{\lambda_{t-1}}\right)^{\alpha_{t-1}-1}\lambda_t^{\gamma\alpha_{t-1}-1}\left(\frac{\lambda_{t-1}}{\gamma} - \lambda_t\right)^{(1-\gamma)\alpha_{t-1}-1}$$

Further, if we assume that at the initial time period $(\lambda_0 \mid H_0)$ follows a gamma distribution with initial parameters (α_0, β_0) and by incorporating the state equations given above, one can analytically track the conditional sequential updating of the model.

By induction, one can show that if

$$(\lambda_{t-1} | H^{t-1}, \gamma) \sim Gamma(\alpha_{t-1}, \beta_{t-1}) \quad (4)$$

then by utilizing (3), the conditional distribution of $(\lambda_t | H^{t-1}, \gamma)$ as

$$(\lambda_t | H^{t-1}, \gamma) \sim Gamma(\gamma\alpha_{t-1}, \gamma\beta_{t-1}) \quad (5)$$

Now, by combining the prior given by (3), and the Poisson model shown in (1) one can obtain the distribution of $(\lambda_t | H^t, \gamma)$ using the Bayes' rule as

$$\text{p}(\lambda_t | H^t, \gamma) \propto \text{p}(X_t | \lambda_t)\,\text{p}(\lambda_t | H^{t-1}, \gamma) \quad (6)$$

which in more detail is

$$\text{p}(\lambda_t | H^t, \gamma) \propto \lambda_t^{\gamma\alpha_{t-1} + X_{t-1}} e^{-(\gamma\beta_{t-1}+1)} \quad (7)$$

Hence, the filtering distribution of λ_t at time t is a gamma density

$$(\lambda_t | H^t, \gamma) \sim Gamma(\alpha_t, \beta_t) \quad (8)$$

where the model parameters are updated by $\alpha_t = \gamma\alpha_{t-1} + X_t$ and $\beta_t = \gamma\beta_{t-1} + 1$. This updating algorithm highlights the benefits from Bayesian analysis in that, the Poisson rate, λ_t, is a function of the observed occurrences over time and should be considered as an evolving entity. Furthermore, one can easily utilize the properties found above in obtaining the one step ahead predictive distribution of counts. Note that by definition of distributions,

$$\text{p}(X_t | H^{t-1}, \gamma) = \int_0^\infty \text{p}(X_t | \lambda_t)\text{p}(\lambda_t | H^{t-1}, \gamma) d\lambda_t \quad (9)$$

where $(X_t | \lambda_t) \sim Poisson(\lambda_t)$ and $(\lambda_t | H^{t-1}, \gamma) \sim Gamma(\gamma\alpha_{t-1}, \gamma\beta_{t-1})$. Hence, the predictive distribution given by

$$\text{p}(X_t | H^{t-1}, \gamma) = \binom{\gamma\alpha_{t-1} + X_t - 1}{X_t} \left(1 - \frac{1}{\gamma\beta_{t-1}+1}\right)^{\gamma\alpha_{t-1}} \left(\frac{1}{\gamma\beta_{t-1}+1}\right)^{X_t} \quad (10)$$

is a negative binomial denoted as

$$(X_t | H^{t-1}, \gamma) \sim Negbin(u_t, v_t) \quad (11)$$

where $\mu_t = \gamma\alpha_{t-1}$ and $v_t = \dfrac{\gamma\beta_{t-1}}{\gamma\beta_{t-1}+1}$. By utilizing (11), one can find one step ahead predictions and compute any prediction intervals easily.

2.2 Multivariate Extension of the Bayesian Model for Counts

We now consider the multivariate extension of the basic model, where we analyze the properties of K different Poisson time series. Several possible extensions have been proposed by Aktekin and Soyer [1] where they consider multiplicative Poisson rates, and in this study we assume similar multiplicative rates for the K Poisson time series. This type of multiplicative multivariate Poisson model could be applied to a vast number of modeling problems. For example, consider the case of modeling customer arrivals to a retail store. In a given week, the arrival rates for each day of the week may be different as more customers are likely to visit the store on weekends compared to weekdays. In addition to the daily differences of the arrival rates, there can also be weekly changes to the arrival rates stemming from seasonality or changes in common economic environment.

In more detail, let us assume that

$$X_{it} \sim Poisson(\lambda_{it}), \text{ for } i = 1, 2, \cdots, K \quad (12)$$

where $\lambda_{it} = \lambda_i\theta_t$, with λ_i being the specific arrival rate for the i-th time series and θ_t is the common term affecting λ_i. For instance, consider X_{it} to be the number of customer visiting the store on a given i day of week t. In such case, λ_i is the day specific arrival rate (Monday, Tuesday, ⋯, Sunday) and θ_t is the effect of common economic environment affecting all customers at week t. Economic environment would be termed more favorable in cases where $\theta_t > \theta_{t-1}$, and less favorable if $\theta_t < \theta_{t-1}$.

We start presenting the multivariate model by first considering the assumptions of the basic model from the previous section. Assuming the Markovian structure, (first-order autoregressive form) for the evolution of the common environment rates, we modify (2) to

$$\theta_t = \frac{\theta_{t-1}}{\gamma}\epsilon_t \tag{13}$$

where $(\epsilon_t | H^{t-1}, \gamma, \lambda_1, \lambda_2, \cdots, \lambda_K) \sim \text{Beta}[\gamma\alpha_{t-1}, (1-\gamma)\alpha_{t-1}]$ with $\alpha_{t-1} > 0$, $0 < \gamma < 1$ and $H^{t-1} = \{X_{1(t-1),} X_{2(t-1),} \cdots, X_{K(t-1)}\}$ Note that the past history of the data, H^{t-1}, will now be a matrix of K time series up to time period t-1. In addition we assume that the specific arrival rates, λ_i's, follow a Gamma distribution and that they are independent of each other as well as the initial common rate, θ_0.

$$\lambda_i \sim Gamma(a_i, b_i), \ \text{for } i = 1, 2, \cdots, K \tag{14}$$

As in the basic model we assume that in the starting time period $(\theta_0 | H_0)$ follows a gamma distribution with initial parameters (α_0, β_0), then following a similar induction process we find that

$$(\theta_{t-1} | H^{t-1}, \gamma, \lambda_1, \lambda_2, \cdots, \lambda_K) \sim Gamma(\alpha_{t-1}, \beta_{t-1}) \tag{15}$$

and

$$(\theta_t | H^{t-1}, \gamma, \lambda_1, \lambda_2, \cdots, \lambda_K) \sim Gamma(\gamma\alpha_{t-1}, \gamma\beta_{t-1}) \tag{16}$$

The filtering distribution of θ_t at time t is a gamma density

$$(\theta_t | H^t, \gamma, \lambda_1, \lambda_2, \cdots, \lambda_K) \sim Gamma(\alpha_t, \beta_t) \tag{17}$$

where the model parameters are updated by $\alpha_t = \gamma\alpha_{t-1} + X_{1t} + X_{2t}, \cdots, + X_{Kt}$ and $\beta_t = \gamma\beta_{t-1} + \lambda_1 + \lambda_2 + \cdots + \lambda_K$. Notice that in the multivariate setting the filtering distribution takes advantage of the information obtained from each individual time series. Following a similar argument as the basic model we find the marginal distribution of a given X_{it} for any i as

$$p(X_{it} \mid H^{t-1}, \gamma, \lambda_i) = \binom{\gamma\alpha_{t-1} + X_{it} - 1}{X_{it}} \left(1 - \frac{\lambda_i}{\gamma\beta_{t-1} + \lambda_i}\right)^{\gamma\alpha_{t-1}} \left(\frac{\lambda_i}{\gamma\beta_{t-1} + \lambda_i}\right)^{X_{it}} \tag{18}$$

which is a negative binomial distribution as previously argued. Earlier we assumed that the individual arrival rates $\lambda_i's$ are independent of each other. Hence, the multivariate distribution of $X_{1t}, X_{2t,} \cdots, X_{Kt}'s$ is

$$p(X_{1t}, X_{2t}, \cdots, X_{Kt} | H^{t-1}, \gamma, \lambda_1, \lambda_2, \cdots, \lambda_K) = \frac{\Gamma(\gamma\alpha_{t-1} + \Sigma_i X_{it})}{\Gamma(\gamma\alpha_{t-1}) \Pi_i \Gamma(X_{it} + 1)} \Pi_i \left(\frac{\lambda_i}{\gamma\beta_{t-1} + \Sigma_i \lambda_i}\right)^{X_{it}} \left(\frac{\gamma\beta_{t-1}}{\gamma\beta_{t-1} + \Sigma_i \lambda_i}\right)^{\gamma\alpha_{t-1}} \tag{19}$$

which is a multivariate distribution of negative binomial. Notice that if we look at the bivariate case, the distribution in (19) simplifies to the one proposed by Arbous and Kerrich [4]. Other properties of similar multivariate models are illustrated in Aktekin et al. [2].

Estimation of the parameters is straightforward using the MCMC procedure, where one can take advantage of the forward filtering backward sampling (FFBS) suggested by Fruhwirth-Schnatter [8] with an added Gibbs sampler step for the λ_i's. The full conditional distributions of λ_i's are given by

$$p(\lambda_i \mid H^t, \theta_1, \theta_2, \cdots, \theta_t) \sim Gamma(a_{it}, b_{it}),$$
$$\text{for } i = 1, 2, \cdots, K \qquad (20)$$

where $a_{it} = a_i + X_{i1} + X_{i2} + \cdots + X_{it}$ and $b_{it} = b_i + \theta_1 + \theta_2 + \cdots + \theta_t$. Hence, by iteratively sampling from each of the conditional distributions one can find the full joint multivariate distribution of all model parameters given data, $(\theta_1, \theta_2, \cdots, \theta_t, \lambda_1, \lambda_2, \cdots, \lambda_K | H^t)$.

Following is the illustrative algorithm of the aforementioned Gibbs sampler steps for the bivariate case, where we have two related time series X_{1t} and X_{2t} for time periods t = 1, ⋯, T.

Illustrated Gibbs Algorithm

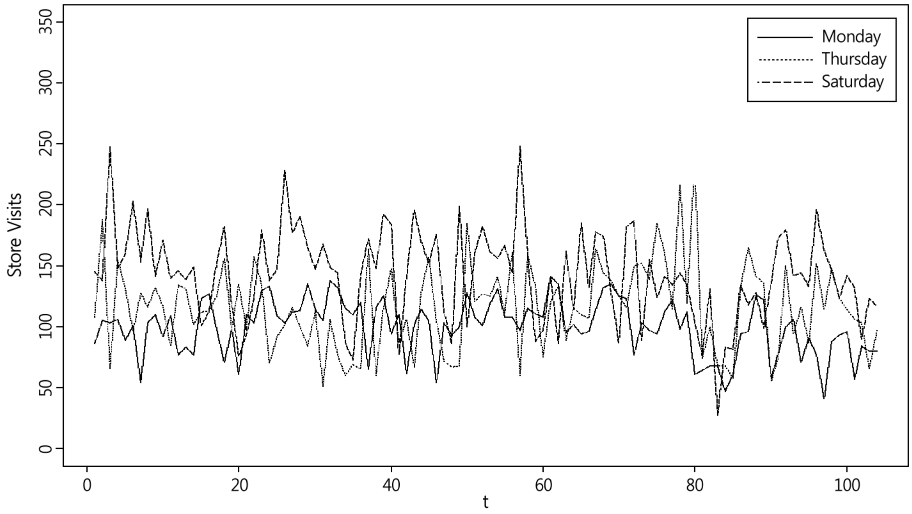
1. Initialize with arbitrary parameters $(\alpha_0^0, \beta_0^0, a_1^0, b_1^0, a_2^0, b_2^0\lambda_1^0, \lambda_2^0, \gamma)$
2. Let t = 1
3. Draw $(\theta_t^0 | H^t, \gamma, \lambda_1^0, \lambda_2^0) \sim Gamma(\alpha_t^0, \beta_t^0)$, where $\alpha_t^0 = \gamma\alpha_{t-1}^0 + X_{1t} + X_{2t}$ and $\beta_t^0 = \gamma\beta_{t-1}^0 + \lambda_1^0 + \lambda_2^0$
4. Draw $(\lambda_1^0 | H^t, \theta_1^0, \theta_2^0, \cdots, \theta_t^0) \sim Gamma(a_{1t}^0, b_{1t}^0)$, where $a_{1t}^0 = a_1^0 + X_{11} + X_{12} + \cdots + X_{1t}$ and $b_{1t}^0 = b_1^0 + \theta_1^0 + \theta_2^0 + \cdots + \theta_t^0$
5. Draw $(\lambda_2^0 | H^t, \theta_1^0, \theta_2^0, \cdots, \theta_t^0) \sim Gamma(a_{2t}^0, b_{2t}^0)$, where $a_{2t}^0 = a_2^0 + X_{21} + X_{22} + \cdots + X_{2t}$ and $b_{2t}^0 = b_2^0 + \theta_1^0 + \theta_2^0 + \cdots + \theta_t^0$
6. Move to step 7 if t = T otherwise return to step 3.
7. Save the updated parameters of the iteration $(\theta_1^1, \theta_2^1, \cdots, \theta_T^1, \lambda_1^1, \lambda_2^1)$

3. Application of the Bayesian Multivariate Model for Count Data

In this section, we show the implementation of the Bayesian multivariate model in business by analyzing a real world dataset. The dataset we use for our analysis is a marketing dataset collected by IRI market consulting firm. It includes variables such as product purchase, coupon usage, price of products and customer arrivals to a largegrocery store in the Chicago Metropolitan area in 2005, where 548 households were followed for a 2 year period, (104 weeks). Of the said dataset, we are interested in multivariate count data, i.e. customer arrivals, and hence we merged customer arrivals by the day of the week, i.e. Monday, Tuesday, ⋯, Sunday, and created 7 different Poisson time series for our Bayesian multivariate count analysis. In sum, we assume these 7 Poisson time series can be modeled by (12) meaning that, customer arrivals to the grocery store will be affected by the same environment, i.e. season, economic situation, climate and so on. In addition, we also assume that customer arrival rates will be different based on the day of the week.

<Figure 1> shows the weekly customer arrivals to the grocery store for Monday, Thursday and Saturday. First thing to note from <Figure 1> is that, there is a marked difference in the overall level of customer arrivals by day. For example, we can observe that customer arrivals are higher for Saturday compared to Monday and also the variation of arrivals is greater for Saturday than Monday. In addition, we also find that the three time series in <Figure 1> exhibit a similar weekly pattern over the observed 104 weeks. Notably, they all show a dip in the number of arrivals around week 80.

One of the most important problems in store operations is the problem of staffing. Based on the estimates and forecasts of customer arrivals, a store manager may plan the number of staff working on a given day and create a differentiated schedule for optimal staff allocations. The Bayesian multivariate model approach might be of interest to store managers who want to identify the weekly pattern or

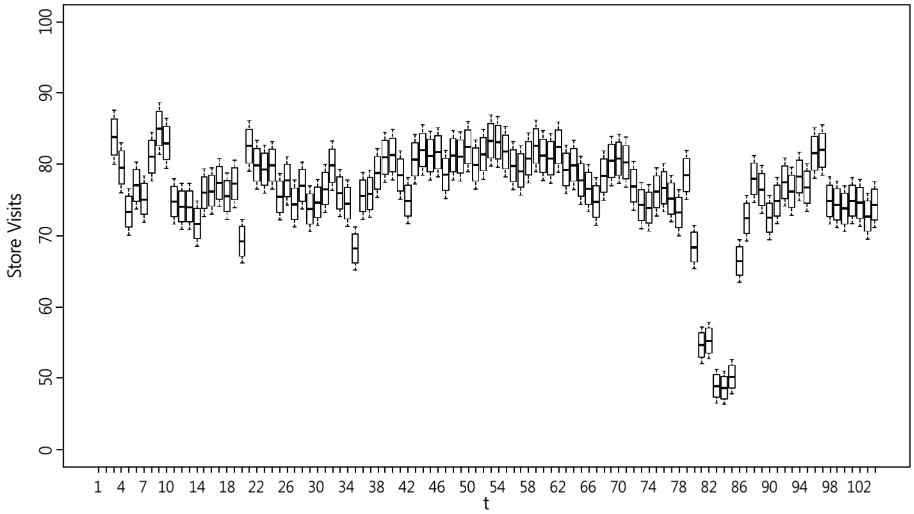


〈Figure 1〉 Weekly Time Series Plot of Customer Arrivals by Day

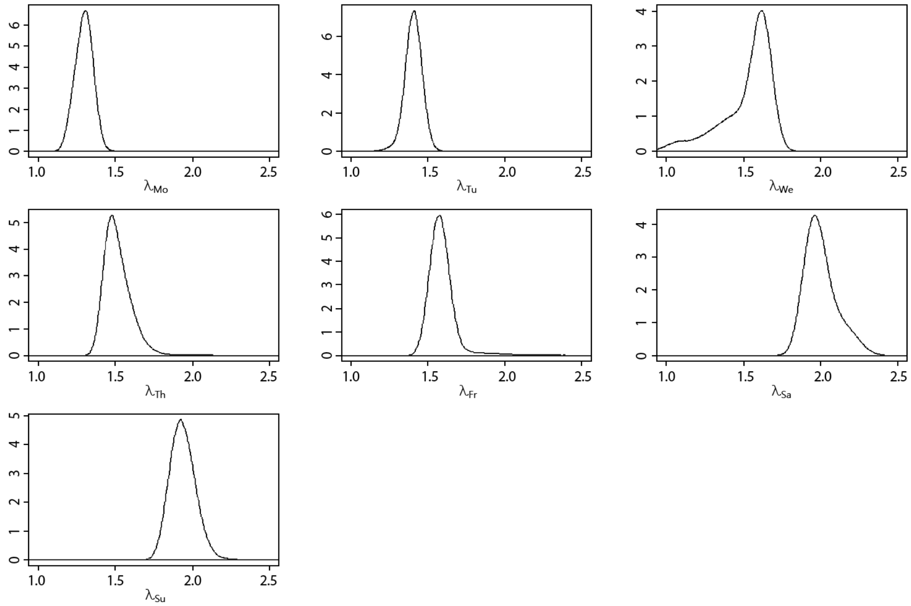
the common environmental effect or customer arrivals as well as separate individual effects for daily customer arrival characteristics. For our analysis, we fixed the discount factor at $\gamma = 0.5$, and used uninformed prior parameters at $a_{i0} = b_{i0} = 0.001$ where i represents each day of the week. Following the FFBS and Gibbs sampler techniques illustrated in the previous section we found the joint distribution of $p(\theta_1, \theta_2, \cdots, \theta_{104}, \lambda_1, \lambda_2, \cdots, \lambda_7 | H^{104})$. Note that these joint distribution results are obtained after 10,000 iterations with a burn in period of 2000, using a standard computer with R statistical package. For readers who are interested in the MCMC sampling and Gibbs technique, Casella and George [5], Albert and Chib [3] and Chib and Greenberg [6] all provide excellent introduction to the aforementioned methods.

First we present the boxplot of the common arrival rates θ_t's obtained from the joint distribution of $p(\theta_1, \theta_2, \cdots, \theta_{104}, | H^{104})$ in <Figure 2>. From the boxplot, one can observe the temporal behavior of the common rates θ_t's after taking into account individual effects of days given the data. As expected from the original time series plot in <Figure 1>, we observe a sudden drop of θ_t's for roughly 2 months in weeks 79~86. Due to the restrictions of our data we do not know the underlying cause for this drop however, from these results a store manager with necessary information may retrospectively study the reasons behind the sudden change in θ_t during this time period. Although the changes are not as significant, we also see that the customer arrival rates during the initial (1~34) and final time periods (72~104) are generally lower than the arrival rates in weeks 35~66. Again, in our study we did not account for common environment related covariates to analyze the reason behind the variations in θ_t's but, one can easily incorporate covariates such as economic climate and seasonal variables in the model for further analysis.

In <Figure 3>, we present the posterior distribution plots of individual rates, λ_i's. One can clearly observe three distinctive groupings for the individual arrival rates in <Figure 3>. We find the arrival rates to be the highest on the weekend, i.e. Saturday and Sunday, and the lowest at the start of the week, i.e. Monday and Tuesday. We also see



〈Figure 2〉 Boxplot of Weekly Common Arrival Rates, θ_t's



〈Figure 3〉 Posterior Density Plots of Individual Arrival Rates, λ_i's

an increasing trend of the individual arrival rates starting from Monday to Sunday. <Table 1> provides the summary statistics of posterior individual rates λ_i's. From the posterior summary statistics we can confirm the observed pattern in <Figure 3>. The mean arrival rate for Sunday is 1.927, which is more than 50% greater than the mean arrival rate for Monday. We also see that the variation of the posterior rates is less for Monday compared to the Weekends, suggesting that the arrival rates for the start of the week is not only low but pretty stable.

〈Table 1〉 Posterior Summary Statistics of Individual Rates λ_i

Day	Mean	St. Dev	5^{th} Percentile	95^{th} Percentile
Monday	1.292	0.067	1.191	1.389
Tuesday	1.401	0.076	1.304	1.497
Wednesday	1.516	0.181	1.116	1.711
Thursday	1.512	0.115	1.392	1.680
Friday	1.587	0.131	1.471	1.735
Saturday	1.992	0.153	1.846	2.214
Sunday	1.927	0.131	1.803	2.080

Store managers may utilize these findings in multiple of ways. First of all, the differences in individual arrival rates identify the busy days for store operation. The store manager could devise an efficient staff schedule by incorporating the findings from the Bayesian multivariate count data analysis. Furthermore, the store manager may devise some type of promotional strategy to lure customers into the store on days when the individual arrival rates are low. For example, we identified the individual arrival rates to be the lowest on Mondays and Tuesdays. Special storewide promotions, such as price discounts and promotional events, on these days may help boosting the number of customers visiting the store. As with the common arrival rates θ_t's, it is possible to extend the analysis and incorporate covariates for the individual arrival rates. This type of covariate analysis will give more insight on the underlying reasons for the heterogeneity in the individual arrival rates. All in all, the Bayesian multivariate analysis is beneficial for multivariate count data as it can clearly separate the common effects from individual effects, as well as taking advantages from the usual benefits from the Bayesian paradigm. One can easily incorporate expert opinion by varying the prior distribution parameters and the joint posterior distribution of all parameters allows the researcher to analyze the problem in more detail by asking probabilistic questions, such as $p(\lambda_1 > \lambda_2 | H^{104})$. To sum, the Bayesian multivariate analysis allows for a more robust analysis of the given problem.

4. Conclusion

In this study, we consider the Bayesian multivariate model for count data with common and individual effects on the arrival rates. We first presented the basic model for Bayesian analysis of univariate count data and showed the extension for the multivariate model and also showed an application of the model using real customer arrival data for a large retail store. In business or economic analysis, researchers are faced with many cases where there exists underlying common factor (economic climate, policy change, season effects etc.) affecting all observations as well as individual factors specific to certain groups of observations. It can be seen from the worked out example that, one can easily separate the temporal (or common) effect and the individual effect from utilizing the Bayesian multivariate model as well as providing a platform for joint probabilistic analysis of the parameters. From these properties, we firmly believe that the Bayesian multivariate model provides a tool for more robust and accurate analysis of stochastic count data.

There are possible extensions to this study. In the Bayesian multivariate analysis, we did not consider any covariates which may possibly explain certain patterns in both the common and the individual arrival rates. This problem was more due to the lack of available covariate data in our analysis, than a model issue. With certain assumptions, it is not too difficult to add covariates in the model and one can further the study of counts with more detailed dataset. Another possible extension may be related to relaxing the first order Markov assumptions we made on the evolution process, which will create potential challenges in the estimation of the parameters.

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