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CONVERGENCE THEOREMS FOR A PAIR OF ASYMPTOTICALLY AND MULTIVALUED NONEXPANSIVE MAPPING IN CAT(0) SPACES

Naknimit Akkasriworn and Kritsana Sokhuma

Abstract. In this paper, we prove △-convergence theorems for Ishikawa iteration of asymptotically and multivalued nonexpansive mapping in CAT(0) spaces. This results we obtain are analogs of Banach spaces results of Sokhuma [13].

1. Introduction

Let (X, d) be a geodesic metric space. We denote by $FB(E)$ the collection of all nonempty closed bounded subsets of X, we also write $K(X)$ to denote the collection of all nonempty compact subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(x,B) = \inf\{d(x,y) : y \in B\}$ is the distance from the point x to the subset B.

A mapping $t : E \to E$ is said to be *nonexpansive* if $d(tx,ty) \le d(x,y)$ for all $x, y \in E$. A point x is called a fixed point of t if $tx = x$. A multi-valued mapping $T : E \to FB(X)$ is said to be *nonexpansive* if $H(Tx,Ty) \le d(x,y)$ for all $x, y \in E$. A point x is called a fixed point for a multivalued mapping T if $x \in Tx$.

Let E be a subset of a metric space X. A mapping $T : E \to 2^X$ with nonempty bounded values is nonexpansive provided $H(Tx,Ty) \le d(x,y)$ for all $x, y \in E$. Let $t : E \to E$ and $T : E \to 2^X$ with $T(x) \cap E \ne \emptyset$ for $x \in E$. Then t and T are said to be commuting mappings if $t(y) \in T(t(x)) \cap E$ for all $y \in T(x) \cap E$ and for all $x \in E$. A point $z \in X$ is called a center [5] for a

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mapping $t : E \to X$ if for each $x \in E, d(z, t(x)) \le d(z, x)$. The set $Z(t)$ denotes the set of all centers of the mapping t.

We use the notation $\mathrm{Fix}(T)$ stands for the set of fixed points of a mapping T and $\mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, s] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(s) = y$, and $d(c(t), c(u)) = |t - u|$ for all $t, u \in [0, s]$. In particular, c is an isometry and $d(x, y) = s$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = (1/2)y_1 \oplus (1/2)y_2$, then the CAT(0) inequality implies that

$$d(x, y_0)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{1}$$

This is the (CN) inequality of Bruhat and Tits [2]. In fact [1], a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

The following results and methods deal with the concept of asymptotic centers. Let E be a nonempty closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

Let

$$r \equiv r(E, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in E\}$$

and

$$A \equiv A(E, \{x_n\}) := \{x \in E : r(x, \{x_n\}) = r\}.$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to E.

It is easy to know that if X is complete CAT(0) spaces and E is a closed convex subset of X, then $A(E, \{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. A bounded sequence $\{x_n\}$ is said to be regular with respect to E if for every subsequence $\{x'_n\}$, we get

$$r(E, \{x_n\}) = r(E, \{x'_n\}).$$

We now give the definition of $\triangle$-convergence.

Definition 1.1 ([7], [11]). A sequence $\{x_n\}$ in a CAT(0) space X is said to $\triangle$-converge to $x \in X$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle\text{-}\lim_n x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results. The following lemma can be found in ([3], [4], [7]).

Lemma 1.2 ([7]). *Every bounded sequence in a complete* CAT(0) *space has a* $\triangle$*-convergent subsequence.*

Lemma 1.3 ([3]). *If E is a closed convex subset of a complete* CAT(0) *space and if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.*

Lemma 1.4 ([4]). *Let (X, d) be a* CAT(0) *space.*

(i) [Lemma 2.1(iv)] *For $x, y \in X$ and $u \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = ud(x, y) \quad \text{and} \quad d(y, z) = (1 - u)d(x, y).$$

We use the notation $(1 - u)x \oplus ty$ for the unique point z satisfying (1).

(ii) [Lemma 2.4] *For $x, y, z \in X$ and $u \in [0, 1]$, we have*

$$d((1 - u)x \oplus uy, z) \leq (1 - u)d(x, z) + ud(y, z).$$

A mapping $t : E \to E$ is called asymptotically nonexpansive if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ such that

$$d(t^n x, t^n y) \leq k_n d(x, y) \text{ for all } n \geq 1,\ x, y \in E.$$

We say that $I - T$ is strongly demiclosed if for every sequence $\{x_n\}$ in C which converges to $x \in C$ and such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, we have $x \in T(x)$.

We note that for every continuous mapping $T : C \to 2^C$, $I - T$ is strongly demiclosed but the converse is not true. Notice also that if T satisfies condition (E), then $I - T$ is strongly demiclosed.

The existence of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces was proved by Kirk [6] as the following result.

Theorem 1.5. *Let E be a nonempty bounded closed and convex subset of a complete* CAT(0) *space X and let $t : E \to E$ be asymptotically nonexpansive. Then t has a fixed point.*

Corollary 1.6 ([4])**.** *Let E be a closed and convex subset of a complete* CAT(0) *space X and let $t : E \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in E such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\triangle$-$\lim_{n\to\infty} x_n = w$. Then $tw = w$.*

Lemma 1.7 ([9])**.** *Let X be a complete* CAT(0) *space and let $x \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, x) = r$ for some $r \ge 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

The following lemma can be found in [15].

Lemma 1.8. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that*

$$a_{n+1} \le (1 + b_n)a_n, \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists. In particular, if there is a subsequence of $\{a_n\}$ which converges to 0, then $\lim_{n\to\infty} a_n = 0$

2. Preliminaries

In 2009, Laokul and Panyanak [8] defined the iterative and proved the $\triangle$-converges for nonexpansive mapping in CAT(0) spaces as follows:

Let C be a nonempty closed convex subset of a complete CAT(0) space and $t : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(t) := \{x \in C : tx = x\} \ne \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

$$x_{n+1} = \alpha_n t[\beta_n t x_n \oplus (1 - \beta_n)x_n] \oplus (1 - \alpha_n)x_n$$

for all $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1] such that one of the following two conditions is satisfied:

(i) $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ for some a, b with $0 < a \le b < 1$,

(ii) $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \le b < 1$.

Then the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of t.

In 2010, Sokhuma and Kaewkhao [14] proved the convergence theorem for a common fixed point in Banach spaces as follow:

Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued nonexpansive mapping and a multivalued nonexpansive mapping, respectively. Assume in addition that $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \ne \emptyset$ and $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. Suppose $\{x_n\}$ is generated iterative by $x_1 \in E$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n t y_n$$

for all $\in \mathbb{N}$ where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2013, Sokhuma [13] proved the convergence theorem for a common fixed point in CAT(0) as follow:

Let E be a nonempty compact convex subset of a complete CAT(0) space X, and $t : E \to E$ and $T : E \to FC(E)$ a single valued nonexpansive mapping and a multivalued nonexpansive mapping, respectively, and $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. Let $\{x_n\}$ is generated iterative by $x_1 \in E$,

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t y_n$$

for all $\in \mathbb{N}$ where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2013, Laowang and Panyanak obtained the following.

Corollary 2.1 ([10]). *Let C be a nonempty bounded closed convex subset of a complete* CAT(0) *spaces X. Let $f : C \to C$ be a pointwise asymptotically nonexpansive mapping, and $g : C \to C$ a quasi-nonexpansive mapping, and let $T : C \to KC(C)$ be a multivalued mapping satisfying conditions (E) and C_λ for some $\lambda \in (0, 1)$. If f, g and T are pairwise commuting, then there exists a point $z \in C$ such that $z = f(z) = g(z) \in T(z)$.*

The purpose of this paper is to study the iterative process, called the modified Ishikawa iteration method with respect to a pair of single valued asymptotically nonexpansive mapping and a multivalued nonexpansive mapping. We also establish the $\triangle$-convergence theorem of a sequence from such process in a nonempty bounded closed convex subset of a complete CAT(0) space.

Now, we introduce an iteration method modifying the above ones and call it the modified Ishikawa iteration method.

Definition 2.2. Let E be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : E \to E$ be a single valued asymptotically nonexpansive mapping, and $T : E \to FB(E)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t^n y_n, \tag{2}$$

where $z_n \in Tt^n x_n$ and $\{\alpha_n\}$, $\{\beta_n\} \in [0, 1]$, $n \geq 1$.

3. Main results

We first prove the following lemmas, which play very important roles in this section.

Lemma 3.1. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2). *Then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.*

Proof. Let $x_1 \in E$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, we have

$$\begin{aligned}
&d(x_{n+1}, w)\\
&= d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w))\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(t^n y_n, t^n w)\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n d(y_n, w)\\
&= (1-\alpha_n)d(x_n, w) + \alpha_n d((1-\beta_n)x_n \oplus \beta_n z_n, w)\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n(1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n d(z_n, w)\\
&= (1-\alpha_n)d(x_n, w) + \alpha_n k_n(1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n \mathrm{dist}(Tt^n x_n, w)\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n(1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n H(Tt^n x_n, Tw)\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n(1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n d(t^n x_n, w)\\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n(1-\beta_n)d(x_n, w) + \alpha_n \beta_n k_n^2 d(x_n, w)\\
&= [1 + \alpha_n(k_n - 1) + \alpha_n\beta_n k_n(k_n - 1)]d(x_n, w)\\
&= [1 + \alpha_n(1 + \beta_n k_n)(k_n - 1)]d(x_n, w).
\end{aligned}$$

By the convergence of k_n and $\alpha_n, \beta_n \in (0, 1)$, then there exists some $M > 0$ such that

$$d(x_{n+1}, w) \leq [1 + M(k_n - 1)]d(x_n, w).$$

By condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and Lemma 1.8, we know that $\lim_{n\to\infty} d(x_n, w)$ exists. □

Lemma 3.2. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2). *Then $\lim_{n\to\infty} d(t^n y_n, x_n) = 0$.*

Proof. From Lemma 3.1, we setting $\lim_{n\to\infty} d(x_n, w) = c$.

Consider,

$$\begin{aligned}
d(ty_n, w) &\leq d(y_n, w)\\
&= d((1-\beta_n)x_n \oplus \beta_n z_n, w)\\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(z_n, w)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n)d(x_n,w) + \beta_n \mathrm{dist}(Tt^n x_n, w) \\
&\leq (1-\beta_n)d(x_n,w) + \beta_n H(Tt^n x_n, Tw) \\
&\leq (1-\beta_n)d(x_n,w) + \beta_n d(t^n x_n, w) \\
&\leq (1-\beta_n)d(x_n,w) + \beta_n k_n d(x_n, w).
\end{aligned}
$$

We have

$$
\begin{aligned}
d(t^n y_n, w) &\leq k_n d(y_n, w) \\
&\leq k_n[(1-\beta_n)d(x_n,w) + \beta_n k_n d(x_n,w)] \\
&= k_n(1-\beta_n)d(x_n,w) + \beta_n k_n^2 d(x_n,w) \\
&= (k_n - k_n\beta_n + \beta_n k_n^2)d(x_n,w) \\
&= [k_n + \beta_n k_n(k_n-1)]d(x_n,w) \\
&\leq [1 + \beta_n k_n(k_n-1)]d(x_n,w).
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
\limsup_{n\to\infty} d(t^n y_n, w) &\leq \limsup_{n\to\infty} k_n d(y_n, w) \\
&\leq \limsup_{n\to\infty} [1 + \beta_n k_n(k_n-1)]d(x_n,w).
\end{aligned}
$$

By $k_n \to 1$ as $n \to \infty$ and $\alpha_n, \beta_n \in (0,1)$, which implies that

$$
\limsup_{n\to\infty} d(t^n y_n, w) \leq \limsup_{n\to\infty} d(y_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c. \tag{3}
$$

Since, $c = \lim_{n\to\infty} d(x_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w)$.

Then by condition of α_n and Lemma 1.7, we have $\lim_{n\to\infty} d(t^n y_n, x_n) = 0$. $\square$

Lemma 3.3. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in$* $\mathrm{Fix(t)} \cap \mathrm{Fix}(T)$ *and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2). *Then* $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Proof. Consider,

$$
\begin{aligned}
d(x_{n+1}, w) &= d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w) \\
&\leq (1-\alpha_n)d(x_n,w) + \alpha_n d(t^n y_n, w) \\
&\leq (1-\alpha_n)d(x_n,w) + \alpha_n k_n d(y_n, w)
\end{aligned}
$$

and hence

$$
\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \leq k_n d(y_n, w) - d(x_n, w).
$$

Therefore, since $0 < a \le \alpha_n \le b < 1$,

$$\left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \le k_n d(y_n, w).$$

Thus,

$$\liminf_{n\to\infty} \left\{ \left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \right\} \le \liminf_{n\to\infty} k_n d(y_n, w).$$

It follows that

$$c \le \liminf_{n\to\infty} d(y_n, w).$$

Since, from (3), $\limsup_{n\to\infty} d(y_n, w) \le c$, we have

$$c = \lim_{n\to\infty} d(y_n, w) = \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n z_n, w).$$

Recall that

$$d(z_n, w) = \operatorname{dist}(z_n, Tw) \le H(Tt^n x_n, Tw) \le d(t^n x_n, w) \le k_n d(x_n, w).$$

Hence we have

$$\limsup_{n\to\infty} d(z_n, w) \le \limsup_{n\to\infty} k_n d(x_n, w) \le \limsup_{n\to\infty} d(x_n, w) = c.$$

Thus,

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

□

Lemma 3.4. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\operatorname{Fix}(t) \cap \operatorname{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2)*. Then* $\lim_{n\to\infty} d(t^n x_n, x_n) = 0$.

Proof. It is easy to see that, $d(t^n x_n, x_n) \le k_n \beta_n d(z_n, x_n) + d(t^n y_n, x_n)$. Then, we have

$$\lim_{n\to\infty} d(t^n x_n, x_n) \le \lim_{n\to\infty} k_n \beta_n) d(z_n, x_n) + \lim_{n\to\infty} d(t^n y_n, x_n).$$

Hence, by Lemma 3.2 and Lemma 3.3, $\lim_{n\to\infty} d(t^n x_n, x_n) = 0$. □

Lemma 3.5. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\operatorname{Fix}(t) \cap \operatorname{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2)*. Then* $\lim_{n\to\infty} d(tx_n, x_n) = 0$.

Proof. Consider,

$$\begin{aligned} d(tx_n, w) &\le d(x_n, t^n x_n) + d(t^n x_n, tx_n) \\ &\le d(x_n, t^n x_n) + k_1[d(t^{n-1}x_n, t^{n-1}x_{n-1}) + d(t^{n-1}x_{n-1}, x_n)] \\ &\le d(x_n, t^n x_n) + k_1 k_{n-1} d(x_n, x_{n-1}) + k_1 d(t^{n-1}x_{n-1}, x_n) \\ &\le d(x_n, t^n x_n) + k_1 k_{n-1}\alpha_{n-1} d(t^{n-1}y_{n-1}, x_{n-1}) \\ &\quad + k_1(1-\alpha_{n-1}) d(x_{n-1}, t^{n-1}x_{n-1}) + k_1 k_{n-1}\alpha_{n-1} d(y_{n-1}, x_{n-1}) \\ &\le d(x_n, t^n x_n) + k_1 k_{n-1}\alpha_{n-1} d(t^{n-1}y_{n-1}, x_{n-1}) \\ &\quad + k_1(1-\alpha_{n-1}) d(x_{n-1}, t^{n-1}x_{n-1}) + k_1 k_{n-1}\alpha_{n-1}\beta_{n-1} d(y_{n-1}, x_{n-1}). \end{aligned}$$

It follows from Lemmas 3.2-3.4, we have $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Theorem 3.6. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2). *Then $\{x_n\}$ $\triangle$-converges to y implies $y \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.*

Proof. Since that $\{x_n\}$ $\triangle$-converges to y. From Lemma 3.5, we have

$$\lim_{n\to\infty} d(tx_n, x_n) = 0.$$

By Corollary 1.6, we have $y \in E$ and $ty = y$; that is $y \in \mathrm{Fix}(t)$. From Lemma 3.3 we have

$$\begin{aligned} dist(y, Ty) &\le d(y, x_n) + \mathrm{dist}(x_n, Tx_n) + H(Tx_n, Ty) \\ &\le d(y, x_n) + d(x_n, z_n) + d(x_n, y) \to 0 \end{aligned}$$

as $n \to \infty$. It follows that $y \in \mathrm{Fix}(T)$. Therefore $y \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ as desired. □

Theorem 3.7. *Let E be a nonempty bounded closed convex subset of a complete* CAT(0) *space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping, respectively, Assume that t and T are commuting and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by* (2). *Then $\{x_n\}$ $\triangle$-converges to a common fixed point of t and T.*

Proof. Since Lemma 3.5 guarantees that $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(tx_n, x_n) = 0$. We now let $\omega_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By

Lemma 1.2 and Lemma 1.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_{n\to\infty} v_n = v \in E$. Since $\lim_{n\to\infty} d(tv_n, v_n) = 0$, then $v \in \mathrm{Fix}(T)$. Since,

$$\begin{aligned} dist(v, Tv) &\le dist(v, Tv_n) + H(Tv_n, Tv) \\ &\le d(v, z_n) + d(v_n, v) \\ &\le d(v, v_n) + d(v_n, z_n) + d(v_n, v) \to 0 \end{aligned}$$

as $n \to \infty$. It follows that $v \in \mathrm{Fix}(T)$. Therefore $v \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$ as desired. We claim that $u = v$. Suppose not, since t is asymptotically nonexpansive mapping and $v \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$, $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.1. Then by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \le \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, v) = \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v) \end{aligned}$$

a contradiction, and hence $u = v \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.

To show that $\{x_n\}$ $\triangle$-converges to a common fixed point, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.2 and Lemma 1.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v \in E$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We can complete the proof by showing that $x = v$. Suppose not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \le \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v) \end{aligned}$$

a contradiction, and hence the conclusion follows. □

Now we present an example to illustrate Theorem 3.7.

Example 3.8. Let $E = [-2, 2]$ with the usual metric. Define $t : E \to E$ and $T : E \to FB(E)$ by:

$$tx = \frac{x}{4} \quad \text{and} \quad Tx = \begin{cases} \left[-\frac{x}{2}, -\frac{x}{5}\right] & \text{if } x \in [0, 2] \\ \left[-\frac{x}{5}, -\frac{x}{2}\right] & \text{if } x \in [-2, 0]. \end{cases}$$

Then t is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ with a unique fixed point 0. It clear that T is a multivalued nonexpansive mapping such that $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) = \{0\} \neq \emptyset$. Also, Example 3.8 satisfies all conditions of Theorem 3.7.

Let $\alpha_n = \frac{3}{4}$, $\beta_n = \frac{1}{4}$ for all $n \ge 1$ and $x_1 = -2$. The modified Ishikawa iteration in Definition 2.2, we have

$$z_n \in Tt^n x_n = T\left(\frac{1}{4}\right)^n x_n = \left[-\frac{\left(\frac{1}{4}\right)^n x_n}{5}, -\frac{\left(\frac{1}{4}\right)^n x_n}{2}\right] \text{ or } \left[-\frac{\left(\frac{1}{4}\right)^n x_n}{2}, -\frac{\left(\frac{1}{4}\right)^n x_n}{5}\right].$$

Let

$$z_n = \frac{3}{4}\left(-\frac{\left(\frac{1}{4}\right)^n x_n}{5} - \frac{\left(\frac{1}{4}\right)^n x_n}{2}\right) = -\left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)\left(\frac{7}{10}\right) x_n.$$

We have

$$\begin{aligned} y_n &= (1-\beta_n)\, x_n + \beta_n z_n \\ &= \left(\frac{3}{4}\right) x_n + \left(\frac{1}{4}\right)\left(-\left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)\left(\frac{7}{10}\right) x_n\right) \\ &= \left(\frac{3}{4}\right) x_n - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n+1}\left(\frac{7}{10}\right) x_n. \end{aligned}$$

Now, we have

$$\begin{aligned} t^n y_n &= \left(\frac{1}{4}\right)^n \left(\left(\frac{3}{4}\right) x_n - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n+1}\left(\frac{7}{10}\right) x_n\right) \\ &= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^n x_n - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{2n+1}\left(\frac{7}{10}\right) x_n \end{aligned}$$

Therefore,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n t^n y_n \\ &= \left(\frac{1}{4}\right) x_n + \left(\frac{3}{4}\right)\left(\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^n x_n - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{2n+1}\left(\frac{7}{10}\right) x_n\right) \\ &= \left(\frac{1}{4}\right) x_n + \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^n x_n - \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{2n+1}\left(\frac{7}{10}\right) x_n \end{aligned}$$

Thus, the modified Ishikawa iteration $\{x_n\}$ in Definition 2.2 converges to the common fixed point $w = 0$.

Table 1. Convergence of the modified Ishikawa iteration

n	x_n	n	x_n
1	-2	23	$-0.2085580268e-012$
2	-0.7689453125	24	$-0.5213950670e-013$
3	-0.2189738857	25	$-0.1303487668e-013$
4	-0.05666278408	26	$-0.3258719170e-014$
5	-0.01429011410	27	$-0.8146797925e-015$
$\vdots$	$\vdots$	28	$-0.2036699481e-015$
21	$-0.3336928428e-011$	29	$-0.5091748702e-016$
22	$-0.8342321070e-012$	$\vdots$	$\vdots$

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Naknimit Akkasriworn
Department of Mathematics
Faculty of Science and Technology
Rambhai Barni Rajabhat University
Chantaburi 22000, Thailand
E-mail address: boyjuntaburi@hotmail.com

Kritsana Sokhuma
Department of Mathematics
Faculty of Science and Technology
Muban Chom Bueng Rajabhat University
Ratchaburi 70150, Thailand
E-mail address: k_ sokhuma@yahoo.co.th