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OPTIMALITY CONDITIONS FOR OPTIMAL CONTROL GOVERNED BY BELOUSOV-ZHABOTINSKII REACTION MODEL

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Abstract. This paper is concerned with the optimality conditions for optimal control problem of Belousov-Zhabotinskii reaction model. That is, we obtain the optimality conditions by showing the differentiability of the solution with respect to the control. We also show the uniqueness of the optimal control.

1. Introduction

Belousov-Zhabotinskii (BZ) reaction model is known as a typical example of self-organization in the chemical reactions ([5]). In order to investigate the mechanics of the BZ reaction model which is considered to consist of more than ten elementary chemical reactions, Keener and Tyson [4] introduced the following model.

$$
(1.1)\qquad
\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} + \frac{1}{\epsilon^2}\Big[y(1-y) - c(\rho+u)\Big(\frac{y-q}{y+q}\Big)\Big] \qquad \text{in } I\times(0,T],\\
&\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + \frac{1}{\epsilon}(y-\rho) \qquad \text{in } I\times(0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial \rho}{\partial x}(0,t) = \frac{\partial \rho}{\partial x}(L,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } I.
\end{aligned}
$$

Here, $I = (0, L)$ is a bounded interval in $\mathbb{R}$. The variables $y(x,t)$ and $\rho(x,t)$ describe the concentrations of $HBrO_2$ and Ce^{4+} at $x \in I$ and a time $t \in [0,T]$, respectively. $a > 0$ and $b > 0$ represent the diffusion rate of each species. Finally, ϵ, q and c are positive constants where $0 < q < 1$ and $0 < \epsilon \le 1$. The control term $u(t)$ denotes a light induced bromide production rate to intensity of illumination at a time $t \in [0,T]$ ([9], [10], [11], [12]).

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In this paper we are concerned with the optimality conditions for the following optimal control problem:

$$(\mathbf{P}) \qquad \text{minimize} \;\; \mathcal{J}(u)$$

with the cost functional $\mathcal{J}(u)$ of the form

$$\begin{aligned}\mathcal{J}(u) = &\int_0^T \|y(u)-y_d\|^2_{H^1(I)}dt + \int_0^T \|\rho(u)-\rho_d\|^2_{H^1(I)}dt \\ &+ \gamma\|u\|^2_{H^1(0,T)}, \quad u \in H^1(0,T),\end{aligned}$$

where $y = y(u)$ and $\rho = \rho(u)$ is governed by (1.1).

The optimal control problem for the reaction diffusion model are studied in many papers. In [2], Garvie and Trenchea studied the optimal control problem for a nutrient-phytoplankton-zooplankton-fish system. Hoffman and Jiang [3] considered the optimal control problem of a phase field model for solidification. In particular, Ryu and Yagi [8] studied the optimal control problem for the chemotaxis model. Recently, Ryu [7] showed the existence of the global weak solution and the optimal control for (1.1). In this paper, we obtain the optimality conditions by showing the differentiability of the solution with respect to the control. We also show the uniqueness of the optimal control.

The paper is organized as follows. Section 2 is a preliminary section. In Section 3, we obtain the optimality conditions for the prolem (P). Section 4 show the uniqueness of the optimal control under the some assumption.

Notations. Let J be an interval in the real line $\mathbb{R}$. $L^p(J;\mathcal{H})$, $1 \le p \le \infty$, denotes the L^p space of measurable functions in J with values in a Hilbert space $\mathcal{H}$. $\mathcal{C}(J;\mathcal{H})$ denotes the space of continuous functions in J with values in $\mathcal{H}$. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by a, b, c, ϵ, γ, m, l and I. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

2. Preliminaries

We shall state some inequalities on the Sobolev spaces ([1]). When $s > \frac{1}{2}$, $H^s(I) \subset \mathcal{C}(\bar{I})$ with

$$\|\cdot\|_{\mathcal{C}} \le C_s\|\cdot\|_{H^s(I)}.$$

In particular, $H^1(I) \subset L^q(I)$ with

$$\|\cdot\|_{L^q(I)} \le C_{p,q}\|\cdot\|^r_{H^1(I)}\|\cdot\|^{1-r}_{L^p(I)}, \tag{2.1}$$

where $1 \le p < q \le \infty$, $r = \frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}+\frac{1}{2}}$.

We take the identification of $L^2(I)$ and $(L^2(I))'$ and consider that $H^1(I) \subset L^2(I) \subset (H^1(I))'$. Then, $L^{q'}(I) \subset (H^1(I))'$ for every $q' \in [1,\infty]$ with

$$\|y\|_{(H^1(I))'} \le C_{q'}\|y\|_{L^{q'}(I)}, \quad y \in L^{q'}(I). \tag{2.2}$$

We set three product Hilbert spaces $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$ as

$$\mathcal{V} = H^1(I) \times H^1(I), \quad \mathcal{H} = L^2(I) \times L^2(I), \quad \mathcal{V}' = (H^1(I))' \times (H^1(I))'.$$

We set also a symmetric bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$a(Y, \widetilde{Y}) = \big(A_1^{1/2} y, A_1^{1/2} \tilde{y}\big)_{L^2(I)} + \big(A_2^{1/2} y, A_2^{1/2} \tilde{y}\big)_{L^2(I)}, \; Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \; \widetilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V}$$

satisfying

$$|a(Y, \widetilde{Y})| \leq M \|Y\|_{\mathcal{V}} \|\widetilde{Y}\|_{\mathcal{V}}, \quad Y, \widetilde{Y} \in \mathcal{V},$$
$$a(Y, Y) \geq \delta \|Y\|_{\mathcal{V}}^2, \quad Y \in \mathcal{V}$$

with some δ and $M > 0$. Here, $A_1 = -a\frac{\partial^2}{\partial x^2} + 1$ and $A_2 = -b\frac{\partial^2}{\partial x^2} + 1$ with the same domain $\mathcal{D}(A_i) = H_n^2(I) = \{z \in H^2(I); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1, 2)$. Then this form defines a linear isomorphism $A = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$.

Then, (1.1) is formulated to the following abstract form

$$\text{(2.3)} \qquad \frac{dY}{dt} + AY = F_u(Y), \quad 0 < t \leq T, \qquad Y(0) = Y_0$$

in the space $\mathcal{V}'$. Here, $F_u(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F_u(Y) = \begin{pmatrix} y + \epsilon^{-2}\Big[y(1-y) - c(\rho + u)\Big(\frac{y-q}{y+q}\Big)\Big] \\ \epsilon^{-1} y + (1 - \epsilon^{-1})\rho \end{pmatrix}$$

and Y_0 is defined by $Y_0 = \left(\begin{smallmatrix} y_0 \\ \rho_0 \end{smallmatrix}\right)$. $U_{ad} = \{u \in H^1(0,T); \|u\|_{H^1(0,T)} \leq m, \; 0 \leq u(t) \leq l\}$ and $\mathcal{K} = \Big\{\left(\begin{smallmatrix} y_0 \\ \rho_0 \end{smallmatrix}\right) \in \mathcal{H}; 0 \leq y_0 \in L^2(I) \text{ and } 0 \leq \rho_0 \in L^2(I)\Big\}$.

Then, we obtain the following result ([7]).

Theorem 2.1. *For any $Y_0 \in \mathcal{K}$ and $u \in U_{ad}$,* (1.1) *has a unique global weak solution*

$$0 \leq Y \in H^1(0,T; \mathcal{V}') \cap \mathcal{C}([0,T]; \mathcal{H}) \cap L^2(0,T; \mathcal{V}),$$

equivalently,

$$0 \leq y \in H^1(0,T; (H^1(I))') \cap \mathcal{C}([0,T]; L^2(I)) \cap L^2(0,T; H^1(I)),$$
$$0 \leq \rho \in H^1(0,T; (H^1(I))') \cap \mathcal{C}([0,T]; L^2(I)) \cap L^2(0,T; H^1(I)).$$

Moreover, we also obtain the stability result with respect to the control.

Theorem 2.2. *For any $Y_0 \in \mathcal{K}$, let $Y_1 = \left(\begin{smallmatrix} y_1 \\ \rho_1 \end{smallmatrix}\right)$ and $Y_2 = \left(\begin{smallmatrix} y_2 \\ \rho_2 \end{smallmatrix}\right)$ be the solutions with respect to $u_1, u_2 \in U_{ad}$. Then, we have the following estimate*

$$\text{(2.4)} \qquad \|Y_1(t) - Y_2(t)\|_{L^\infty(0,T;\mathcal{H})}^2 \leq C \|u_1(t) - u_2(t)\|_{H^1(0,T)}^2$$

for all $t \in [0,T]$.

Proof. Let $\tilde{u} = u_1 - u_2$, $\tilde{y} = y_1 - y_2$ and $\tilde{\rho} = \rho_1 - \rho_2$. Then $\tilde{y}$ and $\tilde{\rho}$ satisfies the following:

$$
\begin{aligned}
&\frac{\partial \tilde{y}}{\partial t} + A_1\tilde{y} = \tilde{y} + \epsilon^{-2}\Big[\tilde{y}(1-\tilde{y}) - c(\rho_1 + u_1)\Big(\frac{2q\tilde{y}}{(y_1+q)(y_2+q)}\Big) \\
&\qquad\qquad - c(\tilde{\rho}+\tilde{u})\Big(\frac{y_2-q}{y_2+q}\Big)\Big], \\
&\frac{\partial \tilde{\rho}}{\partial t} + A_2\tilde{\rho} = \epsilon^{-1}\tilde{y} + (1-\epsilon^{-1})\tilde{\rho}, \qquad (2.5)\\
&\frac{\partial \tilde{y}}{\partial x}(0,t) = \frac{\partial \tilde{y}}{\partial x}(L,t) = \frac{\partial \tilde{\rho}}{\partial x}(0,t) = \frac{\partial \tilde{\rho}}{\partial x}(L,t) = 0, \\
&\tilde{y}(x,0) = 0, \quad \tilde{\rho}(x,0) = 0.
\end{aligned}
$$

Taking the scalar product with $\tilde{y}$ to the first equation of (2.5), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{y}\|^2_{L^2(I)} + \frac{\delta}{2}\|\tilde{y}\|^2_{H^1(I)} \qquad (2.6)\\
&\le C_\delta\big(\|\rho_1\|^2_{H^1(I)} + \|y_1\|^2_{H^1(I)} + \|y_2\|^2_{H^1(I)} + 1\big) \\
&\quad \times \big(\|\tilde{y}\|^2_{L^2(I)} + \|\tilde{\rho}\|^2_{L^2(I)}\big) + C|\tilde{u}|^2
\end{aligned}
$$

and taking the scalar product with $\tilde{\rho}$ to the second equation of (2.5), we have

$$
\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|^2_{L^2(I)} + \frac{\delta}{2}\|\tilde{\rho}\|^2_{H^1(I)} \le C\big(\|\tilde{y}\|^2_{L^2(I)} + \|\tilde{\rho}\|^2_{L^2(I)}\big). \qquad (2.7)
$$

From (2.6) and (2.7), we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\widetilde{Y}\|^2_{\mathcal{H}} + \frac{\delta}{2}\|\widetilde{Y}\|^2_{\mathcal{V}} \le C\big(\|Y_1\|^2_{\mathcal{V}} + \|Y_2\|^2_{\mathcal{V}} + 1\big)\|\widetilde{Y}\|^2_{\mathcal{H}} + C|\tilde{u}|^2.
$$

Using Gronwall's inequality, we obtain that

$$
\begin{aligned}
&\|\widetilde{Y}(t)\|^2_{\mathcal{H}} + \delta\int_0^t \|\widetilde{Y}(t)\|^2_{\mathcal{V}} ds \\
&\le C\|\tilde{u}\|^2_{L^2(0,T)} e^{\int_0^T C(\|Y_1(s)\|^2_{\mathcal{V}} + \|Y_2(s)\|^2_{\mathcal{V}} + 1)ds} \le C\|\tilde{u}\|^2_{H^1(0,T)}
\end{aligned}
$$

for all $t \in [0,T]$. Hence, we obtain the desired result. □

3. Optimality conditions for the optimal control

For each $u \in U_{ad}$, (2.3) has a unique weak solution $Y(u) \in H^1(0,T;\mathcal{V}') \cap \mathcal{C}([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$. Thus, the problem **(P)** is obviously formulated as follows:

$$
(\overline{\mathbf{P}}) \qquad \text{minimize } \mathcal{J}(u),
$$

where

$$
\mathcal{J}(u) = \int_0^T \|Y(u) - Y_d\|^2_{\mathcal{V}} dt + \gamma\|u\|^2_{H^1(0,T)}, \quad u \in U_{ad}.
$$

Here, $Y_d = \binom{y_d}{\rho_d}$ is a fixed element of $L^2(0,T;\mathcal{V})$ with $y_d, \rho_d \in L^2(0,T;H^1(I))$. γ is a positive constant.

Theorem 3.1 ([7])**.** *There exists an optimal control* $\bar{u} \in U_{ad}$ *for* ($\overline{\mathbf{P}}$) *such that* $\mathcal{J}(\bar{u}) = \min_{u \in U_{ad}} \mathcal{J}(u)$.

To derive the differentiability of $Y(u)$ with respect to the control u, we note that the mapping $F_u(\cdot) : \mathcal{V} \to \mathcal{V}'$ must be Fréchet differentiable with the derivative

$$F'_u(Y)Z = \begin{pmatrix} z_1 + \epsilon^{-2}\Big[z_1(1-2y) - cz_2\Big(\frac{y-q}{y+q}\Big) - c(\rho+u)\Big(\frac{2qz_1}{(y+q)^2}\Big)\Big] \\ \epsilon^{-1}z_1 + (1-\epsilon^{-1})z_2 \end{pmatrix},$$

where $Y = \binom{y}{\rho}$, $Z = \binom{z_1}{z_2} \in \mathcal{V}$.

Then, we have the following conditions.

Lemma 3.2. *For each* $\eta > 0$*, there exist increasing continuous functions* $\mu_\eta, \nu :$ $[0,\infty) \to [0,\infty)$ *such that for* $Y, \widetilde{Y}, Z, P \in \mathcal{V}$,

(f.iii)
$$|\langle F'_u(Y)Z, P\rangle_{\mathcal{V}',\mathcal{V}}| \le \begin{cases} \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + (\|Y\|_{\mathcal{V}}+1)\mu_\eta(\|Y\|_{\mathcal{H}})\|Z\|_{\mathcal{H}}\|P\|_{\mathcal{V}}, & \text{a.e. } (0,T), \\ \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + (\|Y\|_{\mathcal{V}}+1)\mu_\eta(\|Y\|_{\mathcal{H}})\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{H}}, & \text{a.e. } (0,T), \end{cases}$$

(f.iv)
$$\|F'_u(\widetilde{Y})Z - F'_u(Y)Z\|_{\mathcal{V}'} \le \nu(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}, \quad \text{a.e. } (0,T).$$

Proof. Indeed, by (2.1), (2.2), it is seen that

$$\begin{aligned} \|z_1(1-2y)\|_{(H^1(I))'} &\le C\|z_1(1-2y)\|_{L^2(I)} \\ &\le C\|z_1\|_{L^2(I)}(1 + \|y\|_{L^\infty(I)}) \\ &\le C\|z_1\|_{L^2(I)}(1 + \|y\|_{H^1(I)}), \quad y, z_1 \in H^1(I) \end{aligned}$$

and

$$\begin{aligned} \Big\|\rho\Big(\frac{2qz_1}{(y+q)^2}\Big)\Big\|_{(H^1(I))'} &\le C\|\rho z_1\|_{L^2(I)} \\ &\le C\|\rho\|_{L^\infty(I)}\|z_1\|_{L^2(I)} \\ &\le C\|\rho\|_{H^1(I)}\|z_1\|_{L^2(I)}, \quad y,\ \rho, z_1 \in H^1(I). \end{aligned}$$

Hence, the condition (f.iii) is fulfilled.

On the other hand, for $y, \bar{y}, \rho, \bar{\rho} \in H^1(I)$,

$$\begin{aligned} \Big\|z_2\Big(\frac{\tilde{y}-q}{\tilde{y}+q} - \frac{y-q}{y+q}\Big)\Big\|_{(H^1(I))'} &= \Big\|z_2\Big(\frac{2q(\tilde{y}-y)}{(\tilde{y}+q)(y+q)}\Big)\Big\|_{(H^1(I))'} \\ &\le C\|z_2\|_{L^\infty(I)}\|\tilde{y}-y\|_{L^2(I)} \\ &\le C\|z_2\|_{H^1(I)}\|\tilde{y}-y\|_{L^2(I)} \end{aligned}$$

and

$$\Big\|\tilde{\rho}\Big(\frac{2qz_1}{(\tilde{y}+q)^2}\Big) - \rho\Big(\frac{2qz_1}{(y+q)^2}\Big)\Big\|_{(H^1(I))'}$$

$$\le C\Big\|\tilde\rho\Big(\frac{2qz_1}{(\tilde y+q)^2}\Big)-\rho\Big(\frac{2qz_1}{(y+q)^2}\Big)\Big\|_{L^1(I)}$$
$$\le C\big(1+\|\rho\|_{L^2(I)}\big)\big(\|\tilde y-y\|_{L^2(I)}+\|\tilde\rho-\rho\|_{L^2(I)}\big)\|z_1\|_{H^1(I)},$$

with the use of (2.1) and (2.2). Hence, the condition (f.iv) is fulfilled. □

Proposition 3.3. *The mapping $u\to Y(u)$ from U_{ad} into $H^1(0,T;\mathcal V')\cap L^2(0,T;\mathcal V)$ is differentiable in the sense*

$$\frac{Y(u+hv)-Y(u)}{h}\to Z \text{ in } H^1(0,T;\mathcal V')\cap L^2(0,T;\mathcal V)$$

as $h\to 0$, where $u,v\in U_{ad}$ and $u+hv\in U_{ad}$. Moreover, $Z=Y'(u)v$ satisfies the linear equation

$$\text{(3.1)}\qquad \begin{aligned}&\frac{dZ}{dt}+AZ-F_u'(Y(u))Z=B_v(Y(u)),\qquad 0<t\le T,\\ &Z(0)=0,\end{aligned}$$

where $B_v(Y(u))=\binom{-\epsilon^{-2}cv\left(\frac{y-q}{y+q}\right)}{0}$.

Proof. Let $u,v\in U_{ad}$ and $0\le h\le 1$. Let $Y_h=Y(u_h)=\binom{y_h}{\rho_h}$ and $Y=Y(u)=\binom{y}{\rho}$ be the solutions of (2.3) corresponding to $u_h=u+hv$ and u, respectively.

Step 1. *$Y_h\to Y$ strongly in $\mathcal C([0,T];\mathcal H)$ as $h\to 0$.* As in the proof of Theorem 2.2, we obtain that

$$\|Y_h(t)-Y(t)\|^2_{\mathcal H}\le C\|u_h(t)-u(t)\|^2_{L^2(0,T)}\le Ch^2\|v(t)\|^2_{H^1(0,T)}$$

for all $t\in[0,T]$. Hence $Y_h\to Y$ strongly in $\mathcal C([0,T];\mathcal H)$ as $h\to 0$.

Step 2. *$\left\{\frac{Y_h-Y}{h}\right\}_{h>0}$ is bounded in $H^1(0,T;\mathcal V')\cap L^2(0,T;\mathcal V)$.* Let $\widetilde Y=\frac{Y_h-Y}{h}=\binom{\tilde y}{\tilde\rho}$. We consider

$$\text{(3.2)}\qquad \begin{aligned}&\frac{d\widetilde Y}{dt}+A\widetilde Y-\frac{F_{u_h}(Y_h)-F_u(Y)}{h}=0,\qquad 0<t\le T,\\ &\widetilde Y(0)=0.\end{aligned}$$

By (2.2) and $y_h\ge 0$, we have

$$\text{(3.3)}\qquad \Big\|\frac{F_{u_h}(Y_h)-F_u(Y_h)}{h}\Big\|_{\mathcal V'}=\Big\|\begin{pmatrix}-cv\left(\frac{y_h-q}{y_h+q}\right)\\0\end{pmatrix}\Big\|_{\mathcal V'}=\Big\|-cv\Big(\frac{y_h-q}{y_h+q}\Big)\Big\|_{(H^1(I))'}\le C|v|$$

and

$$\text{(3.4)}\qquad \Big\|\frac{F_u(Y_h)-F_u(Y)}{h}\Big\|_{\mathcal V'}$$
$$=\Big\|\begin{pmatrix}\tilde y+\epsilon^{-2}\Big[\tilde y(1-y_h-y)-c\tilde\rho\left(\frac{y_h-q}{y_h+q}\right)-c(\rho+u)\left(\frac{2q\tilde y}{(y_h+q)(y+q)}\right)\Big]\\ \epsilon^{-1}\tilde y+(1-\epsilon^{-1})\tilde\rho\end{pmatrix}\Big\|_{\mathcal V'}$$

$$\leq C(\|Y_h\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\|\widetilde{Y}\|_{\mathcal{H}}.$$

Taking the scalar product with $\widetilde{Y}$ to (3.2) and using (3.3), (3.4), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + \frac{\delta}{2}\|\widetilde{Y}(t)\|_{\mathcal{V}}^2 \leq C(\|Y_h(t)\|_{\mathcal{V}}^2 + \|Y(t)\|_{\mathcal{V}}^2 + 1)\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + C|v|^2.$$

Using Gronwall's inequality, we obtain that

$$\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + \delta\int_0^t \|\widetilde{Y}(t)\|_{\mathcal{V}}^2 ds \leq C\|v\|_{L^2(0,T)}^2 e^{\int_0^T C(\|Y_h(s)\|_{\mathcal{V}}^2 + \|Y(s)\|_{\mathcal{V}}^2 + 1)ds}$$

for all $t \in [0,T]$. Hence, $\frac{Y_h - Y}{h}$ is bounded in $H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$.

Step 3. $\frac{Y_h - Y}{h}$ *converges weakly to the solution* Z *of* (3.1) *in* $H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$ *as* $h \to 0$. Since the equation (3.1) is linear, we see from Lemma 3.2 that there exists a unique solution Z of (3.1). Also, we see from Step 2 that

$$\frac{Y_h - Y}{h} \to \overline{Z} \text{ weakly in } H^1(0,S;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$$

and

$$\frac{Y_h - Y}{h} \to \overline{Z} \text{ strongly in } L^2(0,T;\mathcal{H}) \tag{3.5}$$

as $h \to 0$. Let us verify that $\overline{Z} = \binom{\bar{z}_1}{\bar{z}_2}$ is a solution of (3.1). First, we show that for $\Psi = \binom{\psi_1}{\psi_2} \in L^2(0,T;\mathcal{V})$

$$\int_0^T \langle \frac{F_u(Y_h) - F_u(Y)}{h}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^T \langle F_u'(Y)\overline{Z}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \tag{3.6}$$

as $h \to 0$. Indeed, by direct calculation

$$\begin{aligned}
&\frac{F_u(Y_h) - F_u(Y)}{h} - F_u'(Y)\overline{Z} \\
&= \begin{pmatrix} \tilde{w}_1 + \epsilon^{-2}\Big[\tilde{w}_1(1 - y_h - y) - c\tilde{w}_2\Big(\frac{y_h - q}{y_h + q}\Big) - c(\rho + u)\Big(\frac{2q}{(y_h+q)(y+q)}\Big)\tilde{w}_1\Big] \\ \epsilon^{-1}\tilde{w}_1 + (1 - \epsilon^{-1})\tilde{w}_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} \epsilon^{-2}\Big[(y - y_h)\bar{z}_1 - c\bar{z}_2\Big(\frac{y_h - q}{y_h + q} - \frac{y - q}{y + q}\Big) - c(\rho + u)\Big(\frac{2q}{(y_h+q)(y+q)} - \frac{2q}{(y+q)^2}\Big)\bar{z}_1\Big] \\ 0 \end{pmatrix} \\
&= I_1 + I_2,
\end{aligned}$$

where $\tilde{w}_1 = \frac{y_h - y}{h} - \bar{z}_1$ and $\tilde{w}_2 = \frac{\rho_h - \rho}{h} - \bar{z}_2$. For $\psi_1 \in L^2(0,T;H^1(I))$,

$$\begin{aligned}
&\int_0^T \Big\langle \Big(\tilde{w}_1(1 - y_h - y), \psi_1\Big\rangle_{(H^1(I))',H^1(I)} dt \\
&\leq C\int_0^T (\|y_h\|_{L^2(I)} + \|y\|_{L^2(I)} + 1)\|\tilde{w}\|_{L^2(I)}\|\psi_1\|_{H^1(I)} dt \\
&\leq C\big(\|y_h\|_{L^\infty(0,T;L^2(I))} + \|y\|_{L^\infty(0,T;L^2(I))} + 1\big) \\
&\quad \times \|\tilde{w}\|_{L^2(0,T;L^2(I))}\|\psi_1\|_{L^2(0,T;H^1(I))}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \Big\langle \Big(\frac{2q\rho}{(y_h+q)(y+q)}\Big)\tilde{w}_1, \psi_1\Big\rangle_{(H^1(I))',H^1(I)} dt \\
&\le C\int_0^T \|\rho\|_{L^2(I)}\|\tilde{w}\|_{L^2(I)}\|\psi_1\|_{H^1(I)}dt \\
&\le C\|\rho\|_{L^\infty(0,T;L^2(I))}\|\tilde{w}\|_{L^2(0,T;L^2(I))}\|\psi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From (3.5), we see that $\|\tilde{w}\|_{L^2(0,T;L^2(I))} \to 0$. Thus, it is seen that $I_1 \to 0$ weakly in $L^2(0,T;\mathcal{V}')$ as $h \to 0$. Moreover, we have

$$\begin{aligned}
&\int_0^T \Big\langle \Big(\frac{2q\rho}{(y_h+q)(y+q)} - \frac{2q\rho}{(y+q)^2}\Big)\bar{z}_1, \psi_1\Big\rangle_{(H^1(I))',H^1(I)} dt \\
&\le C\int_0^T \int_0^L |\rho(y_h-y)\bar{z}_1|dx\|\psi_1\|_{H^1(I)}dt \\
&\le C\int_0^T \int_0^L |\rho(y_h-y)|dx\|\bar{z}_1\|_{L^\infty(I)}\|\psi_1\|_{H^1(I)}dt \\
&\le C\int_0^T \|\rho\|_{L^2(I)}\|y_h-y\|_{L^2(I)}\|\bar{z}_1\|_{H^1(I)}\|\psi_1\|_{H^1(I)}dt \\
&\le C\|\rho\|_{L^\infty(0,T;L^2(I))}\|y_h-y\|_{L^\infty(0,T;L^2(I))}\|\bar{z}_1\|_{L^2(0,T;H^1(I))}\|\psi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From Step 1, we see that $\|y_h - y\|_{L^\infty(0,T;L^2(I))} \to 0$. Thus, it is seen that $I_2 \to 0$ weakly in $L^2(0,T;\mathcal{V}')$ as $h \to 0$. Therefore, (3.6) is satisfied.

On the other hand, since

$$\begin{aligned}
&\int_0^T \Big\langle v\Big(\frac{y_h-q}{y_h+q} - \frac{y-q}{y+q}\Big), \psi_1\Big\rangle_{(H^1(I))',H^1(I)} dt \\
&\le C\|v\|_{L^2(I)}\|y_h-y\|_{L^\infty(0,T;L^2(I))}\|\psi_1\|_{L^2(0,T;H^1(I))},
\end{aligned}$$

we obtain that

$$\int_0^T \langle \frac{F_{u_h}(Y_h) - F_u(Y_h)}{h}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^T \langle B_v(Y), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

By the uniqueness, we see that $\overline{Z} = Z$. Hence, $\frac{Y_h - Y}{h}$ converges weakly to the unique solution Z of (3.1) in $H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$ as $h \to 0$.

Step 4. *$\frac{Y_h-Y}{h} \to Z$ strongly in $H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$ as $h \to 0$.* $\widetilde{W} = \frac{Y_h - Y}{h} - Z$ satisfies

$$\begin{aligned}
&\frac{d\widetilde{W}}{dt} + A\widetilde{W} - \Big(\frac{F_u(Y_h) - F_u(Y)}{h} - F_u'(Y)Z\Big) \\
&\qquad = \Big(\frac{F_{u_h}(Y_h) - F_u(Y_h)}{h} - B_v(Y)\Big), \qquad 0 < t \le T, \\
&\widetilde{W}(0) = 0.
\end{aligned} \tag{3.7}$$

By using (f.iii) and (f.iv), we obtain that

$$\begin{aligned}(3.8)\qquad &\Big\|\frac{F_u(Y_h)-F_u(Y)}{h}-F'_u(Y)Z\Big\|_{\mathcal{V}'}\\ &\le C(\|Y_h\|_{\mathcal{V}}+\|Y\|_{\mathcal{V}}+1)\|\widetilde{W}\|_{\mathcal{H}}\\ &\quad+\nu(\|Y_h\|_{\mathcal{H}}+\|Y\|_{\mathcal{H}})\|Y_h-Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}\ \text{a.e.}\ (0,T).\end{aligned}$$

And we have

$$\begin{aligned}(3.9)\qquad &\Big\|\frac{F_{u_h}(Y_h)-F_u(Y_h)}{h}-B_v(Y)\Big\|_{\mathcal{V}'}\\ &=\Big\|-cv\Big(\frac{y_h-q}{y_h+q}-\frac{y-q}{y+q}\Big)\Big\|_{(H^1(I))'}\\ &\le C|v|\|y_h-y\|_{L^2(I)}\le C\|Y_h-Y\|_{\mathcal{H}}\ \text{a.e.}\ (0,T).\end{aligned}$$

Taking the scalar product of the equation of (3.7) with $\widetilde{W}$ and using (3.8), (3.9), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{W}(t)\|^2_{\mathcal{H}}+\frac{\delta}{2}\|\widetilde{W}(t)\|^2_{\mathcal{V}}\\ &\le C(\|Y_h(t)\|^2_{\mathcal{V}}+\|Y(t)\|^2_{\mathcal{V}}+1)\|\widetilde{W}(t)\|^2_{\mathcal{H}}\\ &\quad+\nu(\|Y_h\|_{\mathcal{H}}+\|Y\|_{\mathcal{H}})^2(\|Z(t)\|^2_{\mathcal{V}}+1)\|Y_h(t)-Y(t)\|^2_{\mathcal{H}}.\end{aligned}$$

From Gronwall's Lemma,

$$\|\widetilde{W}(t)\|^2_{\mathcal{H}}+\delta\int_0^t\|\widetilde{W}(s)\|^2_{\mathcal{V}}ds\le C\|Y_h(t)-Y(t)\|^2_{L^\infty(0,T;\mathcal{H})}(\|Z\|^2_{L^2(0,T;\mathcal{V})}+1).$$

Since $Y_h\to Y$ strongly in $L^\infty(0,T;\mathcal{H})$, it follows that $\frac{Y_h-Y}{h}$ is strongly convergent to Z in $H^1(0,T;\mathcal{V}')\cap\mathcal{C}([0,T];\mathcal{H})\cap L^2(0,T;\mathcal{V})$. □

Theorem 3.4. *Let $\bar u$ be an optimal control of $(\overline{\mathbf{P}})$ and let $\overline{Y}=\binom{\bar y}{\bar\rho}\in L^2(0,T;\mathcal{V})\cap H^1(0,T;\mathcal{V}')$ be the optimal state, that is $\overline{Y}$ is the solution to* (2.3) *with the control $\bar u$. Then, there exists a unique solution $P=\binom{p_1}{p_2}\in L^2(0,T;\mathcal{V})\cap H^1(0,T;\mathcal{V}')$ to the linear problem*

$$\begin{aligned}(3.10)\qquad &-\frac{dP}{dt}+AP-F'_{\bar u}(\overline{Y})^*P=\Lambda(\overline{Y}-Y_d),\quad 0\le t<T,\\ &P(T)=0\end{aligned}$$

in $\mathcal{V}'$. Here, $\Lambda:\mathcal{V}\to\mathcal{V}'$ is a canonical isomorphism and

$$F'_u(\overline{Y})^*P=\begin{pmatrix}p_1+\epsilon^{-2}\Big[p_1(1-2\bar y)+\epsilon p_2-c(\bar\rho+\bar u)\Big(\frac{2qp_1}{(\bar y+q)^2}\Big)\Big]\\ -\epsilon^{-2}c\Big(\frac{\bar y-q}{\bar y+q}\Big)p_1+(1-\epsilon^{-1})p_2\end{pmatrix}.$$

Moreover, $\bar u$ satisfies

$$\int_0^T\langle P,B_{v-\bar u}(\overline{Y})\rangle_{\mathcal{V},\mathcal{V}'}dt+\gamma\langle\bar u,v-\bar u\rangle_{H^1(0,T)}\ge 0\quad\textit{for all }\ v\in U_{ad}.$$

Proof. Since $\mathcal{J}$ is Gâteaux differentiable at $\bar{u}$ and U_{ad} is convex, it is seen that

$$\mathcal{J}'(\bar{u})(v-\bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

By direct calculation, we obtain

$$\mathcal{J}'(\bar{u})(v-\bar{u}) = \int_0^T \langle \Lambda(\overline{Y}-Y_d), Z\rangle_{\mathcal{V}',\mathcal{V}} dt + \gamma\langle \bar{u}, v-\bar{u}\rangle_{H^1(0,T)}$$

with $Z = Y'(\bar{u})(v-\bar{u})$. Since the equation (3.10) is linear, we deduce from Lemma 3.2 that there exists a unique solution P of (3.10). Then, we have

$$\int_0^T \langle \Lambda(\overline{Y}-Y_d), Z\rangle_{\mathcal{V}',\mathcal{V}} dt = \int_0^T \langle P, B_{v-\bar{u}}(\overline{Y})\rangle_{\mathcal{V},\mathcal{V}'} dt$$

by using (3.1) and (3.10). Hence, we obtain

$$\int_0^T \langle P, B_{v-\bar{u}}(\overline{Y})\rangle_{\mathcal{V},\mathcal{V}'} dt + \gamma\langle \bar{u}, v-\bar{u}\rangle_{H^1(0,T)} \geq 0 \quad \text{for all } v \in U_{ad}.$$

□

4. Uniqueness of the optimal control

Suppose there exist two solutions u_1, u_2 to the optimal control problem $(\overline{\mathbf{P}})$.

Lemma 4.1. *Let P_1 and P_2 be the corresponding adjoint equation* (3.10) *to u_1 and u_2, respectively. Then, we have*

$$\|P_1(t)-P_2(t)\|^2_{L^2(0,T;\mathcal{V})} \leq C\|u_1(t)-u_2(t)\|^2_{H^1(0,T)}. \tag{4.1}$$

Proof. The proof is similar to that of [6, Lemma 4.2]. □

Theorem 4.2. *If γ is large enough, then there exists a unique solution to* $(\overline{\mathbf{P}})$.

Proof. By Theorem 3.4, we have

$$\int_0^T \langle P_1, B_{u_2-u_1}(Y_1)\rangle_{\mathcal{V},\mathcal{V}'} dt + \gamma\langle u_1, u_2-u_1\rangle_{H^1(0,T)} \geq 0, \tag{4.2}$$

$$\int_0^T \langle P_2, B_{u_1-u_2}(Y_2)\rangle_{\mathcal{V},\mathcal{V}'} dt + \gamma\langle u_2, u_1-u_2\rangle_{H^1(0,T)} \geq 0, \tag{4.3}$$

where P_1, Y_1 is the solution with respect to u_1 and P_2, Y_2 is the solution with respect to u_2.

By adding (4.2) and (4.3), we have

$$\gamma\|u_1-u_2\|^2_{H^1(0,T)} \leq \int_0^T \langle P_1-P_2, B_{u_2-u_1}(Y_1)\rangle_{\mathcal{V},\mathcal{V}'} dt + \int_0^T \langle P_2, B_{u_1-u_2}(Y_2) - B_{u_1-u_2}(Y_1)\rangle_{\mathcal{V},\mathcal{V}'} dt.$$

Since

$$\|B_{u_2-u_1}(Y_1)\|_{\mathcal{V}'} = \left\| -c(u_2-u_1)\left(\frac{y_1-q}{y_1+q}\right)\right\|_{(H^1(I))'} \leq C|u_1-u_2|$$

and

$$\begin{aligned}
&\|B_{u_1-u_2}(Y_2) - B_{u_1-u_2}(Y_1)\|_{\mathcal{V}'} \\
&= \Big\| - c(u_1 - u_2)\Big(\frac{y_2 - q}{y_2 + q} - \frac{y_1 - q}{y_1 + q}\Big)\Big\|_{(H^1(I))'} \\
&\le C|u_1 - u_2|\|y_2 - y_1\|_{L^2(I)} \le C|u_1 - u_2|\|Y_2 - Y_1\|_{\mathcal{H}},
\end{aligned}$$

we have

$$\begin{aligned}
\gamma\|u_1 - u_2\|^2_{H^1(0,T)} \le C\big(&\|P_1 - P_2\|_{L^2(0,T;\mathcal{V})}\|u_1 - u_2\|_{L^2(0,T)} \\
&+ \|P_2\|_{L^2(0,T;\mathcal{V})}\|u_1 - u_2\|_{L^2(0,T)}\|Y_1 - Y_2\|_{L^\infty(0,T;\mathcal{H})}.
\end{aligned}$$

By using (2.4) and (4.1), we obtain

$$\gamma\|u_1 - u_2\|^2_{H^1(0,T)} \le C\|u_1 - u_2\|^2_{H^1(0,T)}.$$

If γ is sufficiently large, we obtain the uniqueness of the optimal control. □

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