Commun. Korean Math. Soc. **30** (2015), No. 3, pp. 227–237
http://dx.doi.org/10.4134/CKMS.2015.30.3.227

π AND OTHER FORMULAE IMPLIED BY HYPERGEOMETRIC SUMMATION THEOREMS

Yong Sup Kim, Arjun Kumar Rathie, and Xiaoxia Wang

Abstract. By employing certain extended classical summation theorems, several surprising π and other formulae are displayed.

1. Introduction

In the usual notation, let $\mathbb{C}$ denote the set of complex numbers. For $\alpha_j \in \mathbb{C}(j = 1, \ldots, p)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (\mathbb{Z}_0^- = \mathbb{Z} \cup \{0\})(j = 1, \ldots, q)$, the generalized hypergeometric function ${}_pF_q$ with p numerator parameters $\alpha_1, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \ldots, \beta_q$ is defined as [3]

$$ {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix} \,\Big|\, z \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \cdot \frac{z^k}{k!}, \tag{1.1} $$

$(p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ p \leq q+1;\ p \leq q$ and $|\, z \,| < \infty;\ p = q+1$ and $|\, z \,| < 1;$ $p = q+1$ and $\mathrm{Re}(\omega) > 0$), where

$$ \omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \tag{1.2} $$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.3} $$

It should be remarked here that whenever generalized hypergeometric functions reduce to quotients of gamma functions, the results are very important from the applications point of view.

Received April 16, 2015.
2010 *Mathematics Subject Classification.* 33C15, 33C20.
Key words and phrases. π formula, Gauss summation theorem, Bailey summation theorem, Watson summation theorem, extension summation theorem.
Y. S. Kim is grateful for support by Wonkwang University Research Fund (2015) and X. Wang acknowledges support of Shanghai Leading Academic Discipline Project S30104.

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role.

Recently, good progress has been done in the direction of generalizing the above mentioned classical summation theorems. For details, we refer the papers [11, 12, 13, 14, 16, 17].

On the other hand, formulas for π–series have been obtained by several mathematicians, see for examples, the papers [1, 4, 5, 6, 7, 8, 9, 15, 18, 19, 20]. But for the history and some introductive information on the formulae for π–series, we especially refer very interesting and useful research papers by Bailey–Borwein [2] and Guillera [10].

In our present investigation, we require the following summation theorems obtained earlier by Kim et al. [11]. These summation theorems are included so that the paper may be self contained.

Gauss summation theorem

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.4} $$

provided $\mathrm{Re}(c-a-b) > 0$.

Extension of Gauss summation theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ & c+1, & d \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b)+\frac{ab}{d}\right] \tag{1.5} $$

for $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(c-a-b) > 0$.

Remark. In (1.5), if we take $d = c$, we recover Gauss summation theorem (1.4).

Gauss second summation theorem

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \,\middle|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}. \tag{1.6} $$

Extension of Gauss second summation theorem

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+3), & d \end{matrix} \,\middle|\, \frac{1}{2}\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2})} \\ &\times \left[\frac{\frac{1}{2}(a+b-1)-\frac{ab}{d}}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})} + \frac{\frac{a+b+1}{d}-2}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}\right]. \end{aligned} \tag{1.7} $$

Remark. In (1.7), if we take $d = \frac{1}{2}(a+b+1)$, we recover Gauss second summation theorem (1.6).

Bailey summation theorem

$$(1.8)\qquad {}_2F_1\left[\begin{matrix} a,\ 1-a \\ c \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}c)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}.$$

Extension of Bailey summation theorem

$$(1.9)\qquad {}_3F_2\left[\begin{matrix} a,\, 1-a,\, d+1 \\ c+1,\quad d \end{matrix} \,\Big|\, \frac{1}{2}\right]$$
$$= 2^{-c}\Gamma(\frac{1}{2})\Gamma(c+1)$$
$$\times\Big[\frac{\frac{2}{d}}{\Gamma(\frac{1}{2}a+\frac{1}{2}c)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} + \frac{1-\frac{c}{d}}{\Gamma(\frac{1}{2}a+\frac{1}{2}c+\frac{1}{2})\Gamma(\frac{1}{2}c-\frac{1}{2}a+1)}\Big].$$

Remark. In (1.9), if we take $d = c$, we recover Bailey summation theorem (1.8).

Watson summation theorem

$$(1.10)\qquad {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+1), 2c \end{matrix} \,\Big|\, 1\right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})},$$

provided $\mathrm{Re}(2c-a-b) > -1$.

Extension of Watson summation theorem

(1.11)
$${}_4F_3\left[\begin{matrix} a,\quad b,\quad c \qquad d+1 \\ \frac{1}{2}(a+b+1), 2c+1,\quad d \end{matrix} \,\Big|\, 1\right]$$
$$= \frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$
$$\times\Big[\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} + (\frac{2c-d}{d})\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+1)}\Big],$$

provided $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(2c-a-b) > -1$.

Remark. In (1.11), if we take $d = 2c$, we recover Watson summation theorem (1.10).

2. Main results

Our main results are given in the following theorems, corollaries and examples.

2.1. Summation formulae implied by Gauss summation theorem (1.4)

Letting $a = \frac{1}{2}+m$, $b = \frac{1}{2}-m$ and $c = \frac{3}{2}+m$ in (1.4), we achieve the identity.

Theorem 1. *For $m \in \mathbb{N}_0$, there holds the summation formulae for π*

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}+m,\ \frac{1}{2}-m\\ \frac{3}{2}+m\end{matrix}\ \Big|\ 1\right] = \frac{\pi}{2^{2m+1}}\frac{(\frac{3}{2})_m}{m!}. $$

Example 1 ($m = 0$)**.**

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2}\end{matrix}\ \Big|\ 1\right] = \frac{\pi}{2}. $$

Example 2 ($m = 1$)**.**

$$ {}_2F_1\left[\begin{matrix}\frac{3}{2},\ -\frac{1}{2}\\ \frac{5}{2}\end{matrix}\ \Big|\ 1\right] = \frac{3\pi}{16}. $$

Example 3 ($m = 2$)**.**

$$ {}_2F_1\left[\begin{matrix}\frac{5}{2},\ -\frac{3}{2}\\ \frac{7}{2}\end{matrix}\ \Big|\ 1\right] = \frac{15\pi}{256}. $$

Example 4 ($m = 3$)**.**

$$ {}_2F_1\left[\begin{matrix}\frac{7}{2},\ -\frac{5}{2}\\ \frac{9}{2}\end{matrix}\ \Big|\ 1\right] = \frac{35\pi}{2048}. $$

Example 5 ($m = 4$)**.**

$$ {}_2F_1\left[\begin{matrix}\frac{9}{2},\ -\frac{7}{2}\\ \frac{11}{2}\end{matrix}\ \Big|\ 1\right] = \frac{315\pi}{65536}. $$

Remark. Results given in Examples 2, 3 and 4 are obtained recently by Wei et al. [18] by employing Whipple summation theorem.

2.2. Summation formulae implied by extension of Gauss summation theorem (1.5)

Letting $a = \frac{1}{2} + m$, $b = \frac{1}{2} - m$ and $c = \frac{3}{2} + m$ in (1.5), we achieve the following identity.

Theorem 2. *For $m \in \mathbb{N}_0$ and $\mathrm{Re}(d) > 0$, there holds the summation formulae for π*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}+m,\ \frac{1}{2}-m,\ d+1\\ \frac{5}{2}+m,\quad d\end{matrix}\ \Big|\ 1\right] = (1+\frac{1-2m}{2d})\frac{3\pi}{2^{2m+3}m!}(\frac{5}{2})_m. $$

Corollary 1 $(m = 0)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ d+1\\ \frac{5}{2},\quad d\end{matrix}\ \Big|\ 1\right] = \frac{3\pi}{8}\left(1+\frac{1}{2d}\right). $$

Example 6 $(d = 1)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 2\\ \frac{5}{2},\ 1\end{matrix}\ \Big|\ 1\right] = \frac{9\pi}{16}. $$

Example 7 $(d = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 3\\ \frac{5}{2},\ 2\end{matrix}\ \Big|\ 1\right] = \frac{15\pi}{32}. $$

Example 8 $(d = 3)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 4\\ \frac{5}{2},\ 3\end{matrix}\ \Big|\ 1\right] = \frac{7\pi}{16}. $$

Corollary 2 $(m = 1)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2},\ -\frac{1}{2},\ d+1\\ \frac{7}{2},\quad d\end{matrix}\ \Big|\ 1\right] = \frac{15\pi}{64}\left(1-\frac{1}{2d}\right). $$

Example 9 $(d = 1)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2},\ -\frac{1}{2},\ 2\\ \frac{7}{2},\ 1\end{matrix}\ \Big|\ 1\right] = \frac{15\pi}{128}. $$

Example 10 $(d = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2},\ -\frac{1}{2},\ 3\\ \frac{7}{2},\ 2\end{matrix}\ \Big|\ 1\right] = \frac{45\pi}{256}. $$

Example 11 $(d = 3)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2},\ -\frac{1}{2},\ 4\\ \frac{9}{2},\ 3\end{matrix}\ \Big|\ 1\right] = \frac{75\pi}{384}. $$

Corollary 3 $(m = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{5}{2},\ -\frac{3}{2},\ d+1\\ \frac{9}{2},\quad d\end{matrix}\ \Big|\ 1\right] = \frac{105\pi}{1024}\left(1-\frac{3}{2d}\right). $$

Example 12 $(d = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}\frac{5}{2},\ -\frac{3}{2},\ 3\\ \frac{9}{2},\ 2\end{matrix}\ \Big|\ 1\right] = \frac{105\pi}{4096}. $$

Example 13 ($d = 3$)**.**

$$ {}_3F_2\left[\begin{matrix}\frac{5}{2}, -\frac{3}{2}, 4\\ \frac{9}{2}, \ 3\end{matrix}\ \Big|\ 1\right] = \frac{105\pi}{2048}. $$

Example 14 ($d = 4$)**.**

$$ {}_3F_2\left[\begin{matrix}\frac{5}{2}, -\frac{3}{2}, 5\\ \frac{9}{2}, \ 4\end{matrix}\ \Big|\ 1\right] = \frac{525\pi}{8192}. $$

2.3. Summation formulae implied by extension of Gauss second summation theorem (1.7)

Letting $a = 1 + 2m$ and $b = 1 + 2n$ in (1.7), we get the following identity.

Theorem 3. *For* $m, n \in \mathbb{N}_0$ *and* $\mathrm{Re}(d) > 0$*, there holds the summation formula*

$$ {}_3F_2\left[\begin{matrix}1+2m, \quad 1+2n, \quad d+1\\ m+n+\frac{5}{2}, \quad d\end{matrix}\ \Big|\ \frac{1}{2}\right] $$
$$ = \frac{2\pi(\frac{3}{2})_{m+n+1}}{(2m-2n+1)(2m-2n-1)} $$
$$ \times \Big[\frac{(m+n+\frac{1}{2}) - \frac{1}{d}(2m+1)(2n+1)}{m!\,n!} + \frac{\frac{1}{d}(2m+2n+3) - 2}{\pi(\frac{1}{2})_m(\frac{1}{2})_n}\Big]. $$

Corollary 4 ($m = n = 0$)**.**

$$ {}_3F_2\left[\begin{matrix}1, 1, d+1\\ \frac{5}{2}, \quad d\end{matrix}\ \Big|\ \frac{1}{2}\right] = 3\pi(\frac{1}{d} - \frac{1}{2}) - 3(\frac{3}{d} - 2). $$

Example 15 ($d = 2$)**.**

$$ {}_3F_2\left[\begin{matrix}1, 1, 3\\ \frac{5}{2}, 2\end{matrix}\ \Big|\ \frac{1}{2}\right] = \frac{3}{2}. $$

Example 16 ($d = 3/2$)**.**

$$ {}_2F_1\left[\begin{matrix}1, 1\\ \frac{3}{2}\end{matrix}\ \Big|\ \frac{1}{2}\right] = \frac{\pi}{2}. $$

Corollary 5 ($m = 1, n = 0$)**.**

$$ {}_3F_2\left[\begin{matrix}3, 1, d+1\\ \frac{7}{2}, \quad d\end{matrix}\ \Big|\ \frac{1}{2}\right] = \frac{5\pi}{2}\big[3(\frac{1}{2} - \frac{1}{d}) + \frac{2}{\pi}(\frac{5}{d} - 2)\big]. $$

Example 17 ($d = 2$)**.**

$$ {}_3F_2\left[\begin{matrix}3, 1, 3\\ \frac{7}{2}, 2\end{matrix}\ \Big|\ \frac{1}{2}\right] = \frac{5}{2}. $$

Example 18 ($d = 5/2$).

$$ {}_2F_1\left[\begin{matrix} 3, \ 1 \\ \frac{5}{2} \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{3\pi}{4}. $$

Corollary 6 ($m = 1, n = 1$).

$$ {}_3F_2\left[\begin{matrix} 3, \ 3, \ d+1 \\ \frac{9}{2}, \quad d \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{105\pi}{4}(\frac{9}{d} - \frac{5}{2}) - 105(\frac{7}{d} - 2). $$

Example 19 ($d = 7/2$).

$$ {}_2F_1\left[\begin{matrix} 3, \ 3 \\ \frac{7}{2} \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{15\pi}{8}. $$

Corollary 7 ($m = 2, n = 0$).

$$ {}_3F_2\left[\begin{matrix} 5, \ 1, \ d+1 \\ \frac{9}{2}, \quad d \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{7\pi}{4}\Big[\frac{5}{2}(\frac{1}{2} - \frac{1}{d}) + \frac{4}{3\pi}(\frac{7}{d} - 2)\Big]. $$

Example 20 ($d = 2$).

$$ {}_3F_2\left[\begin{matrix} 5, \ 1, \ 3 \\ \frac{9}{2}, \ 2 \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{7}{2}. $$

Example 21 ($d = 7/2$).

$$ {}_2F_1\left[\begin{matrix} 5, \ 1 \\ \frac{7}{2} \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{15\pi}{16}. $$

Remark. The results given in Examples 16, 18 and 21 are also obtained recently by Wei et al. [18].

2.4. Summation formulae implied by extension of Bailey summation theorem (1.9)

Letting $a = \frac{1}{2} + m$ and $c = \frac{3}{2} + m + 2n$ in (1.9), we achieve the following identity.

Theorem 4. *For* $m, n \in \mathbb{N}_0$ *and* $\mathrm{Re}(d) > 0$*, there holds the summation formula*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}+m, \quad \frac{1}{2}-m, \quad d+1 \\ m+2n+\frac{5}{2}, \quad d \end{matrix} \Big| \ \frac{1}{2}\right] $$

$$ = \frac{\pi}{2^{m+2n+\frac{5}{2}}}(\frac{3}{2})_{m+2n+1}\Big[\frac{\frac{2}{d}}{(m+n)!\, n!} + \frac{1 - \frac{3/2+m+2n}{d}}{\pi(\frac{1}{2})_{m+n+1}(\frac{1}{2})_{n+1}}\Big]. $$

Corollary 8 ($m = n = 0$).

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \ \frac{1}{2}, \ d+1 \\ \frac{5}{2}, \quad d \end{matrix} \Big| \ \frac{1}{2}\right] = \frac{3\pi\sqrt{2}}{8d} + \frac{3\sqrt{2}}{4}(1 - \frac{3}{2d}). $$

Example 22 $(d = 3/2)$.

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{\pi\sqrt{2}}{4}. $$

Corollary 9 $(m = 1, n = 0)$.

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2}, -\frac{1}{2}, d+1 \\ \frac{7}{2}, \quad d\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{15\pi\sqrt{2}}{32d} + \frac{5\sqrt{2}}{8}(1 - \frac{5}{2d}). $$

Example 23 $(d = 5/2)$.

$$ {}_2F_1\left[\begin{matrix}\frac{3}{2}, -\frac{1}{2} \\ \frac{5}{2}\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{3\pi\sqrt{2}}{16}. $$

Corollary 10 $(m = 2, n = 0)$.

$$ {}_3F_2\left[\begin{matrix}\frac{5}{2}, -\frac{3}{2}, d+1 \\ \frac{9}{2}, \quad d\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{105\pi\sqrt{2}}{256d} + \frac{7\sqrt{2}}{16}(1 - \frac{7}{2d}). $$

Example 24 $(d = 7/2)$.

$$ {}_2F_1\left[\begin{matrix}\frac{5}{2}, -\frac{3}{2} \\ \frac{7}{2}\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{15\pi\sqrt{2}}{128}. $$

Corollary 11 $(m = 3, n = 0)$.

$$ {}_3F_2\left[\begin{matrix}\frac{7}{2}, -\frac{5}{2}, d+1 \\ \frac{11}{2}, \quad d\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{315\pi\sqrt{2}}{1024d} + \frac{9\sqrt{2}}{32}(1 - \frac{9}{2d}). $$

Example 25 $(d = 9/2)$.

$$ {}_2F_1\left[\begin{matrix}\frac{7}{2}, -\frac{5}{2} \\ \frac{9}{2}\end{matrix} \;\Big|\; \frac{1}{2}\right] = \frac{35\pi\sqrt{2}}{512}. $$

2.5. Summation formulae implied by extension of Watson summation theorem (1.11)

Letting $a = 1 + 2m$, $b = 1 + 2n$ and $c = 1 + m + n + s$ in (1.11), we obtain the following identity.

Theorem 5. *For $m, n, s \in \mathbb{N}_0$ and $\mathrm{Re}(d) > 0$, there holds the summation formula*

$$ {}_4F_3\left[\begin{matrix}1+2m, \quad 1+2n, \quad 1+m+n+s, \quad d+1 \\ m+n+\frac{3}{2}, 3+2m+2n+2s, \quad d\end{matrix} \;\Big|\; 1\right] $$
$$ = \frac{\pi}{4}\frac{(\frac{3}{2})_{m+n+s}(\frac{3}{2})_{m+n}(\frac{1}{2})_s}{(\frac{1}{2})_m(\frac{1}{2})_n m!\; n!}\Big[\pi\frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(m+s)!(n+s)!} + \frac{1}{\pi}\frac{(\frac{2+2m+2n+2s}{d} - 1)m!\, n!}{(\frac{1}{2})_{m+s+1}(\frac{1}{2})_{n+s+1}}\Big]. $$

Corollary 12 $(m = n = s = 0)$**.**

$$ {}_4F_3\left[\begin{matrix}1,\ 1,\ 1,\ d+1\\ \frac{3}{2},\ 3,\quad d\end{matrix}\ \Big|\ 1\right] = \frac{\pi^2}{4} + (\frac{2}{d} - 1). $$

Example 26 $(d = 1)$**.**

$$ {}_3F_2\left[\begin{matrix}1,\ 1,\ 2\\ \frac{3}{2},\ 3\end{matrix}\ \Big|\ 1\right] = \frac{\pi^2}{4} + 1. $$

Example 27 $(d = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\ 2\end{matrix}\ \Big|\ 1\right] = \frac{\pi^2}{4}. $$

Example 28 $(d = 3)$**.**

$$ {}_4F_3\left[\begin{matrix}1,\ 1,\ 1,\ 4\\ \frac{3}{2},\ 3,\ 3\end{matrix}\ \Big|\ 1\right] = \frac{\pi^2}{4} - \frac{1}{3}. $$

Corollary 13 $(m = 1, n = s = 0)$**.**

$$ {}_4F_3\left[\begin{matrix}3,\ 1,\ 2,\ d+1\\ \frac{5}{2},\ 5,\quad d\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16} + 3(\frac{4}{d} - 1). $$

Example 29 $(d = 1)$**.**

$$ {}_3F_2\left[\begin{matrix}3,\ 2,\ 2\\ \frac{5}{2},\ 5\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16} + 9. $$

Example 30 $(d = 2)$**.**

$$ {}_3F_2\left[\begin{matrix}3,\ 1,\ 3\\ \frac{5}{2},\ 5\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16} + 3. $$

Example 31 $(d = 3)$**.**

$$ {}_3F_2\left[\begin{matrix}1,\ 2,\ 4\\ \frac{5}{2},\ 5\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16} + 1. $$

Example 32 $(d = 4)$**.**

$$ {}_3F_2\left[\begin{matrix}3,\ 1,\ 2\\ \frac{5}{2},\ 4\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16}. $$

Example 33 $(d = 5)$**.**

$$ {}_4F_3\left[\begin{matrix}3,\ 1,\ 2,\ 6\\ \frac{5}{2},\ 5,\ 5\end{matrix}\ \Big|\ 1\right] = \frac{9\pi^2}{16} - \frac{3}{5}. $$

Corollary 14 $(m = 2, n = s = 0)$.

$$ {}_4F_3\left[\begin{matrix} 5,\ 1,\ 3,\ d+1 \\ \frac{7}{2},\ 7,\ \ d \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + 5(\frac{6}{d} - 1). $$

Example 34 $(d = 1)$.

$$ {}_3F_2\left[\begin{matrix} 5,\ 3,\ 2 \\ \frac{7}{2},\ 7 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + 25. $$

Example 35 $(d = 2)$.

$$ {}_4F_3\left[\begin{matrix} 5,\ 1,\ 3,\ 3 \\ \frac{7}{2},\ 7,\ 2 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + 10. $$

Example 36 $(d = 3)$.

$$ {}_3F_2\left[\begin{matrix} 5,\ 1,\ 4 \\ \frac{7}{2},\ 7 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + 5. $$

Example 37 $(d = 4)$.

$$ {}_4F_3\left[\begin{matrix} 5,\ 1,\ 3,\ 5 \\ \frac{7}{2},\ 7,\ 4 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + \frac{5}{2}. $$

Example 38 $(d = 5)$.

$$ {}_3F_2\left[\begin{matrix} 1,\ 3,\ 6 \\ \frac{7}{2},\ 7 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} + 1. $$

Example 39 $(d = 6)$.

$$ {}_3F_2\left[\begin{matrix} 5,\ 1,\ 3 \\ \frac{7}{2},\ 6 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256}. $$

Example 40 $(d = 7)$.

$$ {}_4F_3\left[\begin{matrix} 5,\ 1,\ 3,\ 8 \\ \frac{7}{2},\ 7,\ 7 \end{matrix} \,\Big|\, 1\right] = \frac{225\pi^2}{256} - \frac{5}{7}. $$

References

[1] V. Adamchik and S. Wagon, *A simple formula for π*, Amer. Math. Monthly **104** (1997), no. 9, 852–855.

[2] D. H. Bailey and J. M. Borwein, *Experimental mathematics: examples, methods and implications*, Notices Amer. Math. Soc. **52** (2005), no. 5, 502–514.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.

[4] W. N. Bailey, P. B. Borwein, and S. Plouffe, *On the rapid computation of various polylogarithmic constants*, Math. Comp. **66** (1997), no. 218, 903–913.

[5] J. M. Borwein and P. B. Borwein, *π and the AGM*, John Wiley & Sons, Inc., New York, 1987.

[6] H. C. Chan, *More formulas for π*, Amer. Math. Monthly **113** (2006), no. 5, 452–455.

[7] W. Chu, *π-formulae implied by Dougall's summation theorem for ${}_5F_4$-series*, Ramanujan J. **26** (2011), no. 2, 251–255.

[8] B. Gourévitch and J. Guillera, *Construction of binomial sums for π and polylogarithmic constant inspired by BBP formulas*, Appl. Math. E-Notes **7** (2007), 237–246.

[9] J. Guillera, *Hypergeometric identities for 10 extended Ramanujan type series*, Ramanujan J. **15** (2008), no. 2, 219–234.

[10] ———, *History of the formulas and algorithms for π*, Gems in experimental mathematics, 173–188, Contemp. Math., 517, Amer. Math. Soc., Providence, RI, 2010.

[11] Y. S. Kim, M. A. Rakha, and A. K. Rathie, *Extensions of certain classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations*, Int. J. Math. Math. Sci. **2010** (2010), 309503, 26 pp.

[12] J. L. Lavoie, F. Grondin, and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a ${}_3F_2$*, Indian J. Math. **32** (1992), no. 1, 23–32.

[13] ———, *Generalizations of Whipple's theorem on the sum of a ${}_3F_2$*, J. Comput. Appl. Math. **72** (1996), no. 2, 293–300.

[14] J. L. Lavoie, F. Grondin, A. K. Rathie, and K. Arora, *Generalizations of Dixon's theorem on the sum of a ${}_3F_2$*, Math. Comp. **62** (1994), no. 205, 267–276.

[15] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Gordon and Breach Science, New York, 1986.

[16] M. A. Rakha and A. K. Rathie, *Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$*, Integral Transforms Spec. Funct. **22** (2011), no. 11, 823–840.

[17] R. Vidunas, *A generalization of Kummer identity*, Rocky Mountain J. Math. **32** (2002), no. 2, 919–936.

[18] C. Wei, D. Gong, and J. Li, *π-Formulas with free parameters*, J. Math. Anal. Appl. **396** (2012), no. 2, 880–887.

[19] W. E. Weisstein, *Pi Formulas*, MathWorld–A Wolfram Web Resourse, http://math-world. wolfram.com/PiForm-ulas.html.

[20] D. Zheng, *Multisection method and further formulae for π*, Indian J. Pure Appl. Math. **139** (2008), no. 2, 137–156.

Yong Sup Kim
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail address: yspkim@wonkwang.ac.kr

Arjun K. Rathie
Department of Mathematics
School of Mathematical and Physical Sciences
Central University of Kerala
Riverside Transit Campus
Padennakkad P.O. Nileshwar Kasaragod-671 328, Kerala State, India
E-mail address: akrathie@rediffmail.com

Xiaoxia Wang
Department of Mathematics
Shanghai University
Shanghai, 200444, P. R. China
E-mail address: xiaoxiawang@shu.edu.cn