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SPECTRAL CLASSES AND THE PARAMETER DISTRIBUTION SET

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Abstract. The natural projection of a parameter lower (upper) distribution set for a self-similar measure on a self-similar set satisfying the open set condition is the cylindrical lower or upper local dimension set for the Legendre self-similar measure which is derived from the self-similar measure and the self-similar set.

1. Introduction

Recently, we [1] investigated the relation between spectral classes of a self-similar Cantor set in a set theoretical sense. More recently, using the parameter distribution, we find the parallel results for the self-similar set (attractor of the IFS consisting of $n(\geq 2)$ similitudes satisfying the OSC (open set condition)) instead of the self-similar Cantor set (attractor of the IFS consisting of 2 similitudes satisfying the SSC (strong separation condition)), which leads to a generalization of [1]. In this paper, we define the Legendre self-similar measures on the self-similar set which is derived from the self-similar measure and the self-similar set. Using the Legendre self-similar measures on the self-similar set, we give full relationship between the natural projection of a parameter lower (upper) distribution set for a self-similar measure on a self-similar set and the cylindrical lower or upper local dimension set for the Legendre self-similar measures.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the set of real numbers respectively. An attractor K in the d-dimensional Euclidean space $\mathbb{R}^d$ of the IFS $(f_1, \ldots, f_N)$ of contractions where $N \geq 2$ makes each point $v \in K$ have an

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infinite sequence $\omega = (m_1, m_2, \ldots) \in \Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ where

$$\{v\} = \bigcap_{n=1}^{\infty} K_{\omega|n}$$

for $K_{\omega|n} = K_{m_1,\ldots,m_n} = f_{m_1} \circ \cdots \circ f_{m_n}(K)$ [4]. $\omega|n$ denotes the truncation of ω to the nth place. In such case, we sometimes write $\pi(\omega)$ for such v using the natural projection $\pi : \Sigma \to K$ and call $K_{\omega|n}$ the cylinder of v. We note that $K_{\omega|n}$ may be different for the same $v \in K$ since v may have different codes ω. Therefore we write $K_{\omega|n}$ for such distinction for the cylinder of v. We call such $K_{\omega|n}$ the cylinders of K and call K a self-similar set if the IFS $(f_1, \ldots, f_N)$ are similitudes. Each infinite sequence $\omega = (m_1, m_2, \ldots)$ in the coding space Σ has the unique subset $A(x_n(\omega))$ of its accumulation points in the simplex of probability vectors in $\mathbb{R}^N$ of the vector-valued sequence $\{x_n(\omega)\} = \{(u_1, \ldots, u_N)_n\}$ of the probability vectors where u_k for $1 \leq k \leq N$ in the probability vector $(u_1, \ldots, u_N)_n$ for each $n \in \mathbb{N}$ is defined by

$$u_k = \frac{|\{1 \leq l \leq n : m_l = k\}|}{n}.$$

From now on, we assume that the similarity ratios of the similarities $(f_1, \ldots, f_N)$ are $a_1, \ldots, a_N$ and K is the self-similar set for the IFS $(f_1, \ldots, f_N)$ satisfying the open set condition [3, 4, 5] and $\gamma_{\mathbf{p}}$ on K is the self-similar measure associated with $\mathbf{p} = (p_1, \ldots, p_N) \in (0,1)^N$ satisfying $\sum_{k=1}^N p_k = 1$. To avoid the degeneration case, we also assume that $\mathbf{p} = (p_1, \ldots, p_N) \neq (a_1^s, \ldots, a_N^s)$ with $\sum_{k=1}^N a_k^s = 1$.

For each $q \in \mathbb{R}$, we define *the Legendre self-similar measure by the self-similar measure* $\gamma_{\mathbf{p}}$ *with respect to* q *on the self-similar set* K by the self-similar measure $\gamma_{\mathbf{r}}$ associated with $\mathbf{r} = (r_1, \ldots, r_N) \in (0,1)^N$ satisfying $r_k = p_k^q a_k^{\beta(q)}$ such that $\sum_{k=1}^N p_k^q a_k^{\beta(q)} = 1$. In particular, if $q = 1$, then $\mathbf{r} = \mathbf{p}$.

We write $\underline{E}_\alpha^{(\mathbf{r})}$ ($\overline{E}_\alpha^{(\mathbf{r})}$) for the set of points at which the cylindrical lower (upper) local dimension of $\gamma_{\mathbf{r}}$ on K is exactly α, so that

$$\underline{E}_\alpha^{(\mathbf{r})} = \pi\{\omega \in \Sigma : \liminf_{n \to \infty} \frac{\log \gamma_{\mathbf{r}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha\},$$

$$\overline{E}_\alpha^{(\mathbf{r})} = \pi\{\omega \in \Sigma : \limsup_{n \to \infty} \frac{\log \gamma_{\mathbf{r}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha\}.$$

Here $|K_{\omega|n}|$ denotes the diameter of the cylinder $K_{\omega|n}$. From now on, we fix distinct i, j respectively satisfying

$$\frac{\log p_i}{\log a_i} = \min_{1 \leq k \leq N} \frac{\log p_k}{\log a_k} < \max_{1 \leq k \leq N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}.$$

In [2], for $\frac{t \log p_i + (1-t) \log p_j}{t \log a_i + (1-t) \log a_j} = \alpha(t)$ and $g(\mathbf{y}, \mathbf{p}) = \frac{\sum_{k=1}^N y_k \log p_k}{\sum_{k=1}^N y_k \log a_k}$ where $\mathbf{y} = (y_1, \ldots, y_N)$ in the $(N-1)$-simplex, the lower and upper parameter distribution sets for the self-similar measure $\mathbf{p}$ on the self-similar set K are represented by

$$\underline{F}(t) = \{\omega : \max_{\mathbf{y}\in A(x_n(\omega))} g(\mathbf{y},\mathbf{p}) = \alpha(t)\}$$

and

$$\overline{F}(t) = \{\omega : \min_{\mathbf{y}\in A(x_n(\omega))} g(\mathbf{y},\mathbf{p}) = \alpha(t)\}.$$

3. Subset relation and multifractal spectrum

In the following theorems, let t_0 be the real number satisfying

$$\frac{t_0 \log p_i + (1-t_0)\log p_j}{t_0 \log a_i + (1-t_0)\log a_j} = g(\mathbf{r}_0,\mathbf{p})$$

for $\mathbf{r}_0 = (a_1^s,\dots,a_N^s)$ with $\sum_{k=1}^N a_k^s = 1$. In the following theorem, given $0 < t < 1$, we have $\alpha(t) = \frac{t\log p_i + (1-t)\log p_j}{t\log a_i + (1-t)\log a_j}$. For $\beta'(q) = -\alpha(t)$, we put $\mathbf{r}(t) = (r_1,\dots,r_N)$ satisfying $r_k = p_k^q a_k^{\beta(q)}$. The following theorem gives the relation between the parameter distribution set and the cylindrical local dimension set for the Legendre self-similar measure by the self-similar measure $\gamma_{\mathbf{p}}$ on the self-similar set K.

Theorem 3.1. *Let $0 \le t \le 1$ and $0 < t' < t_0 < t'' < 1$. Then we have the followings*:

(1) $\pi(\underline{F}(t)) = \underline{E}^{(\mathbf{r}(t'))}_{g(\mathbf{r}(t),\mathbf{r}(t'))} = \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))} = \overline{E}^{(\mathbf{p})}_{g(\mathbf{r}(t),\mathbf{p})}$,

(2) $\pi(\overline{F}(t)) = \overline{E}^{(\mathbf{r}(t'))}_{g(\mathbf{r}(t),\mathbf{r}(t'))} = \underline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))} = \underline{E}^{(\mathbf{p})}_{g(\mathbf{r}(t),\mathbf{p})}$.

Proof. We note that

$$0 < t_0 \le t < 1 \iff q \ge 0$$

and

$$0 < t \le t_0 < 1 \iff q \le 0.$$

Therefore

$$t = t_0 \iff q = 0.$$

From the proof of Theorem 4.1 of [2], it is not to difficult to show that $\pi(\underline{F}(t)) = \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))}$ where $\mathbf{r}(t'') = (p_1^{q''} a_1^{\beta(q'')},\dots,p_N^{q''} a_N^{\beta(q'')})$ such that

$$-\beta'(q'') = \frac{t''\log p_i + (1-t'')\log p_j}{t''\log a_i + (1-t'')\log a_j}$$

for $t_0 < t'' < 1$.

Precisely, if $v \in \pi(\underline{F}(t))$, then $v = \pi(\omega)$ with $g(\mathbf{y},\mathbf{p}) \le \alpha(t)$ for all $\mathbf{y} \in A(x_n(\omega))$, and $g(\mathbf{y},\mathbf{p}) = \alpha(t)$ for some $\mathbf{y} \in A(x_n(\omega))$ since

$$\{\omega : \max_{\mathbf{y}\in A(x_n(\omega))} g(\mathbf{y},\mathbf{p}) = \alpha(t)\} = \underline{F}(t)$$

by Lemma 3.3 of [2]. We note that $\alpha(t) = g(\mathbf{r}(t),\mathbf{p})$ from (11.35) of [4]. Since $q'' \ge 0$, we have $g(\mathbf{y},\mathbf{r}(t'')) \le \alpha(t)q'' + \beta(q'') = g(\mathbf{r}(t),\mathbf{r}(t''))$ for all

$\mathbf{y} \in A(x_n(\omega))$, and $g(\mathbf{y}, \mathbf{r}(t'')) = \alpha(t)q'' + \beta(q'') = g(\mathbf{r}(t), \mathbf{r}(t''))$ for some $\mathbf{y} \in A(x_n(\omega))$. Let

$$G = \{\omega : \max_{\mathbf{y} \in A(x_n(\omega))} g(\mathbf{y}, \mathbf{r}(t'')) = g(\mathbf{r}(t), \mathbf{r}(t''))\}.$$

Then $\pi(G) = \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))}$ by the similar arguments of the proof of Theorem 3.4 of [2]. Therefore $\omega \in G$. This gives $v = \pi(\omega) \in \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))}$.

For (1), since $\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi(\underline{F}(t))$ from Theorem 3.4 of [2], we only need to show that $\overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))} \subset \overline{E}^{(\mathbf{p})}_{\alpha(t)}$. If $v \in \pi(G) = \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))}$, then $v = \pi(\omega)$ with $g(\mathbf{y}, \mathbf{r}(t'')) \le \alpha(t)q'' + \beta(q'') = g(\mathbf{r}(t), \mathbf{r}(t''))$ for all $\mathbf{y} \in A(x_n(\omega))$, and $g(\mathbf{y}, \mathbf{r}(t'')) = \alpha(t)q'' + \beta(q'') = g(\mathbf{r}(t), \mathbf{r}(t''))$ for some $\mathbf{y} \in A(x_n(\omega))$. Since $q'' > 0$ and $g(\mathbf{y}, \mathbf{r}(t'')) = q''g(\mathbf{y}, \mathbf{p}) + \beta(q'')$, we have $g(\mathbf{y}, \mathbf{p}) \le \alpha(t)$ for all $\mathbf{y} \in A(x_n(\omega))$ and $g(\mathbf{y}, \mathbf{p}) = \alpha(t)$ for some $\mathbf{y} \in A(x_n(\omega))$. This gives $v = \pi(\omega) \in \pi(\underline{F}(t)) = \overline{E}^{(\mathbf{p})}_{\alpha(t)}$. We have the rest parts of (1), (2) from the similar arguments above. □

Remark 3.2. $0 < t < 1$ determines $\alpha(t)$, and $\alpha(t)$ determines $q \in \mathbb{R}$ from $\beta'(q) = -\alpha(t)$. Conversely $q \in \mathbb{R}$ determines t from $\beta'(q) = -\alpha(t)$, and t determines $\mathbf{r}(t)$. In particular, if $q = 1$, then $\beta(q) = \beta(1) = 0$ and there is $0 < t_0 < t < 1$ such that $\alpha(t) = -\beta'(1)$. For t such that $\alpha(t) = -\beta'(1)$, $\mathbf{r}(t) = \mathbf{p}$.

Corollary 3.3 ([2]). *We have the followings*:

(1) *if* $t_0 < t'' < 1$, *then*

$$\pi(\underline{F}(t'')) = \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t''),\mathbf{r}(t''))},$$

(2) *if* $0 < t' < t_0$, *then*

$$\pi(\underline{F}(t')) = \underline{E}^{(\mathbf{r}(t'))}_{g(\mathbf{r}(t'),\mathbf{r}(t'))},$$

(3) *if* $t_0 < t'' < 1$, *then*

$$\pi(\overline{F}(t'')) = \underline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t''),\mathbf{r}(t''))},$$

(4) *if* $0 < t' < t_0$, *then*

$$\pi(\overline{F}(t')) = \overline{E}^{(\mathbf{r}(t'))}_{g(\mathbf{r}(t),\mathbf{r}(t'))}.$$

Proof. Putting $t = t'$ or $t = t''$, we have the results from the above theorem. For (1) and (3), we note that $t_0 < t'' < 1 \iff q > 0$. Therefore $q = 1 \iff \mathbf{r}(t'') = \mathbf{p}$. □

Corollary 3.4. *We have the followings*:

(1) *if* $\mathbf{r}(t) = \mathbf{p}$, *then*

$$\pi(\underline{F}(t)) = \overline{E}^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})},$$

(2) *if* $\mathbf{r}(t) = \mathbf{p}$, *then*

$$\pi(\overline{F}(t)) = \underline{E}^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}.$$

Proof. We note that if $\mathbf{r}(t) = \mathbf{p}$, then $0 < t_0 < t < 1$. From (1) and (3) of the above corollary, it follows. □

In the following corollary, Dim means the packing dimension and dim means the Hausdorff dimension.

Corollary 3.5. *We have the followings*:

(1) *if* $t_0 < t'' < 1$, *then*

$$\mathrm{Dim}(\cup_{t'' \le t \le 1} \pi(\underline{F}(t)) = g(\mathbf{r}(t''), \mathbf{r}(t'')),$$

(2) *if* $0 < t' < t_0$, *then*

$$\dim(\cup_{0 \le t \le t'} \pi(\underline{F}(t)) = g(\mathbf{r}(t'), \mathbf{r}(t')),$$

(3) *if* $t_0 < t'' < 1$, *then*

$$\dim(\cup_{t'' \le t \le 1} \pi(\overline{F}(t)) = g(\mathbf{r}(t''), \mathbf{r}(t'')),$$

(4) *if* $0 < t' < t_0$, *then*

$$\mathrm{Dim}(\cup_{0 \le t \le t'} \pi(\overline{F}(t)) = g(\mathbf{r}(t'), \mathbf{r}(t')).$$

Proof. We note that

$$g(\mathbf{r}(t), \mathbf{r}(t'')) = \alpha(t) q'' + \beta(q''),$$

where $\alpha(t) = \frac{t \log p_i + (1-t) \log p_j}{t \log a_i + (1-t) \log a_j}$ and $\beta'(q'') = -\alpha(t'')$ where $\mathbf{r}(t'') = (r_1, \ldots, r_N)$ satisfying $r_k = p_k^{q''} a_k^{\beta(q'')}$.

We also note that

$$0 < t_0 \le t'' < 1 \iff q'' \ge 0$$

and

$$0 < t'' \le t_0 < 1 \iff q'' \le 0.$$

If $t_0 < t'' < 1$, then from the above theorem

$$\mathrm{Dim}(\cup_{t'' \le t \le 1} \pi(\underline{F}(t)) = \mathrm{Dim}(\cup_{t'' \le t \le 1} \overline{E}^{(\mathbf{r}(t''))}_{g(\mathbf{r}(t),\mathbf{r}(t''))}) \le \sup_{t'' \le t \le 1} g(\mathbf{r}(t), \mathbf{r}(t'')).$$

Since $\sup_{t'' \le t \le 1} g(\mathbf{r}(t), \mathbf{r}(t'')) = g(\mathbf{r}(t''), \mathbf{r}(t''))$ and Theorem 4.2(1) of [2], (1) follows.

If $0 < t' < t_0$, then from the above theorem

$$\dim(\cup_{0 \le t \le t'} \pi(\underline{F}(t)) = \dim(\cup_{0 \le t \le t'} \underline{E}^{(\mathbf{r}(t'))}_{g(\mathbf{r}(t),\mathbf{r}(t'))}) \le \sup_{0 \le t \le t'} g(\mathbf{r}(t), \mathbf{r}(t')).$$

Since $\sup_{0 \le t \le t'} g(\mathbf{r}(t), \mathbf{r}(t')) = g(\mathbf{r}(t'), \mathbf{r}(t'))$ and Theorem 4.3(1) of [2], (2) follows. Similarly (3) and (4) follow. □

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