Commun. Korean Math. Soc. **30** (2015), No. 3, pp. 191–200
http://dx.doi.org/10.4134/CKMS.2015.30.3.191

CERTAIN NEW GENERATING RELATIONS FOR PRODUCTS OF TWO LAGUERRE POLYNOMIALS

Junesang Choi and Arjun Kumar Rathie

Abstract. Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. Exton [13] presented a very general double generating relation involving products of two Laguerre polynomials. Motivated essentially by Exton's derivation [13], the authors aim to show how one can obtain nineteen new generating relations associated with products of two Laguerre polynomials in the form of a single result. We also consider some interesting and potentially useful special cases of our main findings.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

It is well known that generating functions involving Laguerre polynomials play an important role due to their appearance in various branches of pure and applied mathematics (see, *e.g.*, [11, 15]). The Laguerre polynomials have been fully discussed by many authors (see [11, 12, 15]).

By using a two dimensional extension of a very general series transform given by Bailey, Exton [13] deduced a very general double generating relation for products of two Laguerre polynomials whose several interesting special cases were also presented.

Here, in this paper, we aim at showing how one can obtain nineteen new generating relations involving products of two Laguerre polynomials in the form of a single result. We also consider some interesting and potentially useful specials cases of our main findings, including the results due to Exton in corrected forms. The results are derived with the help of a general formula for

Received January 2, 2015.
2010 *Mathematics Subject Classification.* Primary 33B20, 33C20; Secondary 33B15, 33C05.
Key words and phrases. gamma function, hypergeometric function, generalized hypergeometric function, Kampé de Fériet function, Kummer's second summation theorem, Dixon and Whipple's summation theorems.

the reducibility of Kampé de Fériet function recently obtained by the authors [9].

For our purpose, we begin by recalling the generalized hypergeometric series ${}_pF_q$ is defined by (see [3], [14, p. 73], [16] and [17, pp. 71–75]):

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z), \tag{1} $$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [17, p. 2 and p. 5]):

$$ (\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{2} $$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1), that is,

$$ (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\; j = 1, \ldots, q). \tag{3} $$

We also recall a double hypergeometric function which is defined and introduced by Kampé de Fériet and subsequently abbreviated by Burchnall and Chaundy [5, 6]. Here, we use a slightly modified notation of the function given by Srivastava and Panda [20, p. 423, Eq. (26)]. For this, let (h_H) denote the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and, for $n \in \mathbb{N}_0$, define the Pochhammer symbol

$$ ((h_H))_n := (h_1)_n \cdots (h_H)_n, $$

where, when $n = 0$, the product is understood to reduce to unity. Therefore, the most convenient generalization of Kampé de Fériet function [1] is defined as follows:

$$ F^{H:\,A;\,B}_{G:\,C;\,D}\left[\begin{matrix}(h_H)\,:\,(a_A)\,;\,(b_B);\\ (g_G)\,:\,(c_C)\,;\,(d_D);\end{matrix}\; x,\, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((h_H))_{m+n}\,((a_A))_m\,((b_B))_n}{((g_G))_{m+n}\,((c_C))_m\,((d_D))_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{4} $$

The symbol (h) is a convenient contraction for the sequence of the parameters $h_1, h_2, \ldots, h_H$ and the Pochhammer symbol $(h)_n$ is the same as defined in (2). For details about the convergence for this function, we refer to [18, 19].

Many authors have investigated the reducibility of Kampé de Fériet function (see, *e.g.*, [7]). The Laguerre polynomials are defined by (see [11])

$$L_n^{(a)}(x) = \frac{(a)_n}{n!}\, {}_1F_1\left[\begin{matrix} -n\,; \\ a+1\,; \end{matrix}\; x\right]. \tag{5}$$

In a two-dimensional extension of a very general series transform due to Bailey [2], Exton [13] deduced the following interesting double generating relation for a product of two Laguerre polynomials:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, x^m\, s^n}{((g))_{m+n}\, m!\, n!}\, {}_1F_1\left[\begin{matrix} -m\,; \\ p'\,; \end{matrix}\; -y\right] {}_1F_1\left[\begin{matrix} -n\,; \\ p\,; \end{matrix}\; -t\right] \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, (xy)^m\, (st)^n}{((g))_{m+n}\, (p')_m\, (p)_n\, m!\, n!}\, F_{G:\,0;\,0}^{D:\,0;\,0}\left[\begin{matrix} (d) \;:\; —\,;\, —\,; \\ (g) \;:\; —\,;\, —\,; \end{matrix}\; x,\, s\right]. \end{aligned} \tag{6}$$

Moreover, as a simple consequence of the binomial theorem, the inner double series on the right-hand side of (6) is seen to immediately reduce to a single series. Also, if the confluent hypergeometric functions appearing on the left-hand side of (6) are replaced by their representations as Laguerre polynomials (changing y to $-y$ and t to $-t$), we arrive at the following result:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, x^m\, s^n}{((g))_{m+n}\, (p')_m\, (p)_n}\, L_m^{(p'-1)}(y)\, L_n^{(p-1)}(t) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, (-xy)^m\, (-st)^n}{((g))_{m+n}\, (p')_m\, (p)_n\, m!\, n!}\, {}_DF_G\left[\begin{matrix} (d)\,; \\ (g)\,; \end{matrix}\; x+s\right]. \end{aligned} \tag{7}$$

In (7), taking $s = -x$, the inner hypergeometric series reduces to unity and Exton obtained the following very general generating relation for the product of two Laguerre polynomials in terms of Kampé de Fériet function:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, (-1)^n\, x^{m+n}}{((g))_{m+n}\, (p')_m\, (p)_n}\, L_m^{(p'-1)}(y)\, L_n^{(p-1)}(t) \\ &= F_{G:\,1;\,1}^{D:\,0;\,0}\left[\begin{matrix} (d) \;:\; \text{——} \;;\; \text{——}\,; \\ (g) \;:\; p' \;;\; p\;; \end{matrix}\; -xy,\, xt\right] \end{aligned} \tag{8}$$

and deduced the following interesting result:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\, (-1)^n\, x^{m+n}}{((g))_{m+n}\, (p)_m\, (p)_n}\, L_m^{(p-1)}(y)\, L_n^{(p-1)}(y) \\ &= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right),\, \left(\frac{1}{2}d+\frac{1}{2}\right); \\ \left(\frac{1}{2}g\right),\, \left(\frac{1}{2}g+\frac{1}{2}\right),\, p,\, \frac{1}{2}p,\, \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -4^{D-G-1}\, x^2\, y^2\right], \end{aligned} \tag{9}$$

which is a known result due to Exton [13]. Further, in (9), if we set

(i) $D = 1$ and $G = 0$;

(ii) $D = 2$, $G = 0$, $d_1 = p$ and $d_2 = 2p$;

(iii) $D = 2$, $G = 0$, $d_1 = p$ and $d_2 = 2p - 1$,

we, respectively, obtain the following results:

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n}\,L_m^{(p-1)}(y)\,L_n^{(p-1)}(y)\\&= {}_0F_1\left[\begin{matrix}-;\\p\,;\end{matrix}\;-x^2\,y^2\right] = \Gamma(p)\,(xy)^{1-p}\,J_{p-1}(2xy),\end{aligned}\tag{10}$$

where $J_\nu(z)$ is the Bessel function of the first kind having the following connection with ${}_0F_1(\cdot)$ (see, *e.g.*, [4, p. 675]):

$$ {}_0F_1\left[\begin{matrix}\text{---};\\\nu+1\,;\end{matrix}\;-\frac{z^2}{4}\right] = \Gamma(\nu+1)\,\left(\frac{z}{2}\right)^{-\nu}\,J_\nu(z);\tag{11}$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_{m+n}\,(2p)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n}\,L_m^{(p-1)}(y)\,L_n^{(p-1)}(y)\\&= {}_1F_0\left[\begin{matrix}p+\frac{1}{2}\,;\\\text{---}\;;\end{matrix}\;-4\,x^2\,y^2\right] = \left(1+4\,x^2\,y^2\right)^{-p-\frac{1}{2}}\,;\end{aligned}\tag{12}$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_{m+n}\,(2p-1)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n}\,L_m^{(p-1)}(y)\,L_n^{(p-1)}(y)\\&= {}_1F_0\left[\begin{matrix}p-\frac{1}{2}\,;\\\text{---}\;;\end{matrix}\;-4\,x^2\,y^2\right] = \left(1+4\,x^2\,y^2\right)^{-p+\frac{1}{2}}\,.\end{aligned}\tag{13}$$

We remark in passing that the identity (13) is the corrected form of the result obtained by Exton [13].

Very recently, the authors [9] have obtained nineteen interesting formulas for reducibility of Kampé de Fériet function in the form of a single result which is recalled here:

$$\begin{aligned}&F_{G:\,1;\,1}^{D:\,0;\,0}\left[\begin{matrix}(d)\;:\;\text{---};\;\text{---};\\(g)\;:\;p+i\,;\;\;p\;\;;\end{matrix}\;-x,\,x\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(p+\frac{1}{2}(i+|i|)\right)}\\&\times\sum_{n=0}^{\infty}\frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d))_n\,((\frac{1}{2}d)+\frac{1}{2})_n}{n!\,((\frac{1}{2}g))_n\,((\frac{1}{2}g)+\frac{1}{2})_n\,(\frac{1}{2})_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|)+\frac{1}{2})_n}\\&\times\left\{\frac{\mathcal{A}_i'\;(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\,(p+\frac{1}{2}i)_n}+\frac{\mathcal{B}_i'\;(1-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i-\frac{1}{2})\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i])\,(p+\frac{1}{2}i-\frac{1}{2})_n}\right\}\\&+\frac{(d)}{(g)}\,2x\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(1+p+\frac{1}{2}(i+|i|)\right)}\end{aligned}\tag{14}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d)+\frac{1}{2})_n\,((\frac{1}{2}d+\frac{1}{2})+\frac{1}{2})_n}{n!\,((\frac{1}{2}g+\frac{1}{2}))_n\,((\frac{1}{2}g+\frac{1}{2})+\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{1}{2}+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(1+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n}$$

$$\times \left\{ \frac{\mathcal{A}_i''\,(1-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}+\frac{1}{2}i+p)\,\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])\,(\frac{1}{2}+\frac{1}{2}i+p)_n} + \frac{\mathcal{B}_i''\,(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i]-\frac{1}{2})\,(p+\frac{1}{2}i)_n} \right\},$$

where $i = 0, \pm 1, \ldots, \pm 9$. Here, as usual, $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$ and its absolute value is denoted by $|x|$. The coefficients $\mathcal{A}_i'$ and $\mathcal{B}_i'$ can be obtained from the Tables of $\mathcal{A}_i$ and $\mathcal{B}_i$ by simply substituting a and b with $-2n$ and $1-p-2n$, respectively, while the coefficients $\mathcal{A}_i''$ and $\mathcal{B}_i''$ can be obtained from the Tables of $\mathcal{A}_i$ and $\mathcal{B}_i$ by substituting a and b with $-2n-1$ and $-p-2n$, respectively.

TABLE 1. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
9	$-16a^4+72a^3b-108a^2b^2+60ab^3$ $-9b^4-328a^3+972a^2b-792ab^2$ $+150b^3-2240a^2+3612ab-999b^2$ $-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4$ $+248a^3-516a^2b+240ab^2-10b^3$ $+1160a^2-1028ab+35b^2$ $+1576a-50b-24$
8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4$ $+128a^3-312a^2b+176ab^2-10b^3+624a^2$ $+624a^2-672ab+35b^2$ $+896a-50b+24$	$8b^3-40ab^2+48a^2b-16a^3-192a^2$ $+312ab-88b^2-640a+352b-512$
7	$7b^3-28ab^2+28a^2b-8a^3-100a^2$ $+196ab-70b^2-352a+245b-302$	$8a^3-20a^2b+12ab^2-b^3+68a^2$ $-76ab+6b^2+128a-11b+6$
6	$4a^3-12a^2b+9ab^2-b^3+36a^2-51ab$ $+6b^2+74a-11b+6$	$16ab-8a^2-6b^2-48a+34b-52$
5	$10ab-4a^2-5b^2-26a+25b-32$	$4a^2-6ab+b^2+14a-3b+2$
4	$2a^2-4ab+b^2+8a-3b+2$	$4(b-a-2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0

TABLE 2. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-9	$16a^4-72a^3b+108a^2b^2-60ab^3$ $+9b^4-320a^3+972a^2b-828ab^2$ $+174b^3+2240a^2-3936ab+1323b^2$ $-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4$ $-256a^3+564a^2b-300ab^2+26b^3$ $+1376a^2-1568ab+251b^2$ $-2816a+1066b+1680$
-8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4$ $-128a^3+328a^2b-208ab^2+22b^3+688a^2$ $-928ab+179b^2-1408a+638b+840$	$16a^3-48a^2b+40ab^2-8b^3-192a^2$ $+328ab-104b^2+704a-480b-768$
-7	$8a^3-28a^2b+28ab^2-7b^3-96a^2$ $+196ab-77b^2+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3-72a^2$ $+92ab-15b^2+184a-74b-120$
-6	$4a^3-12a^2b+9ab^2-b^3-36a^2+57ab$ $-12b^2+92a-47b-60$	$8a^2-16ab+6b^2-48a+38b+64$
-5	$4a^2-10ab+5b^2-24a+25b+32$	$4a^2-6ab+b^2-16a+7b+12$
-4	$2a^2-4ab+b^2-8a+5b+6$	$4(a-b-2)$
-3	$2a-3b-4$	$2a-b-2$
-2	$a-b-1$	2
-1	1	1

2. A new generating relation

Here, by using (8) and (14), we establish a general generating relation for product of two Laguerre polynomials asserted by the following theorem.

Theorem. *The following generating function holds true*:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p+i)_m\,(p)_n}\,L_m^{(p+i-1)}(y)\,L_n^{(p-1)}(y)=\frac{\Gamma\big(\frac{1}{2}\big)\,\Gamma(p)\,\Gamma(p+i)}{\Gamma\big(p+\frac{1}{2}(i+|i|)\big)} \tag{15}$$

$$\times\sum_{n=0}^{\infty}\frac{4^{n(D-G-1)}\,(-x^2\,y^2)^n\,((\frac{1}{2}d))_n\,((\frac{1}{2}d)+\frac{1}{2})_n}{n!\,((\frac{1}{2}g))_n\,((\frac{1}{2}g)+\frac{1}{2})_n\,(\frac{1}{2})_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|)+\frac{1}{2})_n}$$

$$\times\left\{\frac{\mathcal{A}_i'\,(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\,(p+\frac{1}{2}i)_n}+\frac{\mathcal{B}_i'\,(1-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i-\frac{1}{2})\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i])\,(p+\frac{1}{2}i-\frac{1}{2})_n}\right\}$$

$$+\frac{(d)}{(g)}\,2x\,y\,\frac{\Gamma\big(\frac{1}{2}\big)\,\Gamma(p)\,\Gamma(p+i)}{\Gamma\big(1+p+\frac{1}{2}(i+|i|)\big)}$$

$$\times\sum_{n=0}^{\infty}\frac{4^{n(D-G-1)}\,(-x^2\,y^2)^n\,((\frac{1}{2}d)+\frac{1}{2})_n\,((\frac{1}{2}d+\frac{1}{2})+\frac{1}{2})_n}{n!\,((\frac{1}{2}g+\frac{1}{2}))_n\,((\frac{1}{2}g+\frac{1}{2})+\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{1}{2}+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(1+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n}$$

$$\times \left\{ \frac{\mathcal{A}''_i \; (1-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}+\frac{1}{2}i+p)\,\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])\,(\frac{1}{2}+\frac{1}{2}i+p)_n} + \frac{\mathcal{B}''_i \; (\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i]-\frac{1}{2})\,(p+\frac{1}{2}i)_n} \right\},$$

where $i = 0, \pm 1, \ldots, \pm 9$, *the coefficients* $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{A}'_i$, $\mathcal{B}'_i$, $\mathcal{A}''_i$, $\mathcal{B}''_i$, *and other notations are same as in* (14).

Proof. We can derive our generating relation in a straightforward way. Indeed, if we set $t = y$ and $p' = p + i$ in (8), then, for $i = 0, \pm 1, \ldots, \pm 9$, we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p+i)_m\,(p)_n}\, L_m^{(p+i-1)}(y)\, L_n^{(p-1)}(t) \\ &= F^{D:\,0;\,0}_{G:\,1;\,1}\left[\begin{matrix} (d) & : & \text{———} & ; & \text{———} & ; \\ (g) & : & p+i & ; & p & ; \end{matrix}\; -xy,\, xy\right]. \end{aligned} \tag{16}$$

Replacing x by xy in (14) and applying the resulting identity to (16), we get our desired generating relation (15). This completes the proof of (15). □

3. Special cases

Here, we consider some interesting special cases of our main result.

If we take $i = 1$ in (15), then, after some simplification, we get the following result:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p+1)_m\,(p)_n}\, L_m^{(p)}(y)\, L_n^{(p-1)}(y) \\ &= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right),\ \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -4^{D-G-1}\,x^2\,y^2\right] \\ &\qquad + \frac{(d)}{(g)}\frac{xy}{p(p+1)} \\ &\times {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right)+\frac{1}{2},\ \left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\ p+1,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\,; \end{matrix}\; -4^{D-G-1}\,x^2\,y^2\right]. \end{aligned} \tag{17}$$

Further, in (17), if we take

(i) $D = 1$, $G = 0$ and $d = p + 1$;

(ii) $D = 2$, $G = 0$, $d_1 = p + 1$ and $d_2 = 2p$,

we, respectively, obtain the following results:

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(p+1)_{m+n}\,(-1)^n\,x^{m+n}}{(p+1)_m\,(p)_n}\, L_m^{(p)}(y)\, L_n^{(p-1)}(y) \\ &= {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; -x^2\,y^2\right] + \frac{xy}{p}\,{}_0F_1\left[\begin{matrix} \text{———}\,; \\ p+1\,; \end{matrix}\; -x^2\,y^2\right] \\ &= \Gamma(p)\,(xy)^{1-p}\,\left\{J_{p-1}(2xy) + J_p(2xy)\right\}; \end{aligned} \tag{18}$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p+1)_{m+1}\,(2p)_{m+n}\,(-1)^n\,x^{m+n}}{(p+1)_m\,(p)_n}\,L_m^{(p)}(y)\,L_n^{(p-1)}(y)\\ &= {}_1F_0\left[\begin{matrix}p+\frac{1}{2}\,; \\ \text{---}\;;\end{matrix}\ -4\,x^2\,y^2\right]+2xy\,{}_1F_0\left[\begin{matrix}p+\frac{1}{2}\,; \\ \text{---}\;;\end{matrix}\ -4\,x^2\,y^2\right]\\ &=(1+2xy)\,\left(1+4\,x^2\,y^2\right)^{-p-\frac{1}{2}}.\end{aligned} \tag{19}$$

If we take $i=-1$ in (15), then, after some simplification, we get the following result:

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p-1)_m\,(p)_n}\,L_m^{(p-2)}(y)\,L_n^{(p-1)}(y)\\ &= {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right),\,\left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\,\left(\frac{1}{2}g\right)+\frac{1}{2},\,p-1,\,\frac{1}{2}p,\,\frac{1}{2}p+\frac{1}{2}\,;\end{matrix}\ -4^{D-G-1}\,x^2\,y^2\right]\\ &\qquad -\frac{(d)}{(g)}\frac{xy}{p(p-1)}\\ &\times{}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d+\frac{1}{2}\right),\,\left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g+\frac{1}{2}\right),\,\left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\,p,\,\frac{1}{2}p+\frac{1}{2},\,\frac{1}{2}p+1\,;\end{matrix}\ -4^{D-G-1}\,x^2\,y^2\right].\end{aligned} \tag{20}$$

Further, in (20), if we take

(i) $D=1$, $G=0$ and $d=p$;

(ii) $D=2$, $G=0$, $d_1=p$ and $d_2=2p-2$,

we, respectively, obtain the following results:

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((p))_{m+n}\,(-1)^n\,x^{m+n}}{(p-1)_m\,(p)_n}\,L_m^{(p-2)}(y)\,L_n^{(p-1)}(y)\\ &= {}_0F_1\left[\begin{matrix}\text{---}\;; \\ p-1\,;\end{matrix}\ -x^2\,y^2\right]-\frac{xy}{p-1}\,{}_0F_1\left[\begin{matrix}-\,; \\ p\,;\end{matrix}\ -x^2\,y^2\right]\\ &=\Gamma(p-1)\,(xy)^{2-p}\,\left\{J_{p-2}(2xy)-J_{p-1}(2xy)\right\};\end{aligned} \tag{21}$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((p))_{m+n}\,(2p-2)_{m+n}\,(-1)^n\,x^{m+n}}{(p-1)_m\,(p)_n}\,L_m^{(p-2)}(y)\,L_n^{(p-1)}(y)\\ &= {}_1F_0\left[\begin{matrix}p-\frac{1}{2}\,; \\ \text{---}\;;\end{matrix}\ -4\,x^2\,y^2\right]-2xy\,{}_1F_0\left[\begin{matrix}p-\frac{1}{2}\,; \\ \text{---}\;;\end{matrix}\ -4\,x^2\,y^2\right]\\ &=(1-2xy)\,\left(1+4\,x^2\,y^2\right)^{-p+\frac{1}{2}}.\end{aligned} \tag{22}$$

Remark. It may be easily seen that the results (18), (19), (21) and (22) are closely related to Exton's results (10), (12), and (13), and (presumably) new. Similarly, various other identities can also be obtained.

The results (14) and (15) for $i = 0, \pm 1, \ldots, \pm 5$ are given in [10].

Acknowledgements. The authors are grateful for the reviewers' useful suggestions. This work was supported by the Dongguk University Research Fund of 2015.

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] W. N. Bailey, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **50** (1948), 1–10.

[3] ———, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert Hafner, New York, 1964.

[4] Yu. A. Brychkov, *Handbook of Special Functions*, CRC Press, Taylor & Fancis Group, Boca Raton, London, New York, 2008.

[5] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math. (Oxford Ser.) **11** (1940), 249–270.

[6] ———, *Expansions of Appell's double hypergeometric functions. II*, Quart. J. Math. (Oxford Ser.) **12** (1941), 112–128.

[7] R. G. Buschman and H. M. Srivastava, *Series identities and reducibility of Kampé de Fériet functions*, Math. Proc. Cambridge Philos. Soc. **91** (1982), no. 3, 435–440.

[8] J. Choi, *Notes on formal manipulations of double series*, Commun. Korean Math. Soc. **18** (2003), no. 4, 781–789.

[9] J. Choi and A. K. Rathie, *Reducibility of Kampé de Fériet function*, Preprint, 2015.

[10] ———, *Reducibility of certain Kampé de Fériet function with an application to generating relations for products of two Laguerre polynomials*, Filomat, 2015, accepted for publicatiuon.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vol. I*, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[12] H. Exton, *Multiple Hypergeometric Functions and Applications*, Halsted Press, New York, 1976.

[13] ———, *New generating relations for products of two Laguerre polynomials*, Indian J. Pure Appl. Math. **24** (1993), no. 6, 401–408.

[14] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[15] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1960.

[16] ———, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1966.

[17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[18] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.

[19] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

[20] H. M. Srivastava and R. Panda, *An integral representation for the product of two Jacobi polynomials*, J. London Math. Soc. (2) **12** (1976), no. 4, 419–425.

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 780-714, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Arjun Kumar Rathie
Department of Mathematics
School of Mathematical and Physical Sciences
Central University of Kerala, Riverside Transit Campus
Padennakkad P.O. Nileshwar, Kasaragod- 671 328, Kerala, India
E-mail address: akrathie@gmail.com