Commun. Korean Math. Soc. **30** (2015), No. 3, pp. 201–208
http://dx.doi.org/10.4134/CKMS.2015.30.3.201

REMARKS ON NONLINEAR DIRAC EQUATIONS IN ONE SPACE DIMENSION

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Abstract. This paper reviews recent mathematical progresses made on the study of the initial-value problem for nonlinear Dirac equations in one space dimension. We also prove the global existence of solutions to some nonlinear Dirac equations and propose a model problem (3.6).

1. Introduction

We are interested in the following initial value problem for the one dimensional nonlinear Dirac equations

$$\begin{aligned} &i(\partial_t U_1 + \partial_x U_1) + mU_2 = \partial_{\bar{U}_1} W(U_1,\, U_2), \\ &i(\partial_t U_2 - \partial_x U_2) + mU_1 = \partial_{\bar{U}_2} W(U_1,\, U_2), \\ &U_j(x,\, 0) = u_j(x), \end{aligned} \tag{1.1}$$

where $U_j : \mathbb{R}^{1+1} \to \mathbb{C}$ for $j = 1, 2$ and $m\,(\geq 0)$ is a mass. $\bar{U}$ is a complex conjugate of U. The potential W satisfies the following properties:

1. Symmetry: $W(U_1,\, U_2) = W(U_2,\, U_1)$.
2. Gauge invariance: $W(e^{i\theta}U_1,\, e^{i\theta}U_2) = W(U_1,\, U_2)$ for any $\theta \in \mathbb{R}$.
3. Polynomial in $(U_1,\, U_2)$ and $(\bar{U}_1,\, \bar{U}_2)$.

It is known [11] that fourth order homogeneous polynomial satisfying the above properties takes the form

$$\begin{aligned} W = {}& a_1|U_1|^2|U_2|^2 + a_2(\bar{U}_1U_2 + \bar{U}_2U_1)^2 + a_3(|U_1|^4 + |U_2|^4) \\ & + a_4(|U_1|^2 + |U_2|^2)(\bar{U}_1U_2 + \bar{U}_2U_1), \end{aligned}$$

where a_j are real constants.

The system (1.1) has the charge conservation

$$\int_{\mathbb{R}} |U_1(x,\, t)|^2 + |U_2(x,\, t)|^2 dx = \int_{\mathbb{R}} |u_1(x)|^2 + |u_2(x)|^2 dx. \tag{1.2}$$

Received January 15, 2015.
2010 *Mathematics Subject Classification.* 35L03, 35L40, 35Q40, 35F25.
Key words and phrases. nonlinear Dirac equations, global existence, finite time blow up.

When $W = |U_1|^2|U_2|^2$, the system (1.1) is called Thirring equations and takes the form

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= |U_2|^2 U_1, \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= |U_1|^2 U_2. \end{aligned} \tag{1.3}$$

The Cauchy problem of (1.3) has been studied by several authors [2, 4, 7, 13]. The global existence of solutions to Thirring equations was studied in [4] in terms of Sobolev space H^s $(s \geq 1)$. Low regularity well-posedness was discussed in [2, 7, 13] showing that there exists a time $T > 0$ and solution $U_j \in C([0,T], H^s(\mathbb{R}))$ $(s \geq 0)$. The stability of solitary wave solutions has been studied recently in [3, 12]. In particular, they proved in [12] H^1 orbital stability of Thirring equations by observing a new conserved quantity. They also derived a global uniform bound on the H^1 norm for the small L^2 initial data.

When $W = \frac{1}{4}(\bar{U}_1 U_2 + \bar{U}_2 U_1)^2$, the system (1.1) is called Gross-Neveu equations and takes the form

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= \mathrm{Re}(\bar{U}_1 U_2) U_2, \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= \mathrm{Re}(\bar{U}_1 U_2) U_1. \end{aligned} \tag{1.4}$$

The initial value problem of (1.4) has been studied in [9, 10, 15]. The global existence of a solution in H^s $(s > 1/2)$ was proved in [9] by obtaining L^∞ of the solution. The global solution in critical space L^2 has been proved in [10] recently.

When $W = |U_1|^4 + |U_2|^4$, we have a system

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= |U_1|^2 U_1, \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= |U_2|^2 U_2. \end{aligned} \tag{1.5}$$

The system (1.5) does not have its own name in physical literature but the following system has been considered in [6] where they present an analysis of soliton solutions to the following quasi-one dimensional Dirac system for a Bose-Einstein condensate.

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) &= |U_1|^2 U_1 + 2\mathrm{Re}(U_1 \bar{U}_2) U_2 + |U_2|^2 U_1, \\ i(\partial_t U_2 - \partial_x U_2) &= |U_2|^2 U_2 + 2\mathrm{Re}(U_1 \bar{U}_2) U_1 + |U_1|^2 U_2, \end{aligned}$$

which is a kind of combination of (1.3), (1.4) and (1.5). The initial value problem of (1.5) with $m = 0$ has been studied in [8] in the context of Lebesgue space L^p. Here we will show global well-posedness in H^s $(s > 1/2)$ and existence of standing waves of the system (1.5).

When $W = (|U_1|^2 + |U_2|^2)(\bar{U}_1 U_2 + \bar{U}_2 U_1)$, we have a system

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= U_2(|U_1|^2 + |U_2|^2) + 2\mathrm{Re}(\bar{U}_1 U_2) U_1, \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= U_1(|U_1|^2 + |U_2|^2) + 2\mathrm{Re}(\bar{U}_2 U_1) U_2, \end{aligned} \tag{1.6}$$

which is Dirac equations with pseudoscalar potential [14] and occurs in the context of a nonlinear refractive index [1]. As far as we know, there is no result addressing the initial value problem of (1.6). Here we will show global well-posedness of a related system (3.2) and propose model equation (3.6) which has main difficulties of the original equations (1.6) for proving global well-posedness. The existence of standing waves of the system (1.6) has been studied in [14].

In Section 2 we prove global existence of solution to the system (1.5) (Theorem 2.2) and show existence of standing waves. In Section 3 we prove Theorem 3.1 and propose the model system (3.6).

2. Remarks on the system (1.5)

In this chapter we are interested in the system (1.5). We will show the global well-posedness and existence of standing waves. For the proof of global solvability, we follow the idea of [9]. For the existence of standing waves, we apply the idea of [14].

2.1. Global existence

To begin with, let us recall basic known facts. Local existence and uniqueness of solution to (1.1) in Sobolev space $H^s(\mathbb{R})$ can be proved in the standard way [4, 5].

Theorem 2.1. *For the initial data $u_j(x) \in H^s(\mathbb{R})$ with $s > 1/2$, there exists $T > 0$, depending only on $\|u_j(x)\|_{H^s(\mathbb{R})}$, and a unique solution U_j of (1.1) for $0 \le t \le T$ satisfying*

$$U_j \in C([0,T], H^s(\mathbb{R})),$$

where U_j depends continuously on the initial data.

Our first result is that the local solution in the above theorem can be extended globally.

Theorem 2.2. *For the initial data $u_j(x) \in H^s(\mathbb{R})$ with $s > 1/2$, there exists a unique global solution U_j of (1.5) satisfying*

$$U_j \in C([0,\infty), H^s(\mathbb{R})).$$

Proof. We make use of the idea in [9]. If we obtain a priori estimate $\|U_j(\cdot,\, t)\|_{L^\infty} \le f(t)$ for a bounded function $f(t)$, then energy estimate combined with Gronwall's inequality gives a H^s $(s > 1/2)$ bound of the solution

$$\|U_j(\cdot,\, t)\|_{H^s} \le g(t),$$

for a bounded function $g(t)$ which proves Theorem 2.2. To bound $\|U_j(\cdot,\, t)\|_{L^\infty}$, we multiply (1.5) by $\bar{U}_1$ and $\bar{U}_2$ respectively and take real parts to have

$$\partial_t |U_1|^2 + \partial_x |U_1|^2 = 2m\mathrm{Im}(U_1 \bar{U_2}), \tag{2.1}$$

$$\partial_t |U_2|^2 - \partial_x |U_2|^2 = 2m\mathrm{Im}(U_2 \bar{U_1}), \tag{2.2}$$

which implies

$$\partial_t(|U_1|^2 + |U_2|^2) + \partial_x(|U_1|^2 - |U_2|^2) = 0. \tag{2.3}$$

Integrating (2.3) on the domain

$$D(x_0, t_0) = \{(x,t)|\, 0 < t < t_0,\ x_0 - t_0 + t < x < x_0 + t_0 - t\},$$

we have by applying Green's Theorem

$$\begin{aligned} &2\int_0^{t_0} |U_1|^2(x_0 + t_0 - s, s)\, ds + 2\int_0^{t_0} |U_2|^2(x_0 - t_0 + s, s)\, ds \\ &= \int_{x_0 - t_0}^{x_0+t_0} (\, |u_1(s)|^2 + |u_2(s)|^2\,)\, ds \le M, \end{aligned} \tag{2.4}$$

where we denote $M = \int_{\mathbb{R}}(|u_1(y)|^2 + |u_2(y)|^2)\, dy$. Integrating (2.1) along characteristic curve, we have

$$\frac{d}{dt}|U_1(x+t,t)| \le m|U_2(x+t,t)|.$$

Integrating both sides and considering (2.4), we obtain

$$|U_1(x+t,t)| \le |u_1(x)| + m\frac{M}{2}t^{1/2}.$$

The similar argument applied to (2.2) leads us to

$$|U_2(x-t,t)|^2 \le |u_2(x)| + m\frac{M}{2}t^{1/2}.$$

Therefore we obtain a bound $\|U_j(\cdot,\, t)\|_{L^\infty} \le \|u_j\|_{L^\infty} + m\frac{M}{2}t^{1/2}$ to prove Theorem 2.2. Note that the embedding of $H^s(\mathbb{R})$ $(s > 1/2)$ to the space of bounded continuous functions is used to justify the bound on the L^∞ norm of the initial data. □

2.2. Solitary waves

An explicit standing wave solution to the system (1.5) is found here. We follow the idea in [14] where standing wave solutions of the system (1.6) have been studied. Plugging the ansatz $U_1 = e^{i\omega t}\bar{z}(x)$, $U_2 = e^{i\omega t}z(x)$ into (1.5) leads to

$$z' - i\omega z + im\bar{z} - i|z|^2 z = 0,$$

which can be rewritten, with the notation $z(x) = f(x) + ig(x)$ $(f,\, g \in \mathbb{R})$, as follows

$$f' = -(m+\omega)g - (f^2+g^2)g, \tag{2.5}$$

$$g' = -(m-\omega)f + (f^2+g^2)f. \tag{2.6}$$

Then $g'\times(2.5) - f'\times(2.6)$ gives

$$\frac{d}{dx}\left(\frac{1}{2}(f^2+g^2)^2 - (m-\omega)f^2 + (m+\omega)g^2\right) = 0.$$

Considering a boundary condition $\lim_{|x|\to\infty} f(x) = 0 = \lim_{|x|\to\infty} g(x)$, we have

$$\frac{1}{2}(f^2+g^2)^2 - (m-\omega)f^2 + (m+\omega)g^2 = 0. \tag{2.7}$$

Let us define $q = \frac{g}{f}$. Then we have

$$\begin{aligned} q' &= \frac{1}{f^2}\big((f^2+g^2)^2 - (m-\omega)f^2 + (m+\omega)g^2\big) \\ &= \frac{1}{f^2}\big((m-\omega)f^2 - (m+\omega)g^2\big) \\ &= (m-\omega) - (m+\omega)q^2, \end{aligned} \tag{2.8}$$

where (2.7) is used. We have a solution of (2.8)

$$q(x) = \alpha \tanh(\beta x), \tag{2.9}$$

where $\alpha = (\frac{m-\omega}{m+\omega})^{1/2}$, $\beta = (m^2-\omega^2)^{1/2}$ with $|\omega| < m$. Substituting $g = fq$ into (2.7) and considering (2.9), we have

$$f^2(x) = 2(m-\omega)\frac{1-\tanh^2(\beta x)}{\big(1+\alpha^2\tanh^2(\beta x)\big)^2},$$

where we note that $\lim_{|x|\to\infty} f(x) = 0$.

3. Remarks on the system (1.6)

Here we are interested in the following initial value problem

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= 2\mathrm{Re}(\bar{U}_1 U_2)U_1 + U_2(|U_1|^2+|U_2|^2), \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= \underbrace{2\mathrm{Re}(\bar{U}_2 U_1)U_2}_{\text{(I)}} + \underbrace{U_1(|U_1|^2+|U_2|^2)}_{\text{(II)}}, \end{aligned} \tag{3.1}$$

with initial data $U_j(x,0) = u_j(x)$. We could not prove the global solvability of (3.1) by applying the idea of [9]. A problem occurs in estimating the nonlinear term (II) of (3.1). To make clear the problem, we propose two model systems. The first system is

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= 2\mathrm{Re}(\bar{U}_1 U_2)U_1, \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= 2\mathrm{Re}(\bar{U}_2 U_1)U_2 \end{aligned} \tag{3.2}$$

which consists of nonlinear term (I) of (3.1). The second one is

$$\begin{aligned} i(\partial_t U_1 + \partial_x U_1) + mU_2 &= U_2(|U_1|^2+|U_2|^2), \\ i(\partial_t U_2 - \partial_x U_2) + mU_1 &= U_1(|U_1|^2+|U_2|^2) \end{aligned} \tag{3.3}$$

which consists of nonlinear term (II) of (3.1). We can prove the global existence of solution to the first model equation (3.2).

Theorem 3.1. *For the initial data $u_j(x) \in H^s(\mathbb{R})$ with $s > 1/2$, there exists a unique global solution U_j of (3.2) satisfying*

$$U_j \in C([0,\infty),\, H^s(\mathbb{R})).$$

Proof. As the proof of Theorem 2.2, we will estimate $\|U_j(\cdot,\, t)\|_{L^\infty}$ to obtain global solution. Multiplying (3.2) by $\bar{U}_1$ and $\bar{U}_2$ respectively and taking real parts, we have

$$\partial_t|U_1|^2 + \partial_x|U_1|^2 = 2m\mathrm{Im}(U_1\bar{U_2}),$$
$$\partial_t|U_2|^2 - \partial_x|U_2|^2 = 2m\mathrm{Im}(U_2\bar{U_1}).$$

Note that we obtain the same equations as (2.1), (2.2) although the system (3.2) is different from (1.5). Then we can follow the same argument as proof of Theorem 2.2 to prove Theorem 3.1. □

Let us explain the problem of (3.3) in applying the argument of [9]. For the Gross-Neveu system (1.4), we have the following inequality on the characteristic curve of U_1

$$\frac{d}{dt}|U_1(x+t,t)| \le |U_2(x+t,t)|^2|U_1(x+t,t)| + m|U_2(x+t,t)|. \tag{3.4}$$

Then we can apply the local version of charge conservation (2.4) to control the integrating factor $\int_0^t |U_2(x+s,s)|^2 ds$. For the equations (3.3), however, we have on the characteristic curve of U_1

$$\frac{d}{dt}|U_1(x+t,t)|^2 \le 2|U_2(x+t,t)||U_1(x+t,t)|^3 + 2|U_2(x+t,t)|^3|U_1(x+t,t)| + 2m|U_1(x+t,t)||U_2(x+t,t)|,$$

which implies

$$\frac{d}{dt}|U_1(x+t,t)| \le |U_2(x+t,t)||U_1(x+t,t)|^2 + |U_2(x+t,t)|^3 + m|U_2(x+t,t)|. \tag{3.5}$$

Note that the right hand side of (3.5) has a different algebraic structure from that of (3.4). Considering integrating factor $|U_1||U_2|$, we obtain from (3.5)

$$|U_1(x+t,t)| \le e^{\int_0^t |U_1(x+s,s)||U_2(x+s,s)|ds}\left(|u_1(x)| + \int_0^t |U_2(x+s,s)|^3 + m|U_2(x+s,s)|\, ds\right).$$

If we can control the following quantities, considering the inequality (2.4),

$$\int_0^t |U_1(x+s,s)||U_2(x+s,s)|ds \quad \text{and} \quad \int_0^t |U_2(x+s,s)|^3 ds,$$

then we can prove the global solvability of (3.1).

Let us propose a simple model system of (3.1). Putting an ansatz

$$U_1(x,t) = f(x,t) \quad \text{and} \quad U_2(x,t) = i\, g(x,t),$$

where f, g are real valued functions, the system (3.1) reduces to

$$\begin{aligned} \partial_t f + \partial_x f + mg &= g(f^2+g^2), \\ \partial_t g - \partial_x g - mf &= -f(f^2+g^2). \end{aligned} \tag{3.6}$$

The system (3.6) has the similar structures to the equations (1.6) for proving global well-posedness. We note the conservation of L^2 norm

$$\int_{\mathbb{R}} |f(x,t)|^2 + |g(x,t)|^2 dx = \int_{\mathbb{R}} |f(x,0)|^2 + |g(x,0)|^2 dx.$$

We want to know whether the system (3.6) admits global solution or blows up in finite time for initial data $f(x,0)$, $g(x,0) \in H^1(\mathbb{R})$.

Acknowledgement. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2014R1A1A2053747).

References

[1] D. Agueev and D. Pelinovsky, *Modeling of wave resonances in low-contrast photonic crystals*, SIAM J. Appl. Math. **65** (2005), no. 4, 1101–1129.

[2] T. Candy, *Global existence for an L^2 critical nonlinear Dirac equation in one dimension*, Adv. Differential Equations **16** (2011), no. 7-8, 643–666.

[3] A. Contreras, D. Pelinovsky, and Y. Shimabukuro, *L^2 orbital stability of Dirac solitons in the massive Thirring model*, arXiv:1312.1019.

[4] V. Delgado, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac and other nonlinear Dirac equations in one space dimension*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 289–296.

[5] R. H. Goodman, M. I. Weinstein, and P. J. Holmes, *Nonlinear propagation of light in one-dimensional periodic structures*, J. Nonlinear Sci. **11** (2001), no. 2, 123–168.

[6] L. Haddad, C. Weaver, and L. Carr, *The nonlinear Dirac equations in Bose-Einstein condensates: I. Relativistic solitons in armchair and zigzag geometries*, arxiv: 1305.6532v2.

[7] H. Huh, *Global strong solution to the Thirring model in critical space*, J. Math. Anal. Appl. **381** (2011), no. 2, 513–520.

[8] ______, *Global strong solutions to some nonlinear Dirac equations in super-critical space*, Abstr. Appl. Anal. (2013), Art. ID 602753, 8 pp.

[9] ______, *Global solutions to Gross-Neveu equation*, Lett. Math. Phys. **103** (2013), no. 8, 927–931.

[10] H. Huh and B. Moon, *Low regularity well-posedness for Gross-Neveu equations*, Comm. Pure Appl. Anal. **14** (2015), no. 5, 1903–1913.

[11] D. Pelinovsky, *Survey on global existence in the nonlinear Dirac equations in one spatial dimension*, Harmonic analysis and nonlinear partial differential equations, 37–50, RIMS Kôkyûroku Bessatsu, B26, Kyoto, 2011.

[12] D. Pelinovsky and Y. Shimabukuro, *Orbital stability of Dirac solitons*, Lett. Math. Phys. **104** (2014), no. 1, 21–41.

[13] S. Selberg and A. Tesfahun, *Low regularity well-posedness for some nonlinear Dirac equations in one space dimension*, Differential Integral Equations, **23** (2010), no. 3-4, 265–278.

[14] J. Stubbe, *Exact localized solutions of a family of two-dimensional nonlinear spinor fields*, J. Math. Phys. **27** (1986), no. 10, 2561–2567.

[15] Y. Zhang, *Global strong solution to a nonlinear Dirac type equation in one dimension*, Nonlinear Anal. **80** (2013), 150–155.

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