Commun. Korean Math. Soc. **30** (2015), No. 3, pp. 313–325
http://dx.doi.org/10.4134/CKMS.2015.30.3.313

H-*V*-SUPER MAGIC DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS

Solomon Stalin Kumar and Gurusamy Thevar Marimuthu

Abstract. An H-magic labeling in a H-decomposable graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that for every copy H in the decomposition, $\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is constant. f is said to be H-V-super magic if $f(V(G)) = \{1, 2, \ldots, p\}$. In this paper, we prove that complete bipartite graphs $K_{n,n}$ are H-V-super magic decomposable where $H \cong K_{1,n}$ with $n \geq 1$.

1. Introduction

In this paper we consider only finite and simple undirected bipartite graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$ respectively and we let $|V(G)| = p$ and $|E(G)| = q$. For graph theoretic notations, we follow [2, 3]. A labeling of a graph G is a mapping that carries a set of graph elements, usually vertices and/or edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [6].

Although magic labeling of graphs was introduced by Sedlacek [20], the concept of vertex magic total labeling (VMTL) first appeared in 2002 in [12]. In 2004, MacDougall et al. [13] introduced the notion of super vertex magic total labeling (SVMTL). In 1998, Enomoto et al. [5] introduced the concept of super edge-magic graphs. In 2005, Sugeng and Xie [21] constructed some super edge-magic total graphs. The usage of the word "super" was introduced in [5]. The notion of a V-super vertex magic labeling was introduced by MacDougall et al. [13] as in the name of super vertex-magic total labeling and it was renamed as V-super vertex magic labeling by Marr and Wallis in [16] after referencing the article [14]. Most recently, Tao-ming Wang and Guang-Hui Zhang [22], generalized some results found in [14].

A vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \ldots, p + q$ with the property that for every $u \in V(G)$, $f(u) +$

Received March 18, 2015.
2010 *Mathematics Subject Classification.* 05C78, 05C70.
Key words and phrases. H-decomposable graph, H-V-super magic labeling, complete bipartite graph.

$\sum_{v\in N(u)} f(uv) = k$ for some constant k, such a labeling is V-super if $f(V(G)) = \{1, 2, \dots, p\}$. A graph G is called V-super vertex magic if it admits a V-super vertex labeling. A vertex magic total labeling is called E-super if $f(E(G)) = \{1, 2, \dots, q\}$. A graph G is called E-super vertex magic if it admits a E-super vertex labeling. The results of the article [14] can also be found in [16]. In [13], MacDougall et al., proved that no complete bipartite graph is V-super vertex magic. An edge-magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \dots, p+q$ with the property that for any edge $uv \in E(G)$, $f(u) + f(uv) + f(v) = k$ for some constant k, such a labeling is super if $f(V(G)) = \{1, 2, \dots, p\}$. A graph G is called super edge-magic if it admits a super edge-magic labeling.

Most recently, Marimuthu and Balakrishnan [15], introduced the notion of super edge-magic graceful graphs to solve some kind of network problems. A (p, q) graph G with p vertices and q edges is edge magic graceful if there exists a bijection $f : V(G)\cup E(G) \to \{1, 2, \dots, p+q\}$ such that $|f(u)+f(v)-f(uv)| = k$, a constant for any edge uv of G. G is said to be super edge-magic graceful if $f(V(G)) = \{1, 2, \dots, p\}$.

A covering of G is a family of subgraphs $H_1, H_2, \dots, H_h$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $1 \le i \le h$. Then it is said that G admits an $(H_1, H_2, \dots, H_h)$ covering. If every H_i is isomorphic to a given graph H, then G admits an H-covering. A family of subgraphs $H_1, H_2, \dots, H_h$ of G is a H-decomposition of G if all the subgraphs are isomorphic to a graph H, $E(H_i)\cap E(H_j) = \emptyset$ for $i \ne j$ and $\cup_{i=1}^{h} E(H_i) = E(G)$. In this case, we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_h$ and G is said to be H-decomposable.

The notion of H-super magic labeling was first introduced and studied by Gutiérrez and Lladó [7] in 2005. They proved that some classes of connected graphs are H-super magic. Suppose G is H-decomposable. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \dots, p + q\}$ is called an H-magic labeling of G if there exists a positive integer k (called magic constant) such that for every copy H in the decomposition, $\sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e) = k$. A graph G that admits such a labeling is called a H-magic decomposable graph. An H-magic labeling f is called a H-V-super magic labeling if $f(V(G)) = \{1, 2, \dots, p\}$. A graph that admits a H-V-super magic labeling is called a H-V-super magic decomposable graph. An H-magic labeling f is called a H-E-super magic labeling if $f(E(G)) = \{1, 2, \dots, q\}$. A graph that admits a H-E-super magic labeling is called a H-E-super magic decomposable graph. The sum of all vertex and edge labels on H is denoted by $\sum f(H)$.

In 2007, Lladó and Moragas [11] studied the cycle-magic and cyclic-super magic behavior of several classes of connected graphs. They gave several families of C_r-magic graphs for each $r \ge 3$. In 2010, Ngurah, Salman and Susilowati [18] studied the cycle-super magic labeling of chain graphs, fans, triangle ladders, graph obtained by joining a star $K_{1,n}$ with one isolated vertex, grids and books. Maryati et al. [17] studied the H-super magic labeling of some graphs obtained from k isomorphic copies of a connected graph H. In 2012, Mania Roswitha and Edy Tri Baskoro [19] studied the H-super magic labeling for some trees such as a double star, a caterpillar, a firecracker and banana tree. In 2013, Toru Kojima [9] studied the C_4-super magic labeling of the Cartesian product of paths and graphs. In 2012, Inayah et al. [8] studied magic and anti-magic H-decompositions and Zhihe Liang [10] studied cycle-super magic decompositions of complete multipartite graphs. They are all called a H-magic labeling as a H-super magic if the smallest labels are assigned to the vertices. Note that an edge-magic graph is a K_2-magic graph.

In many of the results about H-magic graphs, the host graph G is required to be H-decomposable. Yoshimi Ecawa et al. [4] studied the decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components. The notion of star-subgraph was introduced by Akiyama and Kano in [1]. A subgraph F of a graph G is called a star-subgraph if each component of F is a star. Here by a star, we mean a complete bipartite graph of the form $K_{1,m}$ with $m \geq 1$. A subgraph F of a graph G is called a n-star-subgraph if $F \cong K_{1,n}$ with $2 \leq n < p$.

2. Main result

In this section, we consider the graphs $G \cong K_{n,n}$ and $H \cong K_{1,n}$, where $n \geq 2$. Clearly $p = 2n$ and $q = n^2$.

Theorem 2.1. *Suppose $\{H_1, H_2, \ldots, H_n\}$ is a n-star-decomposition of G. Then G is n-star-V-super magic decomposable with magic constant $\frac{n^3+6n^2+3n+2}{2}$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be two stable sets of G. Let $\{H_1, H_2, \ldots, H_n\}$ be a n-star decomposition of G, where each H_i is isomorphic to H, such that $V(H_i) = \{u_i, v_1, v_2, \ldots, v_n\}$ and $E(H_i) = \{u_iv_1, u_iv_2, \ldots, u_iv_n\}$ for all $1 \leq i \leq n$. Define a total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ by $f(u_i) = 2i$ and $f(v_i) = 2i-1$ for all $1 \leq i \leq n$.

Case 1: n is even.

Now the edges of G can be labeled as shown in Table 1.

We prove the result for $n = k$ and the result follows for all $1 \leq k \leq n$.

From Table 1 and from definition of f, we get

$$\begin{aligned}\sum f(H_k) &= f(u_k) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_kv_i)\\ &= 2k + (1+3+5+\cdots+(2n-1)) + (3n-(k-1)) + (3n+k)\\ &\quad + (5n-(k-1)) + (5n+k) + \cdots + ((n+1)n-(k-1))\\ &\quad + ((n+2)n-(k-1)).\end{aligned}$$

Now,

$$\sum_{i=1}^{n} f(v_i) = 1+3+5+\cdots+(2n-1)$$

TABLE 1. The edge label of a n-star-decomposition of G if n is even.

f	v_1	v_2	v_3	$\cdots$	v_{n-1}	v_n
u_1	$3n$	$3n+1$	$5n$	$\cdots$	$(n+1)n$	$(n+2)n$
u_2	$3n-1$	$3n+2$	$5n-1$	$\cdots$	$(n+1)n-1$	$(n+2)n-1$
u_3	$3n-2$	$3n+3$	$5n-2$	$\cdots$	$(n+1)n-2$	$(n+2)n-2$
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
u_k	$3n-(k-1)$	$3n+k$	$5n-(k-1)$	$\cdots$	$(n+1)n-(k-1)$	$(n+2)n-(k-1)$
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
u_{n-1}	$2n+2$	$4n-1$	$4n+2$	$\cdots$	$n(n)+2$	$(n+1)n+2$
u_n	$2n+1$	$4n$	$4n+1$	$\cdots$	$n(n)+1$	$(n+1)n+1$

$$
\begin{aligned}
&= (1+2+3+4+5+\cdots+(2n-1)) - (2+4+6+\cdots+(2n-2)) \\
&= \frac{(2n-1)(2n)}{2} - 2(1+2+\cdots+(n-1)) \\
&= 2n^2 - n - \frac{2(n-1)n}{2} \\
&= 2n^2 - n - n^2 + n \\
&= n^2.
\end{aligned}
$$

Also

$$
\begin{aligned}
\sum_{i=1}^{n} f(u_k v_i) &= (3n-(k-1)) + (3n+k) + (5n-(k-1)) + (5n+k) + \cdots \\
&\quad + ((n+1)n-(k-1)) + ((n+2)n-(k-1)) \\
&= (3n + (3n+1) + 5n + (5n+1) + \cdots + (n+1)n + (n+2)n) \\
&\quad - 2(k-1) \\
&= 2(3n+5n+7n+\cdots+(n-1)n) + \frac{n-2}{2}(1) \\
&\quad + n((n+1)+(n+2)) - 2(k-1) \\
&= 2n\{(1+2+3+\cdots+(n-1)) - (2+4+6+\cdots+(n-2)) - 1\} \\
&\quad + \frac{n-2}{2} + n(2n+3) - 2(k-1) \\
&= 2n\{\frac{n(n-1)}{2} - 2\frac{(\frac{n-2}{2})(\frac{n-2}{2}+1)}{2} - 1\} + \{\frac{n-2+4n^2+6n}{2}\} \\
&\quad - 2(k-1) \\
&= 2n\{\frac{n(n-1)}{2} - \frac{n(n-2)}{4} - 1\} + \{\frac{4n^2+7n-2}{2}\} - 2(k-1)
\end{aligned}
$$

TABLE 2. The edge label of a n-star-decomposition of G if n is odd.

f	v_1	v_2	v_3	$\cdots$	v_{n-1}	v_n
u_{n-1}	$2n+1$	$4n$	$4n+1$	$\cdots$	$n(n)+(\frac{n+3}{2})$	$(n+1)n+2$
u_{n-3}	$2n+2$	$4n-1$	$4n+2$	$\cdots$	$n(n)+(\frac{n+5}{2})$	$(n+1)n+4$
u_{n-5}	$2n+3$	$4n-2$	$4n+3$	$\cdots$	$n(n)+(\frac{n+7}{2})$	$(n+1)n+6$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_k	$2n+1$ $+(\frac{n-(k+1)}{2})$	$4n$ $-(\frac{n-(k+1)}{2})$	$4n+1$ $+(\frac{n-(k+1)}{2})$	$\cdots$	$n(n)$ $+(n+1)-\frac{k}{2}$	$(n+2)n$ $-(k-1)$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_2	$2n+(\frac{n-1}{2})$	$3n+(\frac{n+3}{2})$	$4n+(\frac{n-1}{2})$	$\cdots$	$(n+1)n$	$(n+2)n-1$
u_n	$2n+(\frac{n+1}{2})$	$3n+(\frac{n+1}{2})$	$4n+(\frac{n+1}{2})$	$\cdots$	$(n)n+1$	$(n+1)n+1$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_j	$3n$ $-(\frac{(j-1)}{2})$	$3n+1$ $+(\frac{(j-1)}{2})$	$5n$ $-(\frac{(j-1)}{2})$	$\cdots$	$n(n)$ $+(\frac{n+1}{2}-\frac{(j-1)}{2})$	$(n+1)n$ $+(n-(j-1))$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_3	$3n-1$	$3n+2$	$5n-1$	$\cdots$	$n(n)+(\frac{n+1}{2}-1)$	$(n+2)n-2$
u_1	$3n$	$3n+1$	$5n$	$\cdots$	$n(n)+(\frac{n+1}{2})$	$(n+2)n$

$$\begin{aligned}
&= 2n\{\frac{2n(n-1)-n(n-2)-4}{4}\}+\{\frac{4n^2+7n-2}{2}\}-2(k-1)\\
&= n\{\frac{2n^2-2n+2n-n^2-4}{2}\}+\{\frac{4n^2+7n-2}{2}\}-2(k-1)\\
&= n\{\frac{n^2-4}{2}\}+\{\frac{4n^2+7n-2}{2}\}-2(k-1)\\
&= \{\frac{n^3-4n+4n^2+7n-2}{2}\}-2(k-1)\\
&= \{\frac{n^3+4n^2+3n-2}{2}\}-2(k-1).
\end{aligned}$$

Using the above values, we get

$$\begin{aligned}
\sum f(H_k) &= 2k+n^2+\{\frac{n^3+4n^2+3n-2}{2}\}-2(k-1)\\
&= 2+n^2+\{\frac{n^3+4n^2+3n-2}{2}\}\\
&= \frac{n^3+6n^2+3n+2}{2}.
\end{aligned}$$

Thus in this case the graph G is a n-star-V-super magic decomposable graph.

Case 2: n is odd.

Now the edges of G can be labeled as shown in Table 2.

Subcase(i): i is odd, where $1 \leq i \leq n$.

We prove the result for $i = j$ and the result follows for all $1 \leq i \leq n$ and i is odd. From Table 2 and from definition of f, we get

$$\sum f(H_j) = f(u_j) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_j v_i) = 2j + n^2 + \sum_{i=1}^{n} f(u_j v_i).$$

Now,

$$\begin{aligned}
\sum_{i=1}^{n} f(u_j v_i) &= (3n - \frac{(j-1)}{2}) + (3n + 1 + \frac{(j-1)}{2}) + (5n - \frac{(j-1)}{2}) \\
&\quad + (5n + 1 + \frac{(j-1)}{2}) + \cdots + (n(n) - \frac{(j-1)}{2}) \\
&\quad + (n(n) + (\frac{(n+1)}{2} - \frac{(j-1)}{2})) + ((n+2)n - (j-1)) \\
&= (3n + (3n+1) + 5n + (5n+1) + \cdots + (n-2)n \\
&\quad + ((n-2)n + 1)) + (n^2 + (n^2 + \frac{n+1}{2}) + (n+2)n) - 2(j-1) \\
&= (2n(3 + 5 + 7 + \cdots + (n-2))) + \frac{n-3}{2}(1) \\
&\quad + (n^2 + (n^2 + \frac{n+1}{2}) + (n+2)n) - 2(j-1) \\
&= (2n(3 + 5 + 7 + \cdots + (n-2))) + \frac{6n^2 + 4n + n + 1 + n - 3}{2} \\
&\quad - 2(j-1) \\
&= (2n((1 + 2 + 3 + \cdots + (n-2)) - (2 + 4 + 6 + \cdots + (n-3)) - 1)) \\
&\quad + \frac{6n^2 + 6n + -2}{2} - 2(j-1) \\
&= (2n(\frac{(n-2)(n-1)}{2} - 2\frac{(\frac{n-3}{2})(\frac{n-3}{2} + 1)}{2} - 1)) + (3n^2 + 3n - 1) \\
&\quad - 2(j-1) \\
&= (2n(\frac{n^2 - 3n + 2}{2} - \frac{n^2 - 4n + 3}{4} - 1)) + (3n^2 + 3n - 1) \\
&\quad - 2(j-1) \\
&= \frac{n^3 - 2n^2 - 3n + 6n^2 + 6n - 2}{2} - 2(j-1) \\
&= \frac{n^3 + 4n^2 + 3n - 2}{2} - 2(j-1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum f(H_j) &= 2j + n^2 + \{\frac{n^3 + 4n^2 + 3n - 2}{2}\} - 2(j-1) \\
&= 2 + n^2 + \{\frac{n^3 + 4n^2 + 3n - 2}{2}\}
\end{aligned}$$

$$= \frac{n^3 + 6n^2 + 3n + 2}{2}.$$

which is same as in Case 1. So in this case the graph G is a n-star-V-super magic decomposable graph.

Subcase(ii): i is even, where $1 \leq i \leq n$.

We prove the result for $i = k$ and the result follows for all $1 \leq i \leq n$ and i is even. From Table 2 and from definition of f, we get

$$\begin{aligned}\sum f(H_k) &= f(u_k) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_k v_i)\\ &= 2k + n^2 + \sum_{i=1}^{n} f(u_k v_i).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{i=1}^{n} f(u_k v_i) &= (2n + 1 + \frac{(n-(k+1))}{2}) + (4n - \frac{(n-(k+1))}{2})\\ &\quad + (4n + 1 + \frac{(n-(k+1))}{2}) + (6n - \frac{(n-(k+1))}{2}) + \cdots\\ &\quad + (n(n) + (n+1) - \frac{k}{2}) + ((n+2)n - (k-1))\\ &= (2n + \frac{(n-(k-1))}{2}) + (3n + \frac{(n+(k+1))}{2})\\ &\quad + (4n + \frac{(n-(k-1))}{2}) + (5n + \frac{(n-(k+1))}{2}) + \cdots\\ &\quad + ((n-1)n + \frac{(n-(k-1))}{2}) + ((n+1)n - \frac{(k-2)}{2})\\ &\quad + ((n+2)n - 1 - (k-2))\\ &= (2n + 3n + 4n + \cdots + (n-1)n)\\ &\quad + (\frac{n-3}{2})(\frac{n-(k-1)}{2} + \frac{n+(k+1)}{2})\\ &\quad + \frac{n-(k-1)}{2} - \frac{(k-2)}{2} - (k-2) + (n+1)n + ((n+2)n - 1)\\ &= (2n + 3n + 4n + \cdots + (n-1)n) + (\frac{n-3}{2})(\frac{2n+2}{2}) + (\frac{n-1}{2})\\ &\quad + (n+1)n + ((n+2)n - 1) - \frac{(k-2)}{2} - \frac{(k-2)}{2} - (k-2)\\ &= (n(\frac{(n-1)n}{2} - 1)) + \frac{(n-3)(n+1)}{2} + (\frac{n-1}{2})\\ &\quad + n(2n+3) - 1 - 2(k-2)\\ &= \frac{n^3 - n^2 - 2n + n^2 - 2n - 3 + n - 1}{2} + 2n^2 + 3n - 1 - 2(k-2)\end{aligned}$$

TABLE 3. The edge label of a H-decomposition of G.

f	v_1	v_2	v_3	v_4	v_5
u_4	11	20	21	29	32
u_2	12	19	22	30	34
u_5	13	18	23	26	31
u_3	14	17	24	27	33
u_1	15	16	25	28	35

$$= \frac{n^3 - 3n - 4 + 4n^2 + 6n - 2}{2} - 2(k-2)$$
$$= \frac{n^3 + 4n^2 + 3n - 2}{2} - 2 - 2(k-2)$$
$$= \frac{n^3 + 4n^2 + 3n - 2}{2} - 2(k-1).$$

Thus,

$$\sum f(H_k) = 2k + n^2 + \{\frac{n^3 + 4n^2 + 3n - 2}{2}\} - 2(k-1)$$
$$= 2 + n^2 + \{\frac{n^3 + 4n^2 + 3n - 2}{2}\}$$
$$= \frac{n^3 + 6n^2 + 3n + 2}{2}.$$

which is same as in Case 1. So in this case the graph G is a n-star-V-super magic decomposable graph. □

The following example illustrates Theorem 2.1.

Example 2.2. Consider the graphs $G \cong K_{5,5}$ and $H \cong K_{1,5}$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $W = \{v_1, v_2, v_3, v_4, v_5\}$ be two stable sets of G such that $V(G) = U \cup W$. Let $\{H_1, H_2, H_3, H_4, H_5\}$ be a H-decomposition of G, where each H_i is isomorphic to H, such that $V(H_i) = \{u_i, v_1, v_2, v_3, v_4, v_5\}$ and $E(H_i) = \{u_iv_1, u_iv_2, u_iv_3, u_iv_4, u_iv_5\}$, for all $1 \leq i \leq 5$. Define a total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, 35\}$ by $f(u_i) = 2i$ and $f(v_i) = 2i - 1$, for all $1 \leq i \leq 5$. Let us label the edges of G as shown in Table 3.

We find $\sum f(H_3)$ to illustrate Subcase (i) of Theorem 2.1.

Using Table 3 and from the definition of f, we have

$$\sum f(H_3) = f(u_3) + \sum_{i=1}^{5} f(v_i) + \sum_{i=1}^{5} f(u_3v_i)$$
$$= 2(3) + 5^2 + \sum_{i=1}^{5} f(u_3v_i).$$

Now,

$$\begin{aligned}
\sum_{i=1}^{5} f(u_3v_i) &= (3(5) - \frac{(3-1)}{2}) + (3(5) + 1 + \frac{(3-1)}{2}) + (5(5) - \frac{(3-1)}{2}) \\
&\quad + (5(5) + (\frac{(5+1)}{2} - \frac{(3-1)}{2})) + ((5+2)5 - (3-1)) \\
&= (3(5) + (3(5)+1)) + (5^2 + (5^2 + \frac{5+1}{2}) + (5+2)5) - 2(3-1) \\
&= (2(5)(3)) + \frac{5-3}{2}(1) + (5^2 + (5^2 + \frac{5+1}{2}) + (5+2)5) - 2(3-1) \\
&= (2(5)(3)) + \frac{6(5^2) + 6(5) + 1 - 3}{2} - 2(3-1) \\
&= (2(5)((1+2+3) - (2) - 1)) + \frac{6(5^2) + 6(5) - 2}{2} - 2(3-1) \\
&= (2(5)(\frac{(5-2)(5-1)}{2} - 2\frac{(\frac{5-3}{2})(\frac{5-3}{2}+1)}{2} - 1)) \\
&\quad + (3(5^2) + 3(5) - 1) - 2(3-1) \\
&= (2(5)(\frac{5^2 - 3(5) + 2}{2} - \frac{5^2 - 4(5) + 3}{4} - 1)) \\
&\quad + (3(5^2) + 3(5) - 1) - 2(3-1) \\
&= \frac{5^3 - 2(5^2) - 3(5) + 6(5^2) + 6(5) - 2}{2} - 2(3-1) \\
&= \frac{5^3 + 4(5^2) + 3(5) - 2}{2} - 2(3-1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum f(H_3) &= 2(3) + 5^2 + \{\frac{5^3 + 4(5^2) + 3(5) - 2}{2}\} - 2(3-1) \\
&= 2 + 5^2 + \{\frac{5^3 + 4(5^2) + 3(5) - 2}{2}\} \\
&= \frac{5^3 + 6(5^2) + 3(5) + 2}{2}.
\end{aligned}$$

In a similar way we can show that, $\sum f(H_1) = \sum f(H_5) = \frac{5^3+6(5^2)+3(5)+2}{2} = 146$.

We find $\sum f(H_4)$ to illustrate Subcase (ii) of Theorem 2.1.

Using Table 3 and from the definition of f, we have

$$\begin{aligned}
\sum f(H_4) &= f(u_4) + \sum_{i=1}^{5} f(v_i) + \sum_{i=1}^{5} f(u_4v_i) \\
&= 2(4) + 5^2 + \sum_{i=1}^{5} f(u_4v_i).
\end{aligned}$$

Now,

$$\begin{aligned}\sum_{i=1}^{5} f(u_4v_i) &= (2(5)+1+\frac{(5-(4+1))}{2}) + (4(5)-\frac{(5-(4+1))}{2})\\
&\quad + (4(5)+1+\frac{(5-(4+1))}{2}) + (5(5)+(5+1)-\frac{4}{2})\\
&\quad + ((5+2)5-(4-1))\\
&= (2(5)+\frac{(5-(4-1))}{2}) + (3(5)+\frac{(5+(4+1))}{2})\\
&\quad + ((5-1)(5)+\frac{(5-(4-1))}{2}) + ((5+1)(5)-\frac{(4-2)}{2})\\
&\quad + ((5+2)(5)-1-(4-2))\\
&= (2(5)+3(5)+(5-1)(5)) + (\frac{5-3}{2})(\frac{5-(4-1)}{2}+\frac{5+(4+1)}{2})\\
&\quad + \frac{5-(4-1)}{2} - \frac{(4-2)}{2} - (4-2) + (5+1)(5) + ((5+2)(5)-1)\\
&= (2(5)+3(5)+(5-1)(5)) + (\frac{5-3}{2})(\frac{2(5)+2}{2}) + (\frac{5-1}{2})\\
&\quad + (5+1)(5) + ((5+2)(5)-1) - \frac{(4-2)}{2} - \frac{(4-2)}{2} - (4-2)\\
&= ((5)(\frac{(5-1)(5)}{2}-1)) + \frac{(5-3)(5+1)}{2} + (\frac{5-1}{2})\\
&\quad + (5)(2(5)+3) - 1 - 2(4-2)\\
&= \frac{5^3-5^2-2(5)+5^2-2(5)-3+5-1}{2}\\
&\quad + 2(5^2)+3(5)-1-2(4-2)\\
&= \frac{5^3-3(5)-4+4(5^2)+6(5)-2}{2} - 2(4-2)\\
&= \frac{5^3+4(5^2)+3(5)-2}{2} - 2 - 2(4-2)\\
&= \frac{5^3+4(5^2)+3(5)-2}{2} - 2(4-1).\end{aligned}$$

Thus,

$$\begin{aligned}\sum f(H_4) &= 2(4)+5^2+\{\frac{5^3+4(5^2)+3(5)-2}{2}\} - 2(4-1)\\
&= 2+5^2+\{\frac{5^3+4(5^2)+3(5)-2}{2}\}\\
&= \frac{5^3+6(5^2)+3(5)+2}{2}.\end{aligned}$$

In a similar way we can show that, $\sum f(H_2) = \frac{5^3+6(5^2)+3(5)+2}{2} = 146$.

So the graph G is a H-V-super magic decomposable graph.

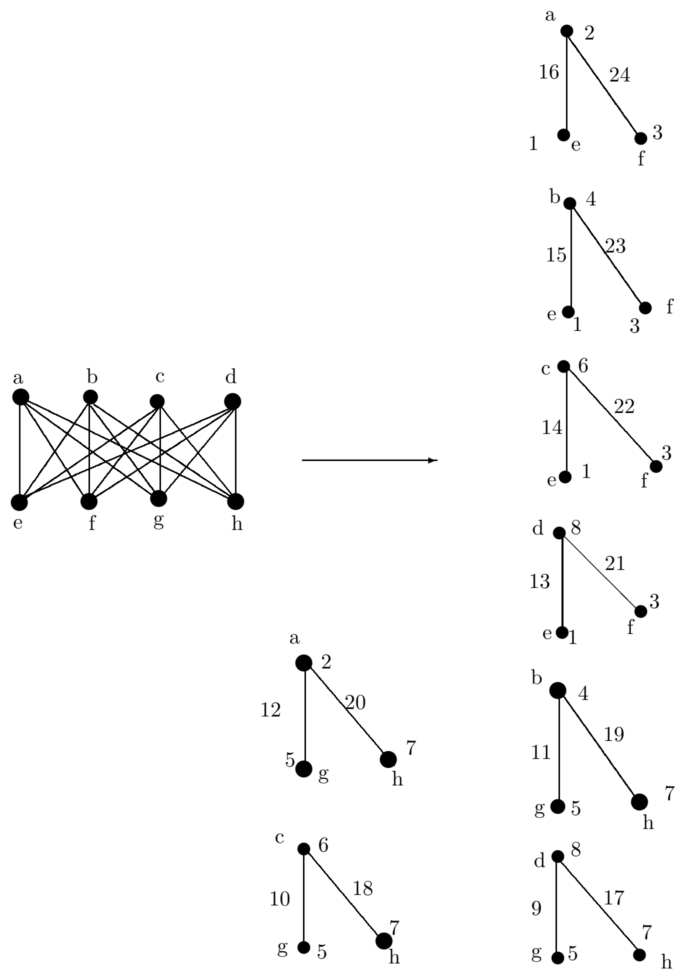

FIGURE 1. 2-star-V-super magic decomposition of $K_{4,4}$

3. Conclusion

In this paper, we studied the n-star-V-super magic decomposition of $K_{n,n}$ with $n \geq 1$. Figure 1 shows that $K_{4,4}$ is a 2-star-V-super magic decomposable graph. Let $U = \{a, b, c, d\}$ and $W = \{e, f, g, h\}$ be two stable sets of $K_{4,4}$ such that $V(G) = U \cup W$. Let $\{H_1 = \{(a, e), (a, f)\}$, $H_2 = \{(b, e), (b, f)\}$, $H_3 = \{(c, e), (c, f)\}$, $H_4 = \{(d, e), (d, f)\}$, $H_5 = \{(a, g), (a, h)\}$, $H_6 = \{(b, g), (b, h)\}$,

$H_7 = \{(c, g), (c, h)\}$, $H_8 = \{(d, g), (d, h)\}\}$ be a H-decomposition of $K_{4,4}$, where each H_i is isomorphic to $H \cong K_{1,2}$ for all $1 \leq i \leq 8$.

It is natural to have the following problem.

Open Problem 3.1. Discuss the m-star-V-super magic decomposition of $K_{n,n}$ with $1 \leq m < n$.

References

[1] J. Akiyama and M. Kano, *Path Factors of a Graph*, Graphs and Applications, Wiley, Newyork, 1985.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd Edition, Chapman and Hall, Boca Raton, London, New-York, Washington, D.C., 1996.

[3] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, Chapman and Hall, CRC, Boca Raton, 2009.

[4] Y. Egawa, M. Urabe, T. Fukuda, and S. Nagoya, *A decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components*, Discrete Math. **58** (1986), no. 1, 93–95.

[5] H. Emonoto, Anna S Lladó, T. Nakamigawa, and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34** (1998), 105–109.

[6] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **16** (2013), $\#DS6$.

[7] A. Gutiérrez and A. Lladó, *Magic coverings*, J. Combin. Math. Combin. Comput. **55** (2005), 43–56.

[8] N. Inayah, A. Lladó, and J. Moragas, *Magic and antimagic H-decompositions*, Discrete Math. **312** (2012), no. 7, 1367–1371.

[9] T. Kojima, *On C_4-Supermagic labelings of the Cartesian product of paths and graphs*, Discrete Math. **313** (2013), no. 2, 164–173.

[10] Z. Liang, *Cycle-supermagic decompositions of complete multipartite graphs*, Discrete Math. **312** (2012), no. 22, 3342–3348.

[11] A. Lladó and J. Moragas, *Cycle-magic graphs*, Discrete Math. **307** (2007), no. 23, 2925–2933.

[12] J. A. MacDougall, M. Miller, Slamin, and W. D. Wallis, *Vertex-magic total labelings of graphs*, Util. Math. **61** (2002), 3–21.

[13] J. A. MacDougall, M. Miller, and K. Sugeng, *Super vertex-magic total labeling of graphs*, Proc. 15th AWOCA (2004), 222–229.

[14] G. Marimuthu and M. Balakrishnan, *E-super vertex magic labelings of graphs*, Discrete Appl. Math. **160** (2012), no. 12, 1766–1774.

[15] ———, *Super edge magic graceful graphs*, Inform. Sci. **287** (2014), 140–151.

[16] A. M. Marr and W. D. Wallis, *Magic Graphs*, 2nd edition, Birkhauser, Boston, Basel, Berlin, 2013.

[17] T. K. Maryati, A. N. M. Salman, E. T. Baskoro, J. Ryan, and M. Miller, *On H-Supermagic labeling for certain shackles and amalgamations of a connected graph*, Util. Math. **83** (2010), 333–342.

[18] A. A. G. Ngurah, A. N. M. Salman, and L. Susilowati, *H-Supermagic labeling of graphs*, Discrete Math. **310** (2010), no. 8, 1293–1300.

[19] M. Roswitha and E. T. Baskoro, *H-Magic covering on some classes of graphs*, AIP Conf. Proc. **1450** (2012), 135–138.

[20] J. Sedlacek, *Problem 27*, Theory of Graphs and its Applications, 163–167, Proceedings of Symposium Smolenice, 1963.

[21] K. A. Sugeng and W. Xie, *Construction of Super edge magic total graphs*, Proc. 16th AWOCA (2005), 303–310.

[22] T.-M. Wang and G.-H. Zhang, *Note on E-super vertex magic graphs*, Discrete Appl. Math. **178** (2014), 160–162.

Solomon Stalin Kumar
Department of Mathematics
The American College
Madurai 625-002, India
E-mail address: sskumbas@gmail.com

Gurusamy Thevar Marimuthu
Department of Mathematics
The Madura College
Madurai 625-011, India
E-mail address: yellowmuthu@yahoo.com