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ON DIFFERENTIAL INVARIANTS OF HYPERPLANE SYSTEMS ON NONDEGENERATE EQUIVARIANT EMBEDDINGS OF HOMOGENEOUS SPACES

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Abstract. Given a complex submanifold M of the projective space $\mathbb{P}(T)$, the hyperplane system R on M characterizes the projective embedding of M into $\mathbb{P}(T)$ in the following sense: for any two nondegenerate complex submanifolds $M \subset \mathbb{P}(T)$ and $M' \subset \mathbb{P}(T')$, there is a projective linear transformation that sends an open subset of M onto an open subset of M' if and only if (M, R) is locally equivalent to (M', R'). Se-ashi developed a theory for the differential invariants of these types of systems of linear differential equations. In particular, the theory applies to systems of linear differential equations that have symbols equivalent to the hyperplane systems on nondegenerate equivariant embeddings of compact Hermitian symmetric spaces. In this paper, we extend this result to hyperplane systems on nondegenerate equivariant embeddings of homogeneous spaces of the first kind.

1. Introduction

Let $L/L' \subset \mathbb{P}(T)$ be a nondegenerate equivariant embedding of homogeneous space L/L', where L is a connected complex Lie group and L' is a closed subgroup of L. We say that $L/L' \subset \mathbb{P}(T)$ is *Griffiths-Harris rigid* if the fundamental forms of $L/L' \subset \mathbb{P}(T)$ determine embedding $L/L' \subset \mathbb{P}(T)$ in the following sense: Let $M \subset \mathbb{P}(T)$ be a (not necessarily closed) complex sub-manifold. If the fundamental forms of M at the general points of M are isomorphic to the fundamental forms of L/L' at the base point, then M is projectively equivalent to an open subset of L/L'.

Theorem 1.1 (Landsberg [4, 5] Hwang-Yamaguchi [3]). *Suppose $L/L' \subset \mathbb{P}(T)$ is a nondegenerate equivariant embedding of a compact Hermitian symmetric space L/L', which contains neither $\mathbb{P}_n$, for $n \geq 1$, nor $\mathbb{Q}_n$, for $n \geq 2$ as an irreducible factor. Then $L/L' \subset \mathbb{P}(T)$ is Griffiths-Harris rigid.*

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A rational homogeneous variety $L/L' \subset \mathbb{P}(T)$ that is not a compact Hermitian symmetric space, has nonzero differential invariants in addition to its fundamental forms (Theorem 4.1 in [6]). Therefore, we cannot expect that they are Griffiths-Harris rigid. In order to obtain similar rigidity results, we must consider additional differential invariants.

We say that $L/L' \subset \mathbb{P}(T)$ is *Fubini-Griffiths-Harris rigid* if the Fubini forms of $L/L' \subset \mathbb{P}(T)$ determine embedding $L/L' \subset \mathbb{P}(T)$ as follows. Let $M \subset \mathbb{P}(T)$ be a (not necessarily closed) complex sub-manifold. Assume there exists a subbundle $\widetilde{\mathcal{F}}_M$ of the bundle $\mathcal{F}_M$ of first-order adapted frames of M, defined over the general points of M, on which the coefficients of the k-th Fubini forms of M agree with the coefficients of the k-th Fubini forms of L/L' for all k. Then M is projectively equivalent to an open subset of L/L'.

Theorem 1.2 (Landsberg-Robles [8, 9])**.** *Let $L/L' \subset \mathbb{P}(T)$ be a nondegenerate equivariant embedding of a rational homogeneous variety L/L' that contains neither $\mathbb{P}_n$, for $n \geq 1$, nor $\mathbb{Q}_n$, for $n \geq 2$ as an irreducible factor. Then $L/L' \subset \mathbb{P}(T)$ is Funibi-Griffiths-Harris rigid.*

Landsberg applied the moving frame method in order to prove Theorem 1.1 for the case in which L/L' has rank 2. For the proof of Theorem 1.1, Hwang-Yamaguchi used Se-ashi theory of differential invariants of linear differential equations, which reduced the problem to a vanishing of the Lie algebra cohomology $H^1(\mathfrak{l}_{-1}, \mathfrak{l}^\perp)$. Here, $\mathfrak{l}^\perp$ represents the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{sl}(S)$ with respect to the Killing form of $\mathfrak{sl}(S)$, and S is the dual representation space of T. The vanishing of $H^1(\mathfrak{l}_{-1}, \mathfrak{l}^\perp)$ follows from the Kostant theory of Lie algebra cohomology.

Subsequently, in [8] and [9], Landsberg and Robles introduced a filtered exterior differential system in order to reduce the rigidity problem in Theorem 1.2 to the vanishing of $H^1(\mathfrak{l}_-, \mathfrak{l}^\perp)$, where $\mathfrak{l}^\perp$ is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{sl}(T)$ with respect to the Killing form of $\mathfrak{sl}(T)$. As in the proof of Theorem 1.1, they applied Kostant theory of Lie algebra cohomology in order to prove the vanishing of $H^1(\mathfrak{l}_-, \mathfrak{l}^\perp)$.

In this paper we generalize these rigidity results to homogeneous spaces of the first kind. A homogeneous space L/L' is said to be *of the first kind* if there is a grading on the Lie algebra $\mathfrak{l}$ of L such that $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let $\tau : L \to GL(T)$ be an irreducible representation of L with a vector $t \in T$ such that the isotropy group of L at $[t] \in \mathbb{P}(T)$ is L'. Then $L/L' = L.[t_1] \subset \mathbb{P}(T)$ is a nondegenerate equivariant embedding of L/L', and any non-degenerate equivariant embedding $L/L' \subset \mathbb{P}(T)$ is obtained in this way (For various examples and their embeddings see Section 2).

Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be the dual representation of τ. Then $\rho(\mathfrak{l}_{-1}) \subset \mathfrak{gl}(S)_{-1}$, where the grading on $\mathfrak{gl}(S)$ is induced by the osculating filtration on T. Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ be the prolongation of $\rho : \mathfrak{l}_{-1} \to \mathfrak{gl}(S)_{-1}$ and let G (G', respectively) be a subgroup of $GL(S)$ with the Lie algebra $\mathfrak{g}$ ($\mathfrak{g}' = \bigoplus_{k \geq 0} \mathfrak{g}_k$, respectively). The representation of $\mathfrak{g}_{-1}$ on the orthogonal complement $\mathfrak{g}^\perp$ of $\mathfrak{g}$ in $\mathfrak{gl}(S)$

induces a complex $(C(\mathfrak{g}^{\perp}),\partial)$ (Section 6). Let $H(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})=\bigoplus_q H^q(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})$ be the cohomology space of $(C(\mathfrak{g}^{\perp}),\partial)$.

Theorem 1.3. *Let $L/L'\subset\mathbb{P}(T)$ be a nondegenerate equivariant embedding of a homogeneous space of the first kind. Assume that*

(1) *there is a positive definite Hermitian inner product $(\ ,\ )$ on $C^1(\mathfrak{g}^{\perp})$ such that the kernel* $\operatorname{Ker}\partial^*$ *of the adjoint operator ∂^* of ∂ with respect to $(\ ,\ )$ is G'-invariant;*
(2) $H^1(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})=0$.

Then $L/L'\subset\mathbb{P}(T)$ is Griffiths-Harris rigid.

In order to prove Theorem 1.3 we apply Se-ashi theory to the hyperplane system on complex manifold $M\subset\mathbb{P}(T)$, as in the proof of Theorem 1.1. Se-ashi studied an integrable system of differential equations, one that has the same symbol as the hyperplane system on $L/L'\subset\mathbb{P}(T)$ when $L/L'\subset\mathbb{P}(T)$ is a nondegenerate equivariant embedding of a compact Hermitian symmetric space L/L', and obtained a complete system of differential invariants on this kind of system of differential equations. We generalize this theory to integrable systems of differential equations that have the same symbols as the hyperplane system on nondegenerate equivariant embedding $L/L'\subset\mathbb{P}(T)$ of homogeneous space L/L' of the first kind.

As in the cases treated in Theorems 1.1 and 1.2, the rigidity of $L/L'\subset\mathbb{P}(T)$ in Theorem 1.3 can be reduced to a vanishing of the Lie algebra cohomology. The difficulty is that there is no uniform way of computing it in our case. When L/L' is a compact Hermitian symmetric space, $\mathfrak{g}$ is the direct sum of $\mathfrak{l}$ and the center of $GL(S)$ and we can apply Kostant theory of Lie algebra cohomology to compute $H^1(\mathfrak{l}_-,\mathfrak{l}^{\perp})$, which works only when $\mathfrak{l}$ is semisimple. However, in general, $\mathfrak{g}$ is not reductive and Kostant theory is not applicable. Therefore, to show the vanishing of $H^1(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})$, we need to develop a new theory, which we will deal with in a forthcoming paper in preparation.

The rest of the paper is organized as follows. In Section 2, we explain horospherical homogeneous spaces and give examples of homogeneous spaces of the first kind in addition to compact Hermitian symmetric spaces, and in Section 3, we recall definitions and properties concerning fundamental forms. Section 4 reviews Se-ashi's theory of differential invariants on linear differential equations. In Section 5 we apply Se-ashi theory to homogenous space L/L' of the first kind, and show that the model system is isomorphic to the hyperplane system on L/L' (Proposition 5.5). A key lemma used to construct a complete system of invariants addresses the existence of a 'good' metric on the complex $(C(\mathfrak{g}^{\perp}),\partial)$ defining Lie algebra cohomology $H^1(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})$. In Section 6, we construct a complete system of invariants under the assumption that there exists such a metric, and prove the rigidity of L/L' under the assumption that $H^1(\mathfrak{g}_{-1},\mathfrak{g}^{\perp})=0$ (Theorem 1.3).

2. Homogeneous spaces of the first kind

2.1. Rational homogeneous varieties

A semisimple graded Lie algebra over $\mathbb{C}$ is a semisimple Lie algebra $\mathfrak{l}$ with gradation $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ such that

$$[\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q} \quad \text{for } p, q \in \mathbb{Z}.$$

Then there exists a unique element $E \in \mathfrak{l}_0$ such that

$$\mathfrak{l}_p = \{X \in \mathfrak{l} : [E, X] = pX\} \quad \text{for } p \in \mathbb{Z}.$$

We call E the *characteristic element* of the semisimple graded Lie algebra $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$. A nilpotent graded Lie algebra $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{l}_p$ such that

$$\mathfrak{l}_p = [\mathfrak{l}_{p+1}, \mathfrak{l}_{-1}] \quad \text{for } p < -1$$

is called a *fundamental graded Lie algebra of* μ-th kind.

To each semisimple graded Lie algebra $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ over $\mathbb{C}$ such that $\mathfrak{l}_- := \bigoplus_{p<0} \mathfrak{l}_p$ is fundamental, there corresponds a homogeneous space L/L', where $L = \mathrm{Int}(\mathfrak{l}) \cdot L_0$, and L_0 is the automorphism group of the graded Lie algebra $\mathfrak{l} = \oplus_{p\in\mathbb{Z}}\mathfrak{l}_p$, that is, the subgroup of $\mathrm{Aut}(\mathfrak{l})$ consisting of elements which preserve the gradation, and L' is the automorphism group of the filtered Lie algebra $\{\mathfrak{f}^{(p)} := \bigoplus_{q\geq p} \mathfrak{l}_p\}$ for $p \in \mathbb{Z}$. A semisimple graded Lie algebra over $\mathbb{C}$ *of the first kind*, i.e., $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$, corresponds to a compact Hermitian symmetric space.

Let $\tau : L \to GL(T)$ be a representation of L with a vector $t \in T$ such that the isotropy group of L at $[t] \in \mathbb{P}(T)$ is L'. Then L/L' can be embedded into $\mathbb{P}(T)$ via $[l] \in L/L' \mapsto l.[t] \in \mathbb{P}(T)$. We call $L/L' \subset \mathbb{P}(T)$ a rational homogeneous variety.

2.2. Horospherical varieties

Let L be a reductive group. A homogeneous space L/L' is called *horospherical* if the map from L/L' to L/P, in which P is the normalizer of L' in L, is a $(\mathbb{C}^\times)^r$-bundle over the rational homogeneous variety L/P. r is called the *rank* of the horospherical space L/L'. A normal variety X is said to be horospherical if there is an open L-orbit that is horospherical.

A smooth horospherical L-variety X of Picard number one is either homogeneous or one of the following ([10]):

(1) $(B_n, \omega_{n-1}, \omega_n)$, $n \geq 3$;
(2) $(B_3, \omega_1, \omega_3)$;
(3) $(C_n, \omega_{i+1}, \omega_i)$, $n \geq 2$ and $i \in \{1, 2, \dots, n-1\}$;
(4) $(F_4, \omega_2, \omega_3)$;
(5) $(G_2, \omega_2, \omega_1)$.

Here, (L, ω_i, ω_j) is the closure of the L-orbit $L[v_i \oplus v_j]$ in $\mathbb{P}(V(\omega_i) \oplus V(\omega_j))$.

Let $Y := L.[v_i]$ and $Z := L.[v_j]$. Let $\widehat{L}$ be the automorphism group $\mathrm{Aut}(X)$ of X. Then, $\widehat{L} = (L \times \mathbb{C}^*) \ltimes H^0(Y, N_{Y/X})$, and $\widehat{L}$ has two orbits: an open orbit $\widehat{L}.[v_i]$ and a closed orbit $\widehat{L}.[v_j] = L.[v_j] = Z$. Here, $\widehat{L}.[v_i]$ is smooth and a simple embedding of $L/H = L.[v_i \oplus v_j]$ with a unique closed L-orbit $L.[v_i] = Y$, giving us isomorphism $L \times_{P_i} V_i = N_{Y/X} \to \widehat{L}.[v_i]$. Furthermore, $H^0(Y, N_{Y/X})$ acts on $L \times_{P_i} V_i = N_{Y/X}$ as translations on fibers. The Lie algebra $\widehat{\mathfrak{l}} = \mathfrak{l} \oplus \mathbb{C} \oplus \mathfrak{u}$ of $\widehat{L}$ is contained in $\mathfrak{gl}(V(\omega_i) \oplus V(\omega_j))$ in the following way:

$$\begin{pmatrix} \mathfrak{l} & 0 \\ \mathfrak{u} & \mathbb{C} \end{pmatrix}.$$

2.3. Smooth projective two-orbit varieties

Proposition 2.1 (Theorem 1 of [1]). *Let $\widehat{X}$ be a smooth complete variety with an effective action of the connected linear non semi-simple group $\widehat{L}$. Suppose that $\widehat{X}$ has two orbits under the action of $\widehat{L}$ and that the closed orbit has codimension 1. Let L be a maximal semi-simple subgroup of $\widehat{L}$. Then there exist a parabolic subgroup L' of L and a L'-module V such that:*

(1) *the action of L' on $\mathbb{P}(V)$ is transitive;*
(2) *there exists an irreducible L-module U and a surjective L'-equivariant morphism $\varepsilon : U \to V$;*
(3) *$\widehat{X} = L \times_{L'} \mathbb{P}(V \oplus \mathbb{C})$ as an L-variety. $L \times_{L'} V$ is the open $\widehat{L}$-orbit.*
(4) *$\widehat{L}$ is either $L \ltimes U$ or $(L \times \mathbb{C}^*) \ltimes U$.*
(5) *$\widehat{X}$ is a horospherical L-variety of rank 1.*
(6) *The L-module U is the set of global sections of the vector bundle $L \times_{L'} V \to L/L'$ and $\varepsilon : U \to V$ is the evaluation map at $[L'] \in L/L'$.*

The open orbit $\widehat{L}/\widehat{L}' = L \times_{L'} V$ of a smooth two-orbit variety $\widehat{X} = L \times_{L'} \mathbb{P}(V \oplus \mathbb{C})$ is a homogeneous space of the first kind if and only if L/L' is a compact Hermitian symmetric space.

In fact, if L/L' is a compact Hermitian symmetric space, then $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a semisimple graded Lie algebra of the first kind. Let E be the characteristic element. Let $\mathfrak{u}$ denote the Lie algebra of U. Then we have the eigenspace decomposition of $\mathfrak{u}$: $\mathfrak{u} = \mathfrak{u}_{(c)} \oplus \mathfrak{u}_{(c+1)}$. Let $\mathfrak{u}_{-1} = \mathfrak{u}_{(c)}$ and $\mathfrak{u}_0 = \mathfrak{u}_{(c+1)}$. Also, set $\widehat{\mathfrak{l}} = \mathfrak{l} \oplus \mathfrak{u}$. Then

$$\widehat{\mathfrak{l}} = (\mathfrak{l}_{-1} \oplus \mathfrak{u}_{-1}) \oplus (\mathfrak{l}_0 \oplus \mathfrak{u}_0) \oplus \mathfrak{l}_1,$$

and the tangent space of $\widehat{L}/\widehat{L}'$ at the base point $[L']$ can be identified with $\widehat{\mathfrak{l}}_- := \mathfrak{l}_{-1} \oplus \mathfrak{u}_{-1}$.

Example. A smooth horospherical variety X of Picard number one in Section 2.2 has an open orbit isomorphic to that of smooth two-orbit variety $\widehat{X}$ in Proposition 2.1. In fact, the blow-up of X along the closed orbit $L.[v_j]$ is one of the $\widehat{X}$ in Proposition 2.1. Details of this can be found in [10]. Among them, the open orbits in $(B_3, \omega_1, \omega_3)$ and $(C_n, \omega_n, \omega_{n-1})$ are homogeneous spaces of the first kind.

3. Fundamental forms

3.1. Fundamental forms of $M \subset \mathbb{P}(T)$

Let $M \subset \mathbb{P}(T)$ be a complex manifold and let $x \in M$ be a point. Assume T_xM denotes the tangent space of M at x and $\hat{T}_xM \subset T$ is the (affine) tangent space of the cone $\hat{M}$ in T at any $p \in T$ with $x = [p]$. Then we have $T_xM = (\hat{T}_xM/\hat{x}) \otimes \hat{x}^*$. Also, suppose $N_xM = T_x\mathbb{P}(T)/T_xM$ represents the normal space of M at x. Then $N_xM = (T/\hat{T}_xM) \otimes \hat{x}^*$.

If E is the restriction of the hyperplane bundle on $\mathbb{P}(T)$ to M, then, from $N_x^*M = (T/\hat{T}_xM)^* \otimes \hat{x} = (\hat{T}_xM)^\perp \otimes E_x^*$, it follows that $N_x^*M \otimes E_x = (\hat{T}_xM)^\perp \subset T^*$.

Define Gauss map $\gamma : M \to Gr(n+1, T)$ to be $\gamma(x) = \hat{T}_xM$. Then the derivative $d\gamma_x$ induces a linear map $\mathbb{FF}^2 : S^2T_xM \to N_xM$, called the *second fundamental form* of M at x. If $\mathbb{FF}^2 : S^2T_xM \to N_xM$ is not surjective, we can refine flag $\hat{x} \subset \hat{T}_xM \subset T$ in order to obtain a refined invariant.

Let $\hat{T}_x^{(2)}M = \hat{T}_xM + \widehat{\mathbb{FF}^2}(S^2T_xM) \otimes E_x^* \subset T$. Define the second order Gauss map $\gamma^{(2)} : M \to Gr(n^{(2)}, T)$ by $\gamma^{(2)}(x) = \hat{T}_x^{(2)}M$. The derivative $d\gamma_x^{(2)}$ defines a linear map $\mathbb{FF}^3 : S^3T_xM \to T_x\mathbb{P}(T)/(T_x^{(2)}M)$, called the *third fundamental form* of M at x.

More generally, the *k-th fundamental form* of M at x is a linear map

$$\mathbb{FF}^k : S^kT_xM \to N_x^kM,$$

which can be obtained by differentiating the $(k-1)$-th Gauss map $\gamma^{(k-1)} : M \to Gr(n^{(k-1)}\ , T)$ defined by $\gamma^{(k-1)}(x) = \hat{T}_x^{(k-1)}M$, where $\hat{T}_x^{(k-1)}M = \hat{T}_x^{(k-2)}M + \mathbb{FF}^{k-1}(S^{k-1}T_xM) \otimes E_x$ is the $(k-1)$-th osculating space and $N_x^kM = (\hat{T}_x^{(k)}M/\hat{T}_x^{(k-1)}M) \otimes E_x$ is the k-th normal space. These osculating spaces determine a flag of T:

$$0 \subset \hat{x} \subset \hat{T}_xM \subset \hat{T}_x^{(2)}M \subset \cdots \subset \hat{T}_x^{(p)}M = T,$$

which is called the *osculating filtration.* The fundamental forms describe the infinitesimal movement of M away from its $(k-1)$-th osculating space $\hat{T}_x^{(k-1)}M \subset T$ at order k.

The osculating filtration induces a filtration of $S := T^*$:

$$S = T^* \supset \hat{x}^\perp \supset (\hat{T}_x^{(1)})^\perp \supset (\hat{T}_x^{(2)})^\perp \supset \cdots \supset 0.$$

Moreover, if we set $S_k := (\hat{T}_x^{(k-1)})^\perp/(\hat{T}_x^{(k)})^\perp$, then $S_k = \left(\hat{T}_x^{(k)}/\hat{T}_x^{(k-1)}\right)^*$ and $S = \oplus_{k=0}^p S_k$.

The dual of $\mathbb{FF}^k \otimes E_x^* : S^k T_x \otimes E_x^* \to N_x^k \otimes E_x^* = (\hat{T}_o^{(k)}/\hat{T}_x^{(k-1)})$ produces injective map

$$S = \oplus_{k=0}^p S_k \to \oplus_{k=0}^p S^k T_x^* \otimes E_x.$$

Let $|\mathbb{FF}_x^k|$ denote the image of S_k in $S^k T_x^* \otimes E_x$.

Let V be a vector space and let $A \subset S^d V^*$ be a linear subspace. Define $A^{(1)}$ to be $(A \otimes V^*) \cap S^{d+1} V^*$. Then $P \in A^{(1)}$ if and only if for all $v \in V$, $w \mapsto P(v, w, \ldots, w)$ belongs to A. Given $v \in V$, let $i(v) : S^k V^* \to S^{k-1} V^*$ denote the interior product. In addition, let $Jac(A) := \{i(v)P : v \in V, P \in A\} \subset S^{d-1} V^*$, the Jacobian space of S. Then $A^{(1)} = \{P \in S^{d+1} V^* : Jac(P) \subset A\}$.

Proposition 3.1. (1) *We have $i(v)|\mathbb{FF}_x^k| \subset |\mathbb{FF}_x^{k-1}|$ for any $v \in T_x$.*

(2) *If $i(v)(\oplus_{k=0}^p |\mathbb{FF}_x^k|) = 0$, $v \in T_x$, then $v = 0$.*

Proof. (1) The statement follows from the fact that $|\mathbb{FF}_x^k| \subset |\mathbb{FF}_x^{k-1}|^{(1)}$ (Section 2.1.3 of [7]).

(2) $S_1 = (\hat{T}_x^{(0)})^\perp / (\hat{T}_x^{(1)})^\perp = (\hat{T}_x / \hat{x})^* \to T_x X^* \otimes (\hat{x})^*$ is an isomorphism. Thus, if $i(v)|\mathbb{FF}_x^1| = 0$, then $v = 0$. □

3.2. Fundamental forms of $L/L' \subset \mathbb{P}(T)$

Let $\tau : L \to GL(T)$ be an irreducible representation of a complex Lie group L, and let $[t_1]$ be a point in $\mathbb{P}(T)$ with isotropy L'. Then the map $L \to \mathbb{P}(T)$ defined by $g \mapsto g.[t_1]$ induces a nondegenerate equivariant embedding of L/L' into $\mathbb{P}(T)$.

Extend t_1 to a basis $\{t_1, \ldots, t_r\}$ of T and take the dual basis $\{s_1, \ldots, s_r\}$ of $S = T^*$. The dual representation $\rho : L \to GL(S)$ of τ is given by

$$\langle \rho(g)s, t \rangle = \langle s, \tau(g^{-1})(t) \rangle$$

for $g \in L$ and $s \in S$ and $t \in T$. Let W be the subspace of S spanned by s_1. Since L' is the stabilizer of $[t_1]$, representation $\rho_W : L' \to GL(W)$ is produced through projection $\pi_0 : S = W \oplus W' \to W$. Let E be the line bundle on L/L' that is induced by representation $\rho_W : L' \to GL(W)$.

Proposition 3.2. *With the notations above.*

1. *There is an injective map ι from S into the space $H^0(L/L', E)$ of holomorphic sections of E.*
2. *Let $\Sigma := \iota(S)$. Let $\phi : L/L' \to \mathbb{P}(\Sigma^*)$ denote the map induced by the subspace Σ of $H^0(L/L', E)$, i.e.,*

   $$\phi([g]) = \text{ the hyperplane } \{\sigma \in \Sigma : \sigma([g]) = 0\} \text{ of } \Sigma.$$

   Then $\phi : L/L' \to \mathbb{P}(\Sigma^*)$ is projectively equivalent to the embedding of L/L' into $\mathbb{P}(T)$ given by $[g] \in L/L' \mapsto [g.t_1] \in \mathbb{P}(T)$.
3. *E is the restriction of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}(\Sigma^*)$ to L/L'.*

Proof. (1) For each $s \in S$ define function $f_s : L \to W$ by $f_s(g) = \pi_0(\rho(g^{-1})s)$. Then $f_s(gg') = \rho_W(g')^{-1} f_s(g)$ for all $g, g' \in L$. Thus f_s defines a section σ_s of

E. Thus we have a map $\iota : S \to H^0(L/L', E)$. By the irreducibility of ρ, ι is injective.

(2) It suffices to show this for $[t_1] \in \mathbb{P}(T)$. $\varphi([t_1]) = \{\sigma_s : s \in S, \sigma_s([t_1]) = 0\} \overset{\iota}{\simeq} W'$, and the hyperplane W' of S corresponds to $[t_1] \in \mathbb{P}(T)$.

(3) follows from (1). □

Suppose $\mathfrak{l}$ is the Lie algebra of L. Let $U(\mathfrak{l}) = \mathfrak{l}^{\otimes}/\{\xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta]\}$ denote the universal enveloping algebra of $\mathfrak{l}$ and assume $U(\mathfrak{l})^k$ signifies the k-th term in the filtration induced by the filtration from the tensor algebra. Then $U(\mathfrak{l})^k/U(\mathfrak{l})^{k-1} = S^k\mathfrak{l}$.

Proposition 3.3 (Proposition 2.3 of [7]). *With the notations above.*

(1) *The k-th osculating space $\hat{T}_x^{(k)}$ of L/L' at $x = [t_1]$ is $U(\mathfrak{l})^k.t_1$.*
(2) *The k-th fundamental form $\mathbb{FF}^k : S^kT_x \to N_x^k$ of L/L' at $x = [t_1]$ is given by*

$$\begin{array}{ccc} U(\mathfrak{l})^k & \longrightarrow & \hat{T}_x^{(k)} \otimes E_x \\ \downarrow & & \downarrow \\ S^kT_x & \xrightarrow{\mathbb{FF}^k} & N_x^k \end{array}$$

where map $U(\mathfrak{l})^k \to \hat{T}_x^{(k)} \otimes E_x$ is defined by $\mathbf{u} \mapsto (\mathbf{u}.t_1) \otimes s_1$ and map $U(\mathfrak{l})^k \to S^kT_x$ is the quotient map induced by $\mathfrak{l} \to \mathfrak{l}/\mathfrak{l}' = T_x$.

Recall that S can be thought of as graded vector space $\oplus_{k=0}^p S_k$, where $S_k = \left(T_x^{(k)}/T_x^{(k-1)}\right)^* = \left(\hat{T}_x^{(k-1)}\right)^\perp / \left(\hat{T}_x^{(k)}\right)^\perp$.

Proposition 3.4. *Assume there is an $\mathfrak{l}'$-invariant complement $\mathfrak{l}_-$ of $\mathfrak{l}'$ in $\mathfrak{l}$ so that $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{l}_-$. Then, we can conclude the following.*

(1) *$\tau(\mathfrak{l}_-)$ maps $T_x^{(k-1)}$ to $T_x^{(k)}$, and $\rho(\mathfrak{l}_-)$ maps S_k to S_{k-1}.*
(2) *Given $v \in T_x = \mathfrak{l}_-$, we have a commutative diagram:*

$$\begin{array}{ccc} S^kT_x^* \otimes E_x & \xleftarrow{(\mathbb{FF}^k \otimes E_x^*)^*} & (N_x^k \otimes E_x^*)^* = S_k \\ i(v) \downarrow & & \downarrow \rho(v) \\ S^{k-1}T_x^* \otimes E_x & \xleftarrow{(\mathbb{FF}^{k-1} \otimes E_x^*)^*} & (N_x^{k-1} \otimes E_x^*)^* = S_{k-1}. \end{array}$$

Proof. The twisted k-th fundamental map $\mathbb{FF}^k \otimes E_x^* : S^kT_x \otimes E_x^* \longrightarrow N_x^k \otimes E_x^*$, sends $[\mathbf{u}] \otimes s_1^* \in S^kT_x \otimes E_x^*$ to $[\mathbf{u}.t_1] \in U(\mathfrak{l})^k.t_1/U(\mathfrak{l})^{k-1}.t_1 = N_x^k \otimes E_x^*$, where $\mathbf{u} \in U(\mathfrak{l})^k$. Therefore, its dual $(\mathbb{FF}^k \otimes E_x^*)^* : (N_x^k \otimes E_x^*)^* \longrightarrow (S^kT_x \otimes E_x^*)^*$ is given by

$$\varphi \in (N_x^k \otimes E_x^*)^* \longmapsto ([\mathbf{u}] \otimes s_1^* \mapsto \varphi([\mathbf{u}.t_1])).$$

If we identify T_x with $\mathfrak{l}_-$, and $E_x = (\hat{x})^*$ with $S_0 = T^*/\hat{x}^\perp$, then $S_k \to S^kT_x^* \otimes E_x$ is given by

$$\varphi \in S_k \mapsto ((v_1, \dots, v_k) \in S^kT_x \mapsto (-1)^k \rho(v_1) \cdots \rho(v_k)\varphi \in E_x),$$

where $v_1, \dots, v_k \in T_x = \mathfrak{l}_-$. □

Proposition 3.5. *Assume that there is a grading on $\mathfrak{l}$ of depth one, i.e., $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and $[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_{i+j}$ for any $i, j \in \{-1, 0, 1\}$, such that $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Then we have*

$$\rho(\mathfrak{l}_i)(S_k) \subset S_{k+i} \quad \textit{for } i \in \{-1, 0, 1\} \textit{ and for } k \in \{0, \dots, p\}.$$

Proof. By Proposition 3.3(1), $T_x^{(k)} = U(\mathfrak{l})^k.t_1 = S^k\mathfrak{l}.t_1 + U(\mathfrak{l})^{k-1}.t_1$. Since $[\mathfrak{l}_0, \mathfrak{l}_{-1}] \subset \mathfrak{l}_{-1}$ and $[\mathfrak{l}_1, \mathfrak{l}_{-1}] \subset \mathfrak{l}_0$, we have $\tau(\mathfrak{l}_0)(U(\mathfrak{l})^k.t_1) \subset U(\mathfrak{l})^k.t_1$ and $\tau(\mathfrak{l}_1)(U(\mathfrak{l})^k.t_1) \subset U(\mathfrak{l})^{k-1}.t_1$. Hence $\rho(\mathfrak{l}_0)(S_k) \subset S_k$ and $\rho(\mathfrak{l}_1)(S_{k-1}) = S_k$. □

4. Linear differential equations

4.1. Symbols of systems of linear differential equations

Given a vector bundle E over a manifold M, we use $J^p(E)$ to denote the bundle of p-jets of E. We obtain an exact sequence

$$0 \to S^pT^* \otimes E \to J^p(E) \to J^{p-1}(E) \to 0.$$

A subbundle R of $J^p(E)$ is called a *system of linear differential equations of order* p on E. A *solution* of R is a local section s of E satisfying $j_x^p(s) \in R_x$ at each $x \in M$, where $j_x^p(s)$ denotes the p-th jet of s at the point $x \in M$. Additionally, $\sigma_r(R) := \pi_r^p(R) \cap (S^rT^* \otimes E)$ is called the *r-th symbol* of R, and $\sigma(R) = \oplus\sigma_r(R)$ is called the *symbol* of R. A system R of linear differential equations of order p is said to be *of finite type* if $\sigma_p(R) = 0$. A finite type linear differential equation R is said to be *integrable* if for every $x \in M$ and every $\eta \in R_x^p$, there is a solution s of R such that $j_x^p(s) = \eta$.

Let (M, E, R^n) and $(M', E', (R^n)')$ be two systems of linear differential equations of order n. A bundle isomorphism $\phi : E \to E'$ is said to be an isomorphism of R onto R' if the induced bundle isomorphism $J^n(\phi) : J^n(E) \to J^n(E')$ maps R^n onto $(R^n)'$.

4.2. Model systems of type S

Let V and W be two vector spaces and let $S = \bigoplus_{r=0}^p S_r \subset \bigoplus_{r=0}^p S^rV^* \otimes W$ be a subspace with $S_0 = W$ and $S_p = 0$. We say that a linear differential equation (M, E, R) is *of type* S, if there exist linear isomorphisms $z_T : V \simeq T_x$ and $z_E : W \simeq E_x$ such that the induced isomorphism $({}^tz_T^{-1}) \otimes z_E : S^p(V^*) \otimes W \simeq S^p(T_x^*) \otimes E_x$ sends S_p onto $\sigma_p(R)_x$ for every p and for every $x \in M$.

Assume that

(A1) The action of V leaves S invariant,

(A2) The action of V on S is faithful.

Define

$$\mathfrak{a}_r = (S^{r+1}V^* \otimes V) \oplus (S^rV^* \otimes \mathfrak{gl}(W)),$$

$$\mathfrak{a} = \bigoplus_{r \geq -1} \mathfrak{a}_r.$$

Then $\mathfrak{a}$ is the Lie algebra of infinitesimal bundle automorphisms of trivial bundle $E_0 = V \times W$ on V. Define

$$\mathfrak{g} = \{X \in \mathfrak{a} : X(S) \subset S\}, \quad \mathfrak{g}_r = \mathfrak{g} \cap \mathfrak{a}_r.$$

Then $\mathfrak{g}$ is the Lie lagebra of infinitesimal automorphisms of system $\widehat{R}_S = V \times S \subset J^p(E_0)$ of linear differential equations. Define

$$\mathfrak{gl}(S)_r = \{X \in \mathfrak{gl}(S) : X(S_k) \subset S_{k+r} \text{ for any } k\}.$$

Proposition 4.1 (Proposition 2.2.2 of [12])**.** *With the notations as above.*

$$\mathfrak{g}_p = \{X \in \mathfrak{gl}(S)_p : [\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{p-1}\}.$$

The group $GL(V) \times GL(W)$ acts on $\mathfrak{a}$ by the adjoint action: $(aX)(s) = (aXa^{-1})(s)$ for $s \in S$. Define

$$\begin{aligned} G_0 &= \{a \in GL(V) \times GL(W) : a(S) \subset S\}, \\ GL^{(0)}(S) &= \{g \in GL(S) : g(S^r) \subset S^r \text{ for any } r\}, \\ G &= \tilde{G} \cdot G_0, \\ G' &= G \cap GL^{(0)}(S). \end{aligned}$$

Define $\rho_W : G' \to GL(W)$ through the projection $S = \bigoplus_{r=0}^{p} S_r \to S_0 = W$. Let $E_S = G \times_{G'} W$. Then each $s \in S$ defines an element σ_s of $H^0(G/G', E_S)$. Put $(R_S)_x = \{j_x^p(\sigma_s) : s \in S\}$. Then $R_S = \cup_{x \in M}(R_S)_x$ is a system of Linear differential equations. $(G/G', R_S)$ is called the *model equation of type S*.

4.3. Canonical $G^{(0)}$ reductions

Define a subgroup $G^{(0)}$ of $GL^{(0)}(S)$ by

$$G^{(0)} = \{a \in GL^{(0)}(S) : gr(a) \in G_0\}.$$

Then the Lie algebra of $G^{(0)}$ is $\mathfrak{g}^{(0)} = \mathfrak{g}_0 \oplus \bigoplus_{r=1}^{p-1} \mathfrak{gl}(S)_r$.

Proposition 4.2 (Section 3.2 of [12])**.** *Consider an integrable system R of linear differential equations of finite type of order p of type S on E, let $F(R)$ denote the frame bundle of R, and assume $\tilde{\omega}$ is the connection form on $F(R)$ induced from the flat connection on R. Then there is a canonical reduction $P(R)$ of $F(R)$ to subgroup $G^{(0)}$ of $GL(S)$.*

Proposition 4.3 (Proposition 3.2.2 of [12])**.** *Let ω be the restriction to $P(R)$ of the connection form $\tilde{\omega}$ on $F(R)$.*

(1) *We have $R_a^* \omega = Ad(a)^{-1}\omega$ for every $a \in G^{(0)}$.*
(2) *$\omega(X^*) = X$ for every $X \in \mathfrak{g}^{(0)}$, where X^* stands for the fundamental vector field corresponding to X.*
(3) *$d\omega + \frac{1}{2}\omega \wedge \omega = 0$.*
(4) *ω_{-1} is a $\mathfrak{g}_{-1}$-valued basic form.*
(5) *$\omega_q = 0$ for $q \leq -2$.*

Furthermore, $(P(R), \omega)$ characterizes the equivalence class of the system R as follows.

Proposition 4.4 (Proposition 3.3.1 of [12])**.** *Let R and R' be integrable systems of type S. Let $(P(R), \omega)$ (resp., $(P(R'), \omega')$ be the natural reduction of $F(R)$ (resp. $F(R')$) to $G^{(0)}$. Then the isomorphism ϕ of R onto R' induces bundle isomorphism $P(\phi) : (P(R), \omega) \to (P(R'), \omega')$, i.e., $P(\phi)$ is a bundle isomorphism of $P(R)$ onto $P(R')$ satisfying $P(\phi)^*\omega' = \omega$. Conversely, for any isomorphism $\Phi : (P(R), \omega) \to (P(R'), \omega')$, there exists a unique isomorphism ϕ of R onto R' such that $\Phi = P(\phi)$.*

5. Hyperplane systems

Let $M \subset \mathbb{P}(T)$ be a nondegenerate complex submanifold. Assume that E is the line bundle on M obtained by restricting the hyperplane line bundle $\mathcal{O}(1)$ of $\mathbb{P}(T)$. Let Σ be the image of the restriction map $\iota : H^0(\mathbb{P}(T), \mathcal{O}(1)) \to H^0(M, E)$. By the nondegeneracy of M, ι is injective. Thus

$$T^* = H^0(\mathbb{P}(T), \mathcal{O}(1)) \overset{\iota}{\simeq} \Sigma.$$

Let $p > 0$ be a positive integer such that an element of Σ that has a zero of order $\geq p$ at a point of M is identically zero. Define $R_x \subset J^p_x(E)$ by

$$R_x := \{j^p_x(s) : s \in \Sigma\}$$

and define a subbundle $R \subset J^p(E)$ over a Zariski open subset M^o of M by

$$R := \bigcup_{x \in M^o} R_x.$$

We call R the *hyperplane system* on M. Given this construction, R is of finite type and integrable.

Proposition 5.1 (Section 4 of [6])**.** *If $M \subset \mathbb{P}(T)$ is a nondegenerate complex submanifold and R is the hyperplane system on M, then the symbol $\sigma_r(R)$ of R equals $|\mathbb{F}\mathbb{F}^r|$.*

Proposition 5.2 (Proposition 1 of [3])**.** *Suppose M and M' are two nondegenerate submanifolds of $\mathbb{P}(T)$. Let R (R', respectively) be the hyperplane system on M (M', respectively), and assume that R and R' are of the same order. Then R and R' are locally equivalent if and only if there exists a projective linear transformation sending an open subset of M onto an open subset of M'.*

Proposition 5.3. *Let M be a complex submanifold of $\mathbb{P}(T)$ and let S be the symbol of the hyperplane system R on M. Then S satisfies* (A1) *and* (A2) *in Section 4.2.*

Proof. This follows directly from Proposition 3.1. □

Let $L/L' \subset \mathbb{P}(T)$ be a non-degenerate equivariant embedding of a homogeneous space L/L'. By Proposition 5.3, the symbol S of the hyperplane system R on L/L' satisfies (A1) and (A2) in Section 4.2. However, $(L/L', R)$ is not necessarily equivalent to the model system $(G/G', R_S)$ of type S, which was constructed in Section 4.2.

Proposition 5.4 (Proposition 4.4.1 of [12]). *Let $M = L/L' \subset \mathbb{P}(T)$ be a nondegenerate equivariant embedding of a compact Hermitian symmetric space. Let S be the symbol of the hyperplane system R on M. Then the hyperplane system (M, R) is isomorphic to the model system $(G/G', R_S)$ of type S.*

Similarly, we obtain the following proposition.

Proposition 5.5. *Let $M = L/L' \subset \mathbb{P}(T)$ be a nondegenerate equivariant embedding of a homogeneous space of the first kind. Let S be the symbol of the hyperplane system R on M. Then M is an open subset of G/G', and the hyperplane system (M, R) is isomorphic to the model system $(G/G', R_S)$ restricted to M.*

Proof. $\mathfrak{l}_{-1} = \mathfrak{g}_{-1} = V$. It suffices to show that $\mathfrak{l}_0 \subset \mathfrak{g}_0$ and $\mathfrak{l}_1 \subset \mathfrak{g}_1$. By Proposition 3.5, $\mathfrak{l}_i \subset \mathfrak{gl}(S)_i$ for $i = 0, 1$. By Proposition 4.1(2), we have $\mathfrak{l}_i \subset \mathfrak{g}_i$ for $i = 0, 1$. □

6. Normal reductions

Assume $L/L' \subset \mathbb{P}(T)$ is a nondegenerate equivariant embedding of a homogeneous space of the first kind. Then the Lie algebra $\mathfrak{l}$ of L has a gradation of depth one, i.e., $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. If $S = \bigoplus_{r \geq 0} S_r \subset S(V) \otimes W$ is the symbol of the hyperplane system on $L/L' \subset \mathbb{P}(T)$, with $V = \mathfrak{l}_{-1}$ and $W = S_0$, then, by Proposition 5.3, $S \subset S(V) \otimes W$ satisfies (A1) and (A2) in Section 4.2. Let $\mathfrak{g} = \bigoplus \mathfrak{g}_r$ and $\mathfrak{gl}(S) = \bigoplus \mathfrak{gl}(S)_r$ be defined according to Section 4.2. Then $\mathfrak{g}_{-1} = V \subset \mathfrak{gl}(S)_{-1}$.

Let $(G/G', R_S)$ be the model system of type S constructed in Section 4.2. Proposition 4.2 ensures canonical reduction $(P(R_S), \omega_S)$ of the frame bundle $F(R_S)$ of R_S to $G^{(0)}$.

Proposition 6.1 (Proposition 3.4.1 of [12]). *Suppose $(G/G', R_S)$ is the model equation of type S. Then there is a canonical G' reduction $(Q(R_S), \chi_S)$ with $\chi_S := \omega_S|_{Q(R_S)}$, which is a flat Cartan connection of type G/G'.*

In general, it is not obvious that there is a G' reduction $(Q(R), \chi)$ of the frame bundle $F(R)$ when R is an integrable system of linear differential equations of type S. To explain the conditions we need to get a G'-reduction of $F(R)$, we introduce a chain complex associated to the graded Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{g}_r \subset \mathfrak{gl}(S)$.

Set $C^{p,q} = \wedge^q(\mathfrak{g}_{-1})^* \otimes \mathfrak{gl}(S)_{p-1}$ and set $C^q = \bigoplus_p C^{p,q}$ and $C = \bigoplus_q C^q$. We define $\partial : C^{p,q} \to C^{p-1,q+1}$ by

$$\partial c(v_0, \ldots, v_q) = \sum_{i=0}^{q} (-1)^i [v_i, c(v_0, \ldots, \hat{v}_i, \ldots, v_q)]$$

for $c \in C^{p,q}$ and $v_0, \ldots, v_q \in \mathfrak{g}_{-1}$.

Moreover, we define a bilinear form Tr on $\mathfrak{gl}(S)$ by

$$Tr(X, Y) = \text{Trace of the endomorphism } XY \text{ of } S,$$

where $X, Y \in \mathfrak{gl}(S)$. The bilinear form Tr is $Ad(G)$ invariant and nondegenerate. Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{gl}(S)$ with respect to Tr. Then $\mathfrak{g}^\perp$ is $Ad(G)$-invariant.

Put $C(\mathfrak{g}^\perp) = \wedge(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}^\perp$. Then $(C(\mathfrak{g}^\perp), \partial)$ is a subcomplex of (C, ∂). The cohomology space $H^q(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)$ of $(C(\mathfrak{g}^\perp), \partial)$ is called the *Lie algebra cohomology associated to the representation* of $\mathfrak{g}_{-1}$ on $\mathfrak{g}^\perp$. Here, we remark that the representation of $\mathfrak{g}_{-1}$ on $\mathfrak{g}^\perp$ is the restriction of the adjoint representation of $\mathfrak{gl}(S)$ on $\mathfrak{gl}(S)$. It is well-defined because $\mathfrak{g}^\perp$ is $Ad(G)$-invariant.

For $a \in G^{(0)}$ and $c \in C^q$, define $ac \in C^q$ by

$$(ac)(v_1, \ldots, v_q) = Ad(a)c(Ad(a_0^{-1})v_1, \ldots, Ad(a_0^{-1})v_q),$$

where $a = a_0 \exp(X_1) \cdots \exp(X_{n-1})$, $a_0 \in G_0, X_p \in \mathfrak{gl}(S)_p$ for $p = 1, \ldots, n-1$, and $v_1, \ldots, v_q \in \mathfrak{g}_{-1}$.

Let (,) be a positive definite Hermitian inner product on C. Let ∂^* denote the adjoint of ∂, i.e., $(\partial c, c') = (c, \partial c^*)$ for any $c \in C^q, c' \in C^{q+1}$. Then we know

$$C^q = \operatorname{Im} \partial \oplus \operatorname{Ker} \partial^*.$$

If we set $\Delta = \partial\partial^* + \partial^*\partial$ and $\mathcal{H}^q(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) := \operatorname{Ker}(\Delta : C^q(\mathfrak{g}^\perp) \to C^q(\mathfrak{g}^\perp))$, then

$$C^q(\mathfrak{g}^\perp) = \operatorname{Im}\partial \oplus \mathcal{H}^q(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) \oplus \operatorname{Im}\partial^*.$$

Thus, the Lie algebra cohomology space $H^q(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ is isomorphic to $\mathcal{H}^q(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$. We use $\mathcal{H}^q : C^q(\mathfrak{g}^\perp) \to \mathcal{H}^q(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ to denote the projection.

Given an integrable system R of differential equations of type S on a complex manifold M, let $(P(R), \omega)$ be the canonical $G^{(0)}$ reduction of the frame bundle $(F(R), \tilde{\omega})$ (Proposition 4.2). Then ω is a $\mathfrak{gl}(S)$-valued 1-form on $P(R)$.

Proposition 6.2 (Proposition 5.1.1 of [12])**.** *Let $(Q(R), \chi)$ be a G'-reduction of $(P(R), \omega)$. Let $\chi_{\mathfrak{g}}$ (respectively, $\chi_{\mathfrak{g}^\perp}$) be the $\mathfrak{g}$-component ($\mathfrak{g}^\perp$-component, respectively) of χ with respect to the decomposition $\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. Then*

1. *$(Q(R), \chi_{\mathfrak{g}})$ is a Cartan connection of type G/G' over M and*
2. *$\chi_{\mathfrak{g}^\perp}$ is a tensorial 1-form on $Q(R)$.*

Define a function $c : Q(R) \to C^1(\mathfrak{g}^\perp)$ by

$$c(u)(X) = \chi_{\mathfrak{g}^\perp}(X_u^*)$$

for $u \in Q(R)$ and $X \in \mathfrak{l}_{-1}$. Then c is called the *structure function* on $Q(R)$. A G'-reduction $(Q(R), \chi)$ is said to be *normal* if the function c is ∂^*-closed.

Proposition 6.3 (Theorem 5.1.2, Theorem 5.3.1 of [12])**.** *Assume that $L/L' \subset \mathbb{P}(T)$ is a nondegenerate equivariant embedding of a homogeneous space of the first kind. Let $S = \bigoplus_{r \geq 0} S_r \subset S(V) \otimes W$ denote the symbol of the hyperplane system on $L/L' \subset \mathbb{P}(T)$, with $V = \mathfrak{l}_{-1}$ and $W = S_0$ and let $(G/G', R_S)$ be the model system of type S constructed in Section 4.2. Assume that there is a positive definite Hermitian inner product $(\ , \ )$ on $C(\mathfrak{g}^{\perp})$ such that the kernel* Ker ∂^* *of the adjoint operator ∂^* of ∂ with respect to $(\ , \ )$ is invariant under the action of G'. Then*

(1) *For any integrable system R of differential equations of type S, there exists a unique normal reduction $(Q(R), \chi)$ of $(P(R), \omega)$.*
(2) *Let R and R' be integrable systems of type S. Then an isomorphism ϕ of R onto R' induces an isomorphism $Q(\phi) : (Q(R), \chi) \to (Q(R'), \chi')$. Conversely, given an isomorphism $\Phi : Q(R), \chi) \to (Q(R'), \chi')$, there exists a unique isomorphism ϕ of R onto R' such that $\Psi = Q(\phi)$.*
(3) *If the structure function c vanishes identically, then R is locally isomorphic with the model system of type S. Furthermore, the harmonica part $\mathcal{H}c$ of c provides a fundamental system of invariants of R, i.e., c vanishes if and only if $\mathcal{H}c$ vanishes.*

We remark that in Theorem 5.1.2 and Theorem 5.3.1 of [12], one assumes that L/L' is a compact Hermitian symmetric space, while in Proposition 6.3 L/L' is a homogeneous space of the first kind. The same arguments in the proof of these Theorems work if we assume that there is a positive definite Hermitian inner product $(\ , \ )$ on $C(\mathfrak{g}^{\perp})$ such that Ker ∂^*, which is the orthogonal complement of $\operatorname{Im} \partial$ in C, is invariant under the action of G'. For details, see the proof of Theorem 5.1.2 and Theorem 5.3.1 of [12].

Proof of Theorem 1.3. Let $L/L' \subset \mathbb{P}(T)$ be a non-degenerate equivariant embedding of a homogeneous space L/L' of the first kind. Then the symbol S of the hyperplane system $R_{L/L'}$ on L/L', satisfies the conditions (A1) and (A2) by Proposition 5.3. Let $(G/G', R_S)$ be the model system of type S constructed in Section 4.2.

Assume that there is a positive definite Hermitian inner product $(\ , \ )$ on $C(\mathfrak{g}^{\perp})$ such that the kernel Ker ∂^* of the adjoint operator ∂^* of ∂ with respect to $(\ , \ )$ is invariant under the action of G'.

Let M be a complex submanifold of $\mathbb{P}(T)$. Assume that the fundamental forms of M at the general points of M are isomorphic to the fundamental forms of L/L' at the base point. Then the hyperplane system R on M has type S. By Proposition 6.3(1), there is a normal G' reduction $Q(R)$. Let c be the structure function of $Q(R)$. Then $\mathcal{H}c \in H^1(\mathfrak{g}_-, \mathfrak{g}^{\perp})$. By the assumption that $H^1(\mathfrak{g}_-, \mathfrak{g}^{\perp}) = 0$, we have $\mathcal{H}c = 0$. By Proposition 6.3(3), R is locally isomorphic to the model system R_S. By Proposition 5.5 R_S is isomorphic to

$R_{L/L'}$. Therefore, M is projectively equivalent to an open subset of L/L' by Proposition 5.1. □

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