

A REMARK ON THE CONJUGATION IN THE STEENROD ALGEBRA

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ABSTRACT. We investigate the Hopf algebra conjugation, χ , of the mod 2 Steenrod algebra, \mathcal{A}_2 , in terms of the Hopf algebra conjugation, χ' , of the mod 2 Leibniz–Hopf algebra. We also investigate the fixed points of \mathcal{A}_2 under χ and their relationship to the invariants under χ' .

1. Introduction

The mod 2 Steenrod algebra \mathcal{A}_2 is the free associative graded algebra generated by the *Steenrod squares* Sq^n [18] of degree n , $n \geq 1$, over \mathbb{F}_2 subject to the *Adem relations*

$$Sq^a Sq^b = \sum_{s=0}^{\lfloor a/2 \rfloor} \binom{b-1-s}{a-2s} Sq^{a+b-s} Sq^s \quad \text{for } 0 < a < 2b.$$

Conventionally, $Sq^0 = 1$, the multiplicative identity. Topologically, \mathcal{A}_2 is the algebra of stable cohomology operations for ordinary cohomology H^* over \mathbb{F}_2 . A monomial in \mathcal{A}_2 can be written in the form $Sq^{j_1} Sq^{j_2} \cdots Sq^{j_r}$, which we shall denote by $Sq^{j_1 \cdot j_2 \cdots j_r}$. Admissible monomials form a vector space basis “admissible basis” for \mathcal{A}_2 . Milnor [16] determined the graded connected Hopf algebra structure of \mathcal{A}_2 by a cocommutative coproduct given by $\Delta(Sq^n) = \sum Sq^i \otimes Sq^{n-i}$. Being a connected graded Hopf algebra, \mathcal{A}_2 inherits the canonical anti-automorphism, called *conjugation* (or antipode), $\chi: \mathcal{A}_2 \rightarrow \mathcal{A}_2$. By Thom’s identity [16, Section 7], a formula for this can be defined inductively by

$$\chi(Sq^0) = 1, \quad \sum_{i=0}^t Sq^i \chi(Sq^{t-i}) = 0 \quad \text{for } t > 0.$$

This conjugation operation is a crucial part of the structure of \mathcal{A}_2 . Many authors studied χ in the literature. Milnor [16, Theorem 5] calculated χ in

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terms of the Milnor basis. In 1974, Davis [8] computed $\chi(\text{Sq}^{2^n-k})$ for $n \geq k$ and $\chi(\text{Sq}^{2^{k-1}-k})$.

Silverman [17] extended works of Davis to give specific conjugation formulas. In 1996, Walker and Wood [22] used conjugation formulas of Davis and Silverman to prove a 20 years old conjecture about \mathcal{A}_2 , which states the element Sq^{2^n} has nilpotence height $2n + 2$. A topological interpretation of χ is given in [11, Section 1].

The *Leibniz–Hopf algebra* \mathcal{F} is the free associative \mathbf{Z} -algebra on one generator, S^n of degree n , in each positive degree, with coproduct $\Delta(S^n) = \sum S^i \otimes S^{n-i}$ (where S^0 denotes 1). \mathcal{F} is also known as the *algebra of non-commutative symmetric functions* [10] and has been studied in [2, 12, 13]. \mathcal{F} is a cocommutative Hopf algebra. \mathcal{A}_2 is naturally defined as a quotient of the mod 2 reduction $\mathcal{F} \otimes \mathbf{Z}/2$, denoted by \mathcal{F}_2 , by the Adem relations. Hence, we have a projection $\pi: \mathcal{F}_2 \rightarrow \mathcal{A}_2$. We approach the concepts concerned throughout the present work mainly via π .

We introduce the organization of this paper together with motivations. Besides the formulas for conjugation in \mathcal{A}_2 mentioned in the preceding paragraphs, various identities among the Steenrod squares and their images under χ are obtained [3, 8, 20]. Most of these are defined iteratively, and a more efficient way of computing conjugates of monomials in Steenrod squares is still an open problem [23, Problem 10.7]. In Section 3 we discuss whether we can compute χ using π together with a conjugation formula on \mathcal{F}_2 which is described as a closed (non-iterative) formula. Considering (7) we see that our approach gives rise to express the conjugates of the Steenrod squares in terms of the admissible basis elements. However, it cannot enable us to solve our problem. Rather, our point of view reveals that understanding the Adem relations leads to understand the conjugate of particular Steenrod squares.

In Section 4, inspired by [23, Problem 4.26], we attempt to calculate the invariants under χ in terms of the admissible monomials. These invariants form the vector subspace $\text{Ker}(\chi - 1)$ (where 1 denotes the identity homomorphism) of \mathcal{A}_2 . Following [7], from now on we denote this by \mathcal{A}_2^χ . A complete description of \mathcal{A}_2^χ has yet to be determined in the literature. What we know currently about the dimension of \mathcal{A}_2^χ is due to the dual case [7]. We give more detailed information about it at the end of Section 2. In Examples 4.2 and 4.3 we discuss whether we can deduce information about \mathcal{A}_2^χ from the conjugation invariants in \mathcal{F}_2 . A dual approach is given in [21, Section 5]. In Theorem 4.6 and Proposition 4.7 we also introduce some specific elements of \mathcal{A}_2^χ .

2. Preliminaries and terminology

\mathcal{F}_2 , being a free $\mathbf{Z}/2$ -algebra on S^1, S^2, \dots , has a basis given by all words $S^{j_1} S^{j_2} \dots S^{j_k}$, which we shall denote by S^{j_1, j_2, \dots, j_k} . A formula for the conjugation operation on \mathcal{F} was introduced in [4, 9, 15]. For a basis element S^{j_1, \dots, j_k}

in \mathcal{F}_2 this simplifies

$$(1) \quad \chi'(S^{j_1, \dots, j_k}) = \sum S^{i_1, \dots, i_l},$$

where the summation is over all refinements i_1, \dots, i_l of the *reversed* word j_k, \dots, j_1 [5, Section 2].

Example 2.1.

$$\begin{aligned} \chi'(S^{2,1}) &= S^{1,2} + S^{1,1,1}, \\ \chi'(S^4) &= S^4 + S^{3,1} + S^{2,2} + S^{2,1,1} + S^{1,2,1} + S^{1,3} + S^{1,1,2} + S^{1,1,1,1}. \end{aligned}$$

Now recall from [5, Section 2] some terminology. Whenever the words ‘higher’ or ‘lower’ is used, the left lexicographical ordering is concerned.

Definition 2.2. A word S^{j_1, j_2, \dots, j_m} is a *palindrome* if $j_h = j_{m-(h-1)}$ for all $h \in \{1, \dots, m\}$. A palindrome is referred to as an *odd-length palindrome*, denoted by OLP, if its length is odd. A non-palindrome S^{j_1, \dots, j_r} is referred to as a *higher non-palindrome*, denoted by HNP, if j_1, \dots, j_r is higher than its reverse j_r, \dots, j_1 .

Example 2.3. $S^{1,9,1}$ is an OLP and $S^{9,4,3}$ is an HNP.

Definition 2.4. In \mathcal{F}_2 , the ρ -image of an OLP $S^{i_1, \dots, i_{2k+1}}$ is defined as

$$(2) \quad \rho(S^{i_1, \dots, i_{2k+1}}) = \sum S^{i_1, \dots, i_k, j_1, \dots, j_l},$$

summed over all refinements j_1, \dots, j_l of i_{k+1}, \dots, i_{2k+1} with $j_1 \geq \frac{i_{k+1}}{2}$.

Example 2.5.

$$\begin{aligned} \rho(S^{2,3,2}) &= S^{2,3,2} + S^{2,3,1,1} + S^{2,2,1,2} + S^{2,2,1,1,1}, \\ \rho(S^5) &= S^5 + S^{4,1} + S^{3,2} + S^{3,1,1}. \end{aligned}$$

Arnon [1, Definition 4] gives some specific monomials.

Definition 2.6. For $n \geq k \geq 0$, a ξ -monomial and a ζ -monomial have the forms

$$X_k^n = \text{Sq}^{2^n, 2^{n-1}, \dots, 2^k}, \quad Z_k^n = \text{Sq}^{2^k, 2^{k+1}, \dots, 2^n}$$

respectively.

Example 2.7. $\text{Sq}^{8,4,2}$ is a ξ -monomial and $\text{Sq}^{2,4,8,16}$ is a ζ -monomial.

We finish this section by recalling some facts about the dimension of \mathcal{A}_2^X from the dual case point of view. Recall that the dual of the mod 2 Steenrod algebra is a graded connected Hopf algebra. Since it is relevant to study the commutativity in ring spectra, Crossley and Whitehouse [7] investigated the conjugation invariants in this Hopf algebra. Particularly, they established bounds for the dimension of the (graded) dual space of \mathcal{A}_2^X in a given degree [7, Theorem 3.1].

In degree d , let the dimensions of \mathcal{A}_2 and \mathcal{A}_2^χ be denoted by D_d and D_d^χ respectively. Due to the fact that the dual of \mathcal{A}_2 is of finite type, the bounds in the dual case gives us that

$$(3) \quad D_d/2 \leq D_d^\chi \leq D_d - (D_{d-1}/2).$$

3. A combinatoric approach to the conjugation

Now we confine our attention to the surjective graded Hopf algebra homomorphism $\pi: \mathcal{F}_2 \rightarrow \mathcal{A}_2$, where $\pi(S^n) = \text{Sq}^n$. Since this preserves conjugation operations, we have the following.

$$(4) \quad \chi \circ \pi = \pi \circ \chi'.$$

This yields

$$(5) \quad (\chi - 1) \circ \pi = \pi \circ (\chi' - 1).$$

We now verify if we can compute χ using (4).

Example 3.1. Consider the odd degree OLP S^3 . Applying the both sides of (4) to S^3 , we obtain

$$(6) \quad \chi(\pi(S^3)) = \pi(\chi'(S^3)).$$

Since $\chi'(S^3) = S^3 + S^{2,1} + S^{1,2} + S^{1,1,1}$, and $\pi(\chi'(S^3)) = \text{Sq}^{2,1}$, (6) turns into $\chi(\text{Sq}^3) = \text{Sq}^{2,1}$.

Fix an integer $n \geq 1$. Considering (4), to compute the conjugate $\chi(\text{Sq}^n)$, we need to have information about the image of the sum of all refinements of S^n under π . Now the natural question arises whether it is possible to find an explicit formula for

$$(7) \quad \pi\left(\sum S^{j_1, \dots, j_y}\right),$$

where the summation is over all refinements j_1, \dots, j_y of the word n . In other words, given a degree n , can we determine which refinements of S^n in the summation (7) vanish after applying the Adem relations? In low degrees this is manageable (see Example 3.1), whereas in higher degrees the Adem relations cause difficulties, since it is hardly controlled. On the other hand, our point of view leads to reprove a property of χ , which also plays an important role in the proof of *Arnon basis* [1, Theorem 5(A)]. Let us explain it. By the same argument as above, we compute the conjugate $\chi(\text{Sq}^{2^n})$, we consider the following

$$(8) \quad \chi(\text{Sq}^{2^n}) = \pi\left(\sum S^{i_1, \dots, i_k}\right),$$

where the summation is over all refinements i_1, \dots, i_k of the word 2^n . Among all the refinements of S^{2^n} , the only one-length refinement is S^{2^n} itself and

$\pi(S^{2^n}) = \text{Sq}^{2^n}$. Following this, by linearity of π , and the fact that Sq^{2^n} is indecomposable [19], (8) turns into

$$(9) \quad \chi(\text{Sq}^{2^n}) = \text{Sq}^{2^n} + L,$$

where L is a polynomial in lower Steenrod squares.

4. Conjugation invariants

We now recall Theorem 2.4 of [5] for the prime 2.

Theorem 4.1. *In the mod 2 Leibniz-Hopf algebra $\mathcal{F} \otimes \mathbf{Z}/2$ the subspace of conjugation invariants $\text{Ker}(\chi' - 1)$ has a basis consisting of:*

- (1) *the $(\chi' - 1)$ -images of all even-degree OLPs,*
- (2) *the $(\chi' - 1)$ -images of all HNP's and*
- (3) *the ρ -images of all odd-degree OLPs.*

We examine whether a description of \mathcal{A}_2^X can be deduced from Theorem 4.1. We start by an example.

Example 4.2. Theorem 4.1 tells us that $(\chi' - 1)(S^2)$ is the only degree 2 basis element of $\text{Ker}(\chi' - 1)$. Hence, applying both sides of (5) to $(\chi' - 1)(S^2)$, we obtain

$$(10) \quad (\chi - 1) \circ \pi((\chi' - 1)(S^2)) = 0.$$

Since $(\chi' - 1)(S^2) = 2S^2 + S^{1,1}$ and the π -images are 0, (10) turns into $\chi(0) = 0$.

The following example shows that we may obtain non-trivial elements of \mathcal{A}_2^X using Theorem 4.1.

Example 4.3. By Theorem 4.1,

$$(\chi' - 1)(S^{2,1}), (\chi' - 1)(S^4), \rho(S^5)$$

are basis elements of $\text{Ker}(\chi' - 1)$ in degrees 3, 4, 5 respectively. The π -images of these elements are

$$(11) \quad \text{Sq}^3 + \text{Sq}^{2,1}, \text{Sq}^{3,1}, \text{Sq}^5 + \text{Sq}^{4,1}.$$

By the same argument as in Example 4.2, (5) shows that the elements in (11) lie in \mathcal{A}_2^X .

In fact, Example 4.3 provides a basis for degrees 3, 4, 5 parts of \mathcal{A}_2^X . We explain it briefly. Since $D_2 = 1, D_3 = D_4 = D_5 = 2$, the bounds in (3) enable us to identify exactly the dimensions of \mathcal{A}_2^X in degrees 3 to 5: $D_3^X = D_4^X = D_5^X = 1$. Furthermore, the invariants in (11) are linearly independent and the number of these (in each degree) are equal to D_3^X, D_4^X, D_5^X , respectively. Thus, they form a basis for \mathcal{A}_2^X in the quoted degrees. It is natural to ask what happens in higher degrees. To answer, we need a computer aided approach. Kaji [14] gives a code for calculating conjugation invariants in the Steenrod algebra. However, it is not related to our argument.

Another observation is that it is also possible to find an element of \mathcal{F}_2 which is not an invariant under χ' , but leads to obtain an element in \mathcal{A}_2^χ .

Example 4.4. Consider S^2 . This is not an invariant under χ' , since $\chi'(S^2) = S^2 + S^{1,1}$. On the other hand, applying the both sides of the identity (5) to S^2 we have

$$(12) \quad (\chi - 1)(\pi(S^2)) = \pi((\chi' - 1)(S^2)).$$

Since $\pi(S^2 + S^{1,1}) = Sq^2 + Sq^{1,1}$, by the Adem relations, (12) turns into $\chi(Sq^2) = Sq^2$.

Recall that the conjugation of \mathcal{A}_2 is an involution, i.e., $\chi^2 = 1$. One may ask that if there is an analogue result of Theorem 4.1 for the algebra \mathcal{A}_2 . Unfortunately, the Adem relations do not help more as seen in the following.

Example 4.5. Consider $Sq^{7,3,1}, Sq^{2,2,2}$. Since $\chi^2 = 1$, we have $(\chi - 1)(\chi - 1) = 0$ from which we can deduce $(\chi - 1)(Sq^{7,3,1}), (\chi - 1)(Sq^{2,2,2}) \in \mathcal{A}_2^\chi$. On the other hand, by the anti-commutativity property of χ , using the Adem relations, we obtain $\chi(Sq^{7,3,1}) = Sq^{7,3,1}, \chi(Sq^{2,2,2}) = Sq^{2,2,2}$, and hence, the $(\chi - 1)$ -images of the above monomials are the trivial elements of \mathcal{A}_2^χ .

We now determine some of non-trivial elements of \mathcal{A}_2^χ in the following.

Theorem 4.6. *The $(\chi - 1)$ -images of ξ -monomials and ζ -monomials for $n > k > 0$ are non-trivial elements of \mathcal{A}_2^χ .*

Proof. We first consider the $(\chi - 1)$ -images of ζ -monomials. Consider a ζ -monomial $Z_k^n = Sq^{2^k, 2^{k+1}, \dots, 2^n}$, where $n > k > 0$. By the same argument in Example 4.5, the $(\chi - 1)$ -image of Z_k^n lies in \mathcal{A}_2^χ . Moreover, property (9) together with the anti-commutativity property of χ gives

$$(13) \quad (\chi - 1)(Z_k^n) = Z_k^n + Sq^{2^n, 2^{n-1}, \dots, 2^k} + \sum Sq^J,$$

summed over non-negative sequences $J = (j_1, \dots, j_r)$, where $J \neq (2^n, 2^{n-1}, \dots, 2^k)$. On the other hand, [21, Proposition 3.5] tells us that for Sq^J has $Sq^{2^n, 2^{n-1}, \dots, 2^k}$ as a summand when expressed as sum of elements in the admissible basis, J must equal to $(2^n, 2^{n-1}, \dots, 2^k)$. This cannot be happen, and hence $Sq^{2^n, 2^{n-1}, \dots, 2^k}$ cannot be cancelled by any terms on the right of (13). Hence, the left of (13) is non-zero. By the same argument as above we can show that the $(\chi - 1)$ -images of ξ -monomials are non-trivial elements of \mathcal{A}_2^χ , which completes the proof. \square

Proposition 4.7.

- (i) $(\chi - 1)(Sq^{2^{n-1}, 2^{n-2}, \dots, 2^2, 2, 1} + Sq^{2^n - 1}) = 0$ for $n \geq 1$,
- (ii) $(\chi - 1)(Sq^{2^{n-1}, 2^{n-2}, \dots, 2^2, 2} + Sq^{2^n - 2}) = 0$ for $n \geq 2$,
- (iii) $(\chi - 1)(Sq^{2^{n-1}, 2^{n-2}, \dots, 2^2, 1} + Sq^{2^n - 3}) = 0$ for $n \geq 3$.

Proof. We first prove (i). In the case $k = 1$, [8, Theorem 2] gives

$$\chi(\mathrm{Sq}^{2^n-1}) = \mathrm{Sq}^{2^{n-1}, 2^{n-2}, \dots, 1}.$$

Since $\chi^2 = 1$, this yields $\chi(\mathrm{Sq}^{2^{n-1}, 2^{n-2}, \dots, 1}) = \mathrm{Sq}^{2^n-1}$, which completes the proof of (i). Similarly, proofs of (ii) and (iii) can be deduced from [8, Theorem 2] in the cases $k = 2$ and $k = 3$, respectively. \square

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