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ON MAXIMAL, MINIMAL OPEN AND CLOSED SETS

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Abstract. We obtain some conditions for disconnectedness of a topological space in terms of maximal and minimal open sets, and some similar results in terms of maximal and minimal closed sets along with interrelations between them. In particular, we show that if a space has a set which is both maximal and minimal open, then either this set is the only nontrivial open set in the space or the space is disconnected. We also obtain a result concerning a minimal open set on a subspace.

1. Introduction

Nakaoka and Oda [1] introduced and studied the concept of minimal open sets (Definition 2.4) in a topological space. Dualizing the concept of minimal open sets, Nakaoka and Oda [2] introduced and studied the idea of maximal open sets (Definition 2.2). Thereafter, as consequences of maximal and minimal open sets, Nakaoka and Oda [3] introduced and studied notions of maximal and minimal closed sets. Nakaoka and Oda [3], also obtained some interrelations among four concepts: maximal open sets, minimal open sets, maximal closed sets, minimal closed sets. In this paper, along with some other properties, we study a topological space in which a set may have double nature such as a set which is both maximal open and minimal open e.g. Theorem 2.8 and we show that the space may be disconnected.

For a subset A of a topological space $(X, \mathscr{T})$, $Cl(A)$ denotes the closure of A with respect to the topological space $(X, \mathscr{T})$. Sometimes the topological space $(X, \mathscr{T})$ is simply denoted by X. By a proper open set of a topological space X, we mean an open set $G \neq \emptyset, X$ and by a proper closed set, we mean a closed set $E \neq \emptyset, X$. For a topological space $(X, \mathscr{T})$ and $A \subset X$, we write $(A, \mathscr{T}_A)$ to denote the subspace on A of $(X, \mathscr{T})$. Throughout the paper, R denotes the set of real numbers.

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2. Maximal and minimal open sets

In this paper, we obtain some results on disconnectedness of a topological space and hence we recall the following definition of disconnectedness (see Willard [4]).

Definition 2.1. A topological space X is disconnected if there exist disjoint nonempty open sets G and H such that $G \cup H = X$.

Definition 2.2 (Nakaoka and Oda [2])**.** A proper nonempty open subset U of X is said to be a maximal open set if any open set which contains U is X or U.

Theorem 2.3 (Nakaoka and Oda [2])**.** *If U is a maximal open set and W is an open set, then either $U \cup W = X$ or $W \subset U$.*

Definition 2.4 (Nakaoka and Oda [1])**.** A nonempty open subset U of X is said to be a minimal open set if any open set which is contained in U is U or $\emptyset$.

Theorem 2.5 (Nakaoka and Oda [1])**.** *If U is a minimal open set and W is an open set, then either $U \cap W = \emptyset$ or $U \subset W$.*

Theorem 2.6. *If G is a maximal open set and H is a minimal open set of a topological space X, then either $H \subset G$ or the space is disconnected.*

Proof. Using the maximality of G by Theorem 2.3, we get either $G \cup H = X$ or $H \subset G$. Using the minimality of H by Theorem 2.5, we get $G \cap H = \emptyset$ or $H \subset G$. The case $G \cup H = X, H \subset G$ implies $G = X$ and the case $H \subset G, G \cap H = \emptyset$ implies $H = \emptyset$. So only feasible cases are $G \cup H = X, G \cap H = \emptyset$ and $H \subset G$. If $G \cup H = X, G \cap H = \emptyset$, then the space is disconnected. □

Remark 2.7. $G \cup H = X, G \cap H = \emptyset$ imply $G = X - H$. In Theorem 2.6, if $H \not\subset G$, then G and H both are also closed. Theorem 2.6 may be stated as follows: if G is a maximal open set and H is a minimal open set of a topological space X, then either $H \subset G$ or $G = X - H$.

Theorem 2.8. *If a topological space X has a set which is both maximal and minimal open, then either this set is the only nontrivial open set in the space or the space is disconnected.*

Proof. Let G be both maximal and minimal open, and H be any open set. Then we get $G \subset G \cup H$. By the maximality of G, we have the following two cases.

Case I: $G = G \cup H$. Then we get $H \subset G$. Since G is minimal, we have $H = \emptyset$ or $H = G$.

Case II: $G \cup H = X$. Considering G as a minimal open set, we get by Theorem 2.5, $G \cap H = \emptyset$ or $G \subset H$. Since G is maximal, $G \subset H$ implies $G = H$ or $H = X$.

Considering all cases, we get $G = H$ or $G \cup H = X$ and $G \cap H = \emptyset$. If $G \cup H = X$ and $G \cap H = \emptyset$, then the space is disconnected. □

It is trivial that if a topological space X has only one proper open set, then that set is both maximal and minimal open. If there are only two proper open sets in a space and the open sets are disjoint, then both are maximal and minimal. If G and H are only two proper open sets in a topological space such that $G \subset H$, then G is a minimal open set and H is a maximal open set in the space. However, there may not exist a set which is both maximal and minimal open in a disconnected space (see Example 2.11).

Corollary 2.9. *If G is both maximal and minimal open, and E is a closed set in a topological space X, then either $G = X - E$ or $G = E$.*

Proof. Given G is both maximal and minimal open, and E is a closed set. So $X - E$ is an open set. Proceeding like the proof of Theorem 2.8, we get $G = X - E$ or $G \cup (X - E) = X$ and $G \cap (X - E) = \emptyset$. $G \cup (X - E) = X$ and $G \cap (X - E) = \emptyset$ imply $G = E$. □

Corollary 2.10. *If G is both maximal and minimal open in a topological space X, then either G is the only proper open set in the space or proper open sets of the space are G and $X - G$ only.*

Proof. Let H be any proper open set of the space. Proceeding like the proof of Theorem 2.8, we get $G = H$ or $G \cup H = X$ and $G \cap H = \emptyset$. $G \cup H = X$ and $G \cap H = \emptyset$ imply $H = X - G$. □

Example 2.11. For $a \in R$, we define

$$\mathscr{T} = \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a], [a, \infty)\}.$$

The topological space $(R, \mathscr{T})$ is disconnected with a separation

$$((-\infty, a), [a, \infty)).$$

But the space has no open set which is both maximal and minimal open.

Theorem 2.12. *If A and B are two different maximal open sets of a topological space X with $A \cap B$ is a closed set, then X is disconnected.*

Proof. Since A and B are maximal, we have $A \cup B = X$. We put $G = A - A \cap B, H = B$ or $G = A, H = B - A \cap B$. We note that G, H are disjoint open sets with $G \cup H = X$. So X is disconnected. □

Theorem 2.13 (Nakaoka and Oda [2]). *If U is a maximal open set, then either $Cl(U) = X$ or $Cl(U) = U$.*

Theorem 2.14. *If there exists a maximal open set which is not dense in a topological space, then the space is disconnected.*

Proof. Let U be a maximal open set which is not dense in X. By Theorem 2.13, $U = Cl(U)$. We write $G = U$ and $H = X - Cl(U)$. So (G, H) is a separation for X. □

3. Maximal and minimal closed sets

Definition 3.1 (Nakaoka and Oda [3]). A proper nonempty closed subset E of X is said to be a maximal closed set if any closed set which contains E is X or E.

Theorem 3.2 (Nakaoka and Oda [3]). *If E is a maximal closed set and F is any closed set, then either $E \cup F = X$ or $F \subset E$.*

Definition 3.3 (Nakaoka and Oda [3]). A proper nonempty closed subset E of X is said to be a minimal closed set if any closed set which is contained in E is E or $\emptyset$.

Theorem 3.4 (Nakaoka and Oda [3]). *If E is a minimal closed set and F is any closed set, then either $E \cap F = \emptyset$ or $E \subset F$.*

By Theorem 2.5 and Theorem 3.4, we note that if G is both minimal open and minimal closed, and E is clopen, then either $G \subset E$ or $G \cap E = \emptyset$.

Analogous to Theorem 2.6, Theorem 2.8, Corollary 2.9 and Corollary 2.10, we have Theorem 3.5, Theorem 3.6, Corollary 3.7 and Corollary 3.8 respectively. Since proofs of theorems and corollaries are similar to the proof of corresponding theorems and corollaries already establish, we omit proofs.

Theorem 3.5. *If E is a maximal closed set and F is a minimal closed set of a topological space X, then either $F \subset E$ or $E \cup F = X$, $E \cap F = \emptyset$.*

Theorem 3.6. *If a topological space X has a set E which is both maximal and minimal closed, then either of the following is true.*

(i) *E is the only proper closed set in the space.*

(ii) *If there exists another proper closed set F, then $E \cup F = X$ and $E \cap F = \emptyset$.*

Corollary 3.7. *If E is both maximal and minimal closed, and G is an open set in a topological space X, then either $E = X - G$ or $E = G$.*

Corollary 3.8. *If E is both maximal and minimal closed in a topological space X, then either E is the only proper closed set in the space or proper closed sets of the space are E and $X - E$ only.*

It is trivial that if a topological space X has only one proper closed set, then that set is both maximal and minimal closed. If there are only two proper closed sets in a space and the closed sets are disjoint, then both are maximal and minimal closed. If E and F are only two proper closed sets in a topological space such that $F \subset E$, then F is a minimal closed and E is a maximal closed set in the space. In the topological space $(R, \mathscr{T})$ of Example 2.11, there exist disjoint proper closed sets $[a, \infty)$ and $(-\infty, a)$ such that $[a, \infty) \cup (-\infty, a) = R$. But no proper closed set in the space is both maximal and minimal closed. So we conclude that there may exist closed sets E, F in X such that $E \cup F = X$ and $E \cap F = \emptyset$ but there may not exists a set which is both maximal and minimal closed.

Theorem 3.9. *If G is both maximal open and minimal closed, H is open and E is closed, then either of the following is true.*

(i) $H \subset G \subset E$.
(ii) $H \subset G$ *and* $G \cap E = \emptyset$.
(iii) $G \cup H = X$ *and* $G \subset E$.
(iv) $G \cup H = X, G \cap E = \emptyset$.

Proof. Considering G as a maximal open set, By Theorem 2.3 we get $H \subset G$ or $G \cup H = X$. Considering G as a minimal closed set, by Theorem 3.4, $G \subset E$ or $G \cap E = \emptyset$. $H \subset G$ and $G \subset E$ imply $H \subset G \subset E$. The remaining probable combinations are $H \subset G$, $G \cap E = \emptyset$; $G \cup H = X$, $G \subset E$ and $G \cup H = X$, $G \cap E = \emptyset$. □

Corollary 3.10. *If G is both maximal open and minimal closed, then G and $X - G$ are only proper clopen sets in the space.*

Proof. Let E be clopen in X. Putting $H = E$ in Theorem 3.9, we get $G = E$ or $G = X - E$. □

Theorem 3.11. *If G is both maximal open and maximal closed, and E is clopen, then either $E \subset G$ or $G \cup E = X$.*

Proof. Similar to the proof of Theorem 3.9. □

Corresponding to Theorem 3.9 and Corollary 3.10, we have Theorem 3.12 and Corollary 3.13 respectively. The proofs of the theorem and the corollary are omitted as they are similar to proofs already establish.

Theorem 3.12. *If G is both minimal open and maximal closed, H is open and E is closed, then either of the following is true.*

(i) $E \subset G \subset H$.
(ii) $G \subset H, G \cup E = X$.
(iii) $G \cap H = \emptyset, E \subset G$.
(iv) $G \cup E = X, G \cap H = \emptyset$.

Corollary 3.13. *If G is both minimal open and maximal closed, then G and $X - G$ are only proper clopen sets in the space.*

Theorem 3.14. *Let A, G be open sets in X such that $A \cap G \neq \emptyset, A$. Then $A \cap G$ is a minimal open set in $(A, \mathscr{T}_A)$ if G is a minimal open set in $(X, \mathscr{T})$.*

Proof. If $A \cap G$ is not a minimal open set in $(A, \mathscr{T}_A)$, there exists an open set $U \neq \emptyset$ in $(A, \mathscr{T}_A)$ such that $U \subsetneqq A \cap G$. Since G is a minimal open set in X and $A \cap G \neq \emptyset$, we have by Theorem 2.5, $G \subset A$ which implies $A \cap G = G$. A being open in X, U is also open in X. So we get a set U open in X such that $\emptyset \neq U \subsetneqq G$ which is a contradiction to our assumption that G is a minimal open set in X. □

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