Commun. Korean Math. Soc. **30** (2015), No. 3, pp. 169–176
http://dx.doi.org/10.4134/CKMS.2015.30.3.169

NUMERICAL RANGE AND SOT-CONVERGENCY

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Abstract. A sequence of composition operators on Hardy space is considered. We prove that, by numerical range properties, it is SOT-convergence but not converge.

1. introduction

Let φ be a holomorphic self-map of the unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The function φ induces the *composition operator* C_φ, defined on the space of holomorphic functions on $\mathbb{U}$ by $C_\varphi f = f \circ \varphi$. The restriction of C_φ to various Banach spaces of holomorphic functions on $\mathbb{U}$ has been an active subject of research for more than three decades and it will continue to be for decades to come (see [11], [12] and [4]). Let H^2 denote the *Hardy space* of analytic functions on the open unit disc with square summable Taylor coefficients. In recent years the study of composition operators on the Hardy space has received considerable attention.

In this paper we consider the *numerical range* of elliptic composition operators on H^2. The numerical range of a bounded linear operator A on a Hilbert space $\mathcal{H}$ is the set of complex numbers

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$.

In [9] V. Matache determined the shape $W(C_\varphi)$ when the symbol of the composition operator is a monomial or an inner function fixing 0. Also he gave some properties of the numerical range of composition operators in some cases. In [2] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disc specially parabolic and hyperbolic were investigated. In [2], Bourdon and Shapiro have considered the question of when the numerical range of a composition operator is a disc centered at the origin and have shown that this happens whenever the inducing map is a non elliptic conformal automorphism of the unit disc. They also have shown that the numerical range of elliptic automorphism with order 2 is an

Received November 7, 2014.
2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 47B33, 30H10.
Key words and phrases. numerical range, composition operator, Hardy space.

ellipse with focus at ± 1. In [1], the author has completed their results by finding the exact value of the major axis of the ellipses. However, for the elliptic automorphisms with finite order $k > 2$, this is an open problem yet.

2. Notations and preliminaries

Let $\mathbb{U}$ denote the open unit disc in the complex plane, and the *Hardy space* H^2 the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $\langle f, g \rangle := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\partial \mathbb{U}} f(z)\overline{g(z)} \frac{dz}{z},$$

where $\partial\mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits.

For each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a *composition operator* C_φ, defined by the equation $C_\varphi f = f \circ \varphi$ $(f \in H^2)$. A consequence of a famous theorem of J. E. Littlewood [8] asserts that C_φ is a bounded operator (see also [11] and [4]). In fact (see [4])

$$\sqrt{\frac{1}{1 - |\varphi(0)|^2}} \le \|C_\varphi\| \le \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

In the case $\varphi(0) \neq 0$ Joel H. Shapiro has been shown that the second inequality changes to equality if and only if φ is an inner function (see [11]).

With each point $\lambda \in \mathbb{U}$ we associate the *reproducing kernel*

$$K_\lambda(z) = \frac{1}{1 - \overline{\lambda} z} = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n \quad (z \in \mathbb{U}).$$

Each kernel function K_λ is holomorphic in a neighborhood of $\overline{\mathbb{U}}$ and hence belongs to H^2. Moreover for each $\lambda \in \mathbb{U}$ and $f \in H^2$, $\langle f, K_\lambda \rangle = f(\lambda)$.

A *conformal automorphism* is a univalent holomorphic mapping of $\mathbb{U}$ onto itself. Each such map is a linear fractional, and can be represented as a product $w.\alpha_p$, where

$$\alpha_p(z) := \frac{p - z}{1 - \overline{p} z}, \ (z \in \mathbb{U}),$$

for some fixed $p \in \mathbb{U}$ and $w \in \partial\mathbb{U}$ (see [10]).

The map α_p interchanges the point p and the origin and it is a self-inverse automorphism of $\mathbb{U}$.

Each conformal automorphism is a bijection map from the sphere $\mathbb{C} \bigcup \{\infty\}$ to itself with two fixed points (counting multiplicity). An automorphism is called:

- *elliptic* if it has one fixed point in the disc and one outside the closed disc,
- *hyperbolic* if it has two distinct fixed point on the boundary $\partial\mathbb{U}$, and
- *parabolic* if there is one fixed point of multiplicity 2 on the boundary $\partial\mathbb{U}$.

For $r \in \mathbb{U}$, an *r-dilation* is a map of the form $\delta_r(z) = rz$. We call r the dilation parameter of δ_r and in the case that $r > 0$, δ_r is called *positive dilation.* A *conformal r-dilation* is a map that is conformally conjugate to an r-dilation, i.e., a map $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where $r \in \mathbb{U}$ and α is a conformal automorphism of $\mathbb{U}$.

For $w \in \partial\mathbb{U}$, an *w-rotation* is a map of the form $\rho_w(z) = wz$. We call w the rotation parameter of ρ_w. A straightforward calculation shows that every elliptic automorphism φ of $\mathbb{U}$ must have the form

$$\varphi = \alpha_p \circ \rho_w \circ \alpha_p$$

for some $p \in \mathbb{U}$ and some $w \in \partial\mathbb{U}$.

Let A be a (bounded linear) operator on a complex Hilbert space $\mathcal{H}$. The *numerical range* of A is the set

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

in the complex plane, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{H}$. In other words, $W(A)$ is the image of the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ of $\mathcal{H}$ under the (bounded) quadratic form $x \mapsto \langle Ax, x\rangle$.

Some properties of the numerical range follow easily from the definition. For one thing, the numerical range is unchanged under the unitary equivalence of operators: $W(A) = W(U^*AU)$ for any unitary U. It also behaves nicely under the operation of taking the adjoint of an operator: $W(A^*) = \{\overline{z} : z \in W(A)\}$. One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz-Hausdorff Theorem (see [5] and [6]). Other important property of W(A) is that its closure contains the spectrum of the operator. W(A) is a connected set and, in the finite dimensional case, is compact.

One more important property of the numerical range map, $A \to \overline{W(A)}$, that we need in this paper, is the continuity of it. For the convergence of compact subsets of the plane, we use the topology induced by the Hausdorff metric [6].

Let K_1 and K_2 be two compact subsets of plane. The Hausdorff distance $\Delta(K_1, K_2)$ is the minimal number r such that the closed r-neighborhood of any x in K_1 contains at least one point y of K_2 and vice versa. In other words,

$$\Delta(K_1, K_2) = \max\{\sup_{x\in K_1} \inf_{y\in K_2} |x - y|, \sup_{y\in K_2} \inf_{x\in K_1} |x - y|\}.$$

The next theorem says that the closure of the numerical range induces a continuous map on operators when the latter is endowed with the norm topology.

Theorem 2.1. *If $\mathcal{H}$ is a Hilbert space and $\{A_n\}$ is a sequence of bounded linear operators on $\mathcal{H}$ which converges to the bounded linear operator A in norm, then $\overline{W(A_n)}$ converges to $\overline{W(A)}$ in the Hausdorff metric.*

In [9] V. Matache determined the shape of $W(C_\varphi)$ in the case when φ is a monomial or an inner function fixing 0. Also he gave some properties of the numerical range of composition operators in some cases. He also showed that if $\varphi = a, 0 < |a| < 1$, then $W(C_\varphi)$ is a closed elliptical disc whose boundary is the ellipse of foci 0 and 1, having horizontal axis of length $\frac{1}{\sqrt{1-|a|^2}}$. Also, the numerical ranges of some compact composition operators was presented.

In [2] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disc specially parabolic and hyperbolic were investigated. The authors proved, among other things, the following results:

(1) If φ is a conformal automorphism of $\mathbb{U}$ is either parabolic or hyperbolic, then $W(C_\varphi)$ is a disc centered at the origin; moreover this disc is contained in the essential numerical range $W_e(C_\varphi) = \bigcap \overline{W(C_\varphi + K)}$, the intersection being taken over all K of compact operators.
(2) If φ is a hyperbolic automorphism of $\mathbb{U}$ with antipodal fixed points and it is conformally conjugate to a positive dilation $z \mapsto rz$ $(0 < r < 1)$, then $W(C_\varphi)$ is the open disc of radius $1/\sqrt{r}$ centered at the origin.
(3) If φ is elliptic and conformally conjugate to a rotation $z \mapsto \omega z$ $(|\omega| = 1)$ and ω is not a root of unity, then $\overline{W}(C_\varphi)$ is a disc centered at the origin.
(4) If $T \neq \pm I$ is an operator on the Hilbert space $\mathcal{H}$ with $T^2 = I$, then $\overline{W}(T)$ is a (possibly degenerate) elliptical disc with foci at ± 1. In particular, it is not a circular disc.
(5) If φ is an elliptic automorphism of $\mathbb{U}$ with rotation parameter (multiplier) -1, then $\overline{W}(C_\varphi)$ is a (possibly degenerate) ellipse with foci ± 1. The degenerate case occurs if and only if $C_\varphi(0) = 0$, in which case $C_\varphi(z) \equiv -z$.

In [3] the authors proved that the numerical range of any composition operator, except for the identity, contains the origin in its closure. Using this as motivation, they focus on when $0 \in W(C_\varphi)$. The authors developed methods that produce complete answers to this and related questions for maps φ that fix a point p in $\mathbb{U}$. They showed in this case that $0 \notin W(C_\varphi)$ if and only if there is an $r \in (0, 1]$ such that $|p| \leq \sqrt{r}$ and $\varphi(z) = \alpha_p(r\alpha_p(z))$, where $\alpha_p(z) = (p - z)/(1 - \overline{p}z)$. Further, for such a map φ they proved that C_φ is sectorial, i.e., $W(C_\varphi) \subset \{0\} \bigcup \{z \in \mathbb{C} : |\text{Arg} z| \leq \theta\}$ for some θ, if $|p| < \sqrt{r}$ but not when $|p| = \sqrt{r}$.

Theorem 2.2. *Suppose $w_k = e^{\frac{2\pi i}{k}}$ and $\varphi_k = \alpha_p \circ \rho_{w_k} \circ \alpha_p$. Then $\overline{W(C_{\varphi_k})}$ tends to the $\overline{\mathbb{U}}$=the closed unit disc, as $k \to \infty$.*

Proof. Let $\varphi_p = \alpha_p \circ \rho_w \circ \alpha_p$ and $p = \eta|p|$ for some $\eta \in \partial\mathbb{U}$. It follows easily that $\varphi_p = \rho_\eta \circ \varphi_{|p|} \circ \rho_{\eta^{-1}}$ and hence $C_{\varphi_p} = C_{\rho_{\eta^{-1}}} \circ C_{\varphi_{|p|}} \circ C_{\rho_\eta}$. Since $V = C_{\rho_\eta}$ is the unitary composition operator induced on H^2 by ρ_η, we have

$$W(C_{\varphi_p}) = W(C_{\rho_{\eta^{-1}}} \circ C_{\varphi_{|p|}} \circ C_{\rho_\eta}) = W(C_{\varphi_{|p|}}).$$

Hence without loss of generality we may assume that $0 < p < 1$. Put $e_j(z) = (\alpha_p(z))^j$ for $j = 0, 1, 2, \ldots,\ k-1$. By a straightforward calculation we have $C_{\varphi_k} e_j(z) = w_k^j e_j(z)$, i.e., e_j is the eigenfunction of C_{φ_k} corresponding to eigenvalue w_k^j, which implies $w_k^j \in W(C_{\varphi_k})$. Let T_k be the convex hull of $\{1, w_k, w_k^2, \ldots, w_k^{k-1}\}$. Since $W(C_{\varphi_k})$ is bounded, convex and $r(C_{\varphi_k})$= the numerical radius of $C_{\varphi_k} \leq \|C_{\varphi_k}\|$,

$$T_k \subseteq \overline{W(C_{\varphi_k})} \subseteq \overline{B(0, \|C_{\varphi_k}\|)}.$$

Then

$$\Delta(\overline{W(C_{\varphi_k})}, T_k) = \sup_{x \in \overline{W(C_{\varphi_k})}} \inf_{y \in T_k} |x - y| \leq \|C_{\varphi_k}\| - \cos\frac{\pi}{k}. \tag{1}$$

On the other hand

$$\Delta(T_k, \overline{\mathbb{U}}) \to 0 \quad \text{as} \quad k \to 0. \tag{2}$$

Also, since φ_k is an inner function,

$$\|C_{\varphi_k}\| = \sqrt{\frac{|1 - p^2 w_k| + p|1 - w_k|}{|1 - p^2 w_k| - p|1 - w_k|}} \to 1 \quad \text{as} \quad k \to \infty. \tag{3}$$

The proof is completed by combining (1), (2) and (3). □

3. Continuity with respect to symbols φ

An elliptic automorphism φ of $\mathbb{U}$ that does not fix the origin must have the form $\varphi = \alpha_p \circ \rho_w \circ \alpha_p$, where

$$\rho_w(z) = wz \ (z \in \mathbb{U})$$

for some fixed $p \in \mathbb{U} - \{0\}$ and $w \in \partial\mathbb{U}$. Let us call w the rotation parameter(multiplier) of φ. If we wish to show this dependence of φ on p and w, we will denote the elliptic automorphism $\alpha_p \circ \rho_w \circ \alpha_p$ by $\varphi_{p,w}$.

If such a map φ is not periodic, then the closure of $W(C_\varphi)$ is a disc centered at the origin [2]. If φ is periodic then, surprisingly, the situation seems even murkier: For period 2 has been shown the closure of $W(C_\varphi)$ is an elliptical disc with foci at ± 1 (Corollary 4.4 of [2]). It is easy to see that $W(C_\varphi)$ is open, also in [1], the author completely determined $W(C_\varphi)$ for period 2. But for period $k > 2$ then all we can say is that the numerical range of C_φ has k-fold symmetry and we strongly suspect that in this case the closure is not a disc.

Recall that a sequence of bounded linear operators $\{A_n\}$ on Hilbert space $\mathcal{H}$ is said to converge to A in the strong operator topology or SOT if $\|(A_n - A)x\| \to 0$ for any vector x in $\mathcal{H}$.

In this section we prove that the composition operators is continuous as a function of the symbol φ. In fact, we prove the following theorem:

Theorem 3.1. *Let $\{\varphi_n\}$ be a sequence of conformal automorphism of the unit disc. If $\varphi_n \to \varphi$ uniformly on compact subsets of $\mathbb{U}$ as $n \to \infty$, then for each $f \in H^2$,*

$$C_{\varphi_n} f \to C_\varphi f \quad as \quad n \to \infty.$$

Indeed

$$C_{\varphi_n} \to C_\varphi$$

in SOT.

Proof. Problem 9(a), page 34 of [11] implies that $\varphi_n \to \varphi$ weakly. On the other hand

$$0 \le \|\varphi_n - \varphi\|^2 = \|\varphi_n\|^2 - 2\mathrm{Re}\langle\varphi_n, \varphi\rangle + \|\varphi\|^2 \le 2(1 - \mathrm{Re}\langle\varphi_n, \varphi\rangle) \to 0$$

since $\|\varphi\| = 1$. So $\|\varphi_n - \varphi\| \to 0$.

Let $p_k(z) = z^k$, $k = 0, 1, 2, \ldots$. Then

$$\begin{aligned} \|(C_{\varphi_n} - C_\varphi)p_k\| &= \int_{\partial\mathbb{U}} |(\varphi_n(z))^k - (\varphi(z))^k|^2 dm(z) \\ &\le k^2 \int_{\partial\mathbb{U}} |(\varphi_n(z)) - (\varphi(z))|^2 dm(z) \\ &= k^2 \|\varphi_n - \varphi\|, \end{aligned}$$

which tends to zero as $n \to \infty$. Hence $\|(C_{\varphi_n} - C_\varphi)P\| \to 0$ for any polynomial P. Let $f \in H^2$ and $\{P_k\}$ be a sequence of polynomials which converges to f in H^2. Because φ_n and φ are inner functions and $\lim_{n\to\infty} |\varphi_n(0)| = |\varphi(0)|$, for each positive integer n, we have

$$\|(C_{\varphi_n} - C_\varphi)\| \le \sqrt{\frac{1 + |\varphi_n(0)|}{1 - |\varphi_n(0)|}} + \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} \le M,$$

where M is a positive real number. Now let $\varepsilon > 0$ be given. Then there exists positive integer k_0 such that $\|f - P_{k_0}\| < \varepsilon/2M$. Also there exists a positive integer N such that $n \ge N$ implies $\|(C_{\varphi_n} - C_\varphi)P_{k_0}\| < \varepsilon/2$, which implies

$$\begin{aligned} \|(C_{\varphi_n} - C_\varphi)f\| &= \|(C_{\varphi_n} - C_\varphi)(f \pm P_{k_0})\| \\ &\le \|(C_{\varphi_n} - C_\varphi)(f - P_{k_0})\| + \|(C_{\varphi_n} - C_\varphi)P_{k_0}\| \\ &\le \varepsilon. \end{aligned}$$

Therefore $\|(C_{\varphi_n} - C_\varphi)f\| \to 0$ as $n \to \infty$, and the proof is complete now. □

Corollary 3.2. *Suppose $w_k = e^{\frac{2\pi i}{k}}$ and $\varphi_k = \alpha_p \circ \rho_{w_k} \circ \alpha_p$. Then*

$$C_{\varphi_k} \to C_z$$

as $k \to \infty$ in SOT.

Proof. By a simple computation for each $z \in \partial\mathbb{U}$, we have

$$|\varphi_k(z) - z| = \frac{|1-w_k||p-z||1-pz|}{|1-p^2w_k - p(1-w_k)z}| \leq \frac{|1-w_k|(1+p)^2}{|1-p^2w_k| - |p(1-w_k)|} \to 0$$

as $k \to \infty$. Since the right side of the above inequality is independent from z, $\varphi_k(z) \to z$ uniformly on $\partial\mathbb{U}$ and by maximum modulo theorem, $\varphi_k(z) \to z$ uniformly on compact subsets of $\mathbb{U}$. Hence the conclusion follows from Theorem 3.1. □

Corollary 3.3. *Let φ_k be as defined in Corollary 3.2. Then $\langle C_{\varphi_k} f, f\rangle \to 1$ as $k \to \infty$ if $f \in H^2$ with $\|f\| = 1$.*

Proof. By corollary 3.2 we have

$$\begin{aligned}|\langle C_{\varphi_k} f, f\rangle - \|f\|^2| &= |\langle C_{\varphi_k} f - f, f\rangle| \\ &\leq \|C_{\varphi_k} f - f\|\|f\| \to 0.\end{aligned}$$

So if $\|f\| = 1$, then $\langle C_{\varphi_k} f, f\rangle \to 1$ as $k \to \infty$ or 1 is the accumulation point. □

Corollary 3.4. *If $\varphi_k = \alpha_p \circ \rho_{w_k} \circ \alpha_p$, then C_{φ_k} is a divergence sequence in norm topology.*

Proof. Since $W(C_z) = \{1\} \neq \overline{\mathbb{U}} = \lim \overline{W(C_{\varphi_k})}$, so the proof easily follows from Theorems 2.1, 2.2 and Corollary 3.2. □

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