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Center of Gravity and a Characterization of Parabolas

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Abstract. Archimedes determined the center of gravity of a parabolic section as follows. For a parabolic section between a parabola and any chord AB on the parabola, let us denote by P the point on the parabola where the tangent is parallel to AB and by V the point where the line through P parallel to the axis of the parabola meets the chord AB. Then the center G of gravity of the section lies on PV called the axis of the parabolic section with $PG = \frac{3}{5}PV$. In this paper, we study strictly locally convex plane curves satisfying the above center of gravity properties. As a result, we prove that among strictly locally convex plane curves, those properties characterize parabolas.

1. Introduction

Archimedes found some interesting area properties of parabolas. Consider the region bounded by a parabola and a chord AB. Let P be the point on the parabola where the tangent is parallel to the chord AB. The parallel line through P to the axis of the parabola meets the chord AB at a point V. Then, he proved that the

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area of the parabolic region is 4/3 times the area of triangle $\triangle ABP$ whose base is the chord and the third vertex is P.

Furthermore, he showed that the center G of gravity of the parabolic section lies on the segment PV called the axis of the parabolic section with $PG = \frac{3}{5}PV$. For the proofs of Archimedes, see Chapter 7 of [25].

Recently, two of the present authors showed that among strictly convex plane curves, the above area properties of parabolic sections characterize parabolas. More precisely, they proved as follows ([18]).

Proposition 1. *Let X be a strictly convex curve in the plane $\mathbb{R}^2$. Then X is a parabola if and only if it satisfies*
(C) : *For a point P on X and a chord AB of X parallel to the tangent of X at P, the area of the region bounded by the curve and AB is $4/3$ times the area of triangle $\triangle ABP$.*

Actually, in [18], they established five characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([25]). In [21], they gave some characterizations of parabolas using area of triangles associated with a plane curve, which are generalizations of some results in [23]. See also [10, 11, 12] for some generalizations of results in [21].

In [16] and [17], two of the present authors proved the higher dimensional analogues of some results in [18]. Some characteristic properties of hyperspheres, ellipsoids, elliptic hyperboloids, hypercylinders and W-curves in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ were given in [1, 4, 6, 7, 13, 15, 22]. In [19], some characteristics for hyperbolic spaces embedded in the Minkowski space were established.

For some characterizations of parabolas or conic sections by properties of tangent lines, see [8] and [20]. In [14], using curvature function κ and support function h of a plane curve, the first and second authors of the present paper gave a characterization of ellipses and hyperbolas centered at the origin.

Among the graphs of functions, Á. Bényi et al. proved some characterizations of parabolas ([2, 3]). B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([24]). In their paper, parabola means the graph of a quadratic polynomial in one variable.

In this paper, we study strictly locally convex plane curves satisfying the above mentioned properties on the center of gravity. Recall that a regular plane curve $X : I \to \mathbb{R}^2$ in the plane $\mathbb{R}^2$, where I is an open interval, is called *convex* if, for all $s \in I$ the trace $X(I)$ of X lies entirely on one side of the closed half-plane determined by the tangent line at s ([5]). A regular plane curve $X : I \to \mathbb{R}^2$ is called *locally convex* if, for each $s \in I$ there exists an open subinterval $J \subset I$ containing s such that the curve $X|_J$ restricted to J is a convex curve.

Hereafter, we will say that a locally convex curve X in the plane $\mathbb{R}^2$ is *strictly locally convex* if the curve is smooth (that is, of class $C^{(3)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $X(s)$ is an arc-length parametrization of X.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect to the upward unit normal N. This condition is equivalent to the positivity of $f''(x)$ on I.

First of all, in Section 2 we prove the following:

Theorem 2. *Let X be a strictly locally convex plane curve in the plane $\mathbb{R}^2$. For a fixed point P on X and a sufficiently small $h > 0$, we denote by l the parallel line through $P + hN(P)$ to the tangent t of the curve X at P. If we let $d_P(h)$ the distance from the center G of gravity of the section of X cut off by l to the tangent t of the curve X at P, then we have*

$$\lim_{h\to 0} \frac{d_P(h)}{h} = \frac{3}{5}. \tag{1.1}$$

Without the help of Proposition 1, in Section 3 we prove the following characterization theorem for parabolas with axis parallel to the y-axis, that is, the graph of a quadratic function.

Theorem 3. *Let X be the graph of a strictly convex function $g : I \to \mathbb{R}$ in the uv-plane $\mathbb{R}^2$ with the upward unit normal N. For a fixed point $P = (u, g(u))$ on X and a sufficiently small $h > 0$, we denote by l (resp., V) the parallel line through $P + hN(P)$ to the tangent t of the curve X at P (resp., the point where the parallel line through P to the v-axis meets l). Then X is an open arc of a parabola with axis parallel to the v-axis if and only if it satisfies*
(D) : *For a fixed point P on X and a sufficiently small $h > 0$, the center G of gravity of the section of X cut off by l lies on the segment PV with*

$$PG = \frac{3}{5}PV, \tag{1.2}$$

where we denote by PV both of the segment and its length.

Note that if X is an open arc of a parabola with axis which is not parallel to the v-axis (for example, the graph of g given in (3.23) with $\alpha \neq 0$), then it does not satisfy Condition (D).

Finally using Proposition 1, in Section 4 we prove the following characterization theorem for parabolas.

Theorem 4. *Let X be a strictly locally convex plane curve in the plane $\mathbb{R}^2$. For a fixed point P on X and a sufficiently small $h > 0$, we denote by l the parallel line through $P + hN(P)$ to the tangent t of the curve X at P. We let $d_P(h)$ the distance from the center G of gravity of the section of X cut off by l to the tangent t of the curve X at P. Then X is an open arc of a parabola if and only if it satisfies*

(E) : *For a fixed point P on X and a sufficiently small $h > 0$, we have*

$$d_P(h) = \frac{3}{5}h. \tag{1.3}$$

In [9], using the results in this paper and the centroids of triangles associated with a strictly convex curve, some characterizations of parabolas were established.

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. Preliminaries and Theorem 2

Suppose that X is a strictly locally convex curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point $P \in X$, and for a sufficiently small $h > 0$, consider the parallel line l through $P + hN(P)$ to the tangent t of X at P. Let's denote by A and B the points where the line l intersects the curve X.

We denote by $S_P(h)$ (respectively, $R_P(h)$) the area of the region bounded by the curve X and chord AB (respectively, of the rectangle with a side AB and another one on the tangent t of X at P with height $h > 0$). We also denote by $L_P(h)$ the length of the chord AB. Then we have $R_P(h) = hL_P(h)$.

We may adopt a coordinate system (x, y) of $\mathbb{R}^2$ in such a way that P is taken to be the origin $(0, 0)$ and the x-axis is the tangent line of X at P. Furthermore, we may assume that X is locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$.

For a sufficiently small $h > 0$, we have

$$\begin{aligned} S_P(h) &= \int_{f(x)<h} \{h - f(x)\}dx, \\ R_P(h) &= hL_P(h) = h \int_{f(x)<h} 1dx. \end{aligned} \tag{2.1}$$

The integration is taken on the interval $I_P(h) = \{x \in \mathbb{R} | f(x) < h\}$.

On the other hand, we also have

$$S_P(h) = \int_{y=0}^{h} L_P(y)dy,$$

which shows that

$$S'_P(h) = L_P(h). \tag{2.2}$$

First of all, we need the following lemma ([18]), which is useful in this article.

Lemma 5. *Suppose that X is a strictly locally convex curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}L_P(h)=\frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{2.3}$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N.

From Lemma 5, we get a geometric meaning of curvature $\kappa(P)$ of a locally strictly convex plane curve X at a point $P\in M$. That is, we obtain

$$\kappa(P)=\lim_{h\to 0}\frac{8h}{L_P(h)^2}. \tag{2.4}$$

Now, we give a proof of Theorem 2.

Let us denote by $d_P(h)$ the distance from the center G of gravity of the section of X cut off by l to the tangent t of the curve X at P. Note that the curve X is of class $C^{(3)}$. If we adopt a coordinate system (x,y) of $\mathbb{R}^2$ as in the beginning of this section, then the curve X is locally the graph of a non-negative strictly convex $C^{(3)}$ function $f:\mathbb{R}\to\mathbb{R}$. Hence, the Taylor's formula of $f(x)$ is given by

$$f(x)=ax^2+f_3(x), \tag{2.5}$$

where $a=f''(0)/2$, and $f_3(x)$ is an $O(|x|^3)$ function. Since $\kappa(P)=2a>0$, we see that a is positive.

From the definition of $d_P(h)$, we have

$$S_P(h)d_P(h)=\phi(h), \tag{2.6}$$

where we put

$$\phi(h)=\frac{1}{2}\int_{f(x)<h}\{h^2-f(x)^2\}dx. \tag{2.7}$$

We decompose $\phi(h)=\phi_1(h)-\phi_2(h)$ as follows:

$$\phi_1(h)=\frac{1}{2}\int_{f(x)<h}h^2dx,\quad \phi_2(h)=\frac{1}{2}\int_{f(x)<h}f(x)^2dx. \tag{2.8}$$

It follows from the definition of $L_P(h)$ that

$$\phi_1(h)=\frac{1}{2}h^2L_P(h). \tag{2.9}$$

Hence, Lemma 5 shows that

$$\lim_{h\to 0}\frac{\phi_1(h)}{h^2\sqrt{h}}=\frac{\sqrt{2}}{\sqrt{\kappa(P)}}. \tag{2.10}$$

Lemma 6. *For the limit of* $\phi_2(h)/(h^2\sqrt{h})$ *as* h *tends to* 0, *we get*

$$\lim_{h\to 0}\frac{\phi_2(h)}{h^2\sqrt{h}} = \frac{\sqrt{2}}{5\sqrt{\kappa(P)}}. \tag{2.11}$$

Proof. If we put $g(x) = f(x)^2$, then we have from (2.5)

$$g(x) = a^2x^4 + f_5(x), \tag{2.12}$$

where $f_5(x)$ is an $O(|x|^5)$ function. We let $x = \sqrt{h}\xi$. Then, together with (2.5), (2.8) gives

$$\begin{aligned}\frac{\phi_2(h)}{h^2\sqrt{h}} &= \frac{1}{2h^2\sqrt{h}}\int_{f(x)<h} g(x)dx\\ &= \frac{1}{2h^2}\int_{a\xi^2+g_3(\sqrt{h}\xi)<1} g(\sqrt{h}\xi)d\xi,\end{aligned} \tag{2.13}$$

where we denote $g_3(\sqrt{h}\xi) = \frac{f_3(\sqrt{h}\xi)}{h}$.

Since $f_3(x)$ is an $O(|x|^3)$ function, we have for some constant C_1

$$|g_3(\sqrt{h}\xi)| \le C_1\sqrt{h}|\xi|^3. \tag{2.14}$$

We also obtain from (2.12) that

$$\frac{|g(\sqrt{h}\xi) - a^2h^2\xi^4|}{h^2} \le C_2\sqrt{h}|\xi|^5 \tag{2.15}$$

where C_2 is a constant.

If we let $h \to 0$, it follows from (2.13)-(2.15) that

$$\begin{aligned}\lim_{h\to 0}\frac{\phi_2(h)}{h^2\sqrt{h}} &= \frac{1}{2}\int_{a\xi^2<1} a^2\xi^4 d\xi\\ &= \frac{1}{5\sqrt{a}}.\end{aligned} \tag{2.16}$$

This completes the proof of Lemma 6. □

Together with (2.10), Lemma 6 shows that

$$\lim_{h\to 0}\frac{\phi(h)}{h^2\sqrt{h}} = \frac{4\sqrt{2}}{5\sqrt{\kappa(P)}}. \tag{2.17}$$

Since $S'_P(h) = L_P(h)$, it follows from Lemma 5 that

$$\lim_{h\to 0}\frac{1}{h\sqrt{h}}S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}. \tag{2.18}$$

Thus, together with (2.17) and (2.18), (2.6) completes the proof of Theorem 2.

3. Proof of Theorem 3

In this section, we give a proof of Theorem 3.

Let X be the graph of a strictly convex function $g: I \to \mathbb{R}$ in the uv-plane $\mathbb{R}^2$ with the upward unit normal N.

For a fixed point $P = (b, c) \in X$ with $c = g(b)$, we denote by θ the angle between the normal $N(P)$ and the positive v-axis. Then we have $g'(b) = \tan\theta$ and $V = (b, c + wh)$ for sufficiently small $h > 0$, where $w = \sqrt{1 + g'(b)^2} = \sec\theta$.

By a change of coordinates in the plane $\mathbb{R}^2$ given by

$$\begin{aligned} u &= x\cos\theta - y\sin\theta + b, \\ v &= x\sin\theta + y\cos\theta + c, \end{aligned} \tag{3.1}$$

the graph $X : v = g(u), u \in I$ is represented by $X : y = f(x), x \in J$, P by the origin and V by the point $(\alpha h, h)$, where $\alpha = \tan\theta$.

Since $f(0) = f'(0) = 0$, the Taylor's formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x), \tag{3.2}$$

where $a = f''(0)/2$, and $f_3(x)$ is an $O(|x|^3)$ function. Since $\kappa(P) = 2a > 0$, we see that a is positive.

For a sufficiently small $h > 0$, it follows from the definition of the center $G = (\bar{x}_P(h), \bar{y}_P(h))$ of gravity of the section of X cut off by the parallel line l through $P + hN(P)$ to the tangent t of X at P that

$$\begin{aligned} \bar{y}_P(h)S_P(h) &= \phi(h), \\ \bar{x}_P(h)S_P(h) &= \psi(h), \end{aligned} \tag{3.3}$$

where we put

$$\phi(h) = \frac{1}{2}\int_{f(x)<h} \{h^2 - f(x)^2\}dx \tag{3.4}$$

and

$$\psi(h) = \int_{f(x)<h} \{x(h - f(x))\}dx. \tag{3.5}$$

First of all, we get

Lemma 7. *If we let $I_P(h) = \{x | f(x) < h\} = (x_1(h), x_2(h))$, then we have*

$$\phi'(h) = h\{x_2(h) - x_1(h)\} = hL_P(h) \tag{3.6}$$

and

$$\psi'(h) = \frac{1}{2}\{x_2(h)^2 - x_1(h)^2\}. \tag{3.7}$$

Proof. If we put $\bar{f}(x) = f(x)^2$ and $k = h^2$, then we have

$$\begin{aligned} 2\phi(h) &= \int_{f(x)^2<h^2} \{h^2 - f(x)^2\}dx \\ &= \int_{\bar{f}(x)<k} \{k - \bar{f}(x)\}dx. \end{aligned} \tag{3.8}$$

We denote by $\bar{S}_P(k)$ the area of the region bounded by the graph of $y = \bar{f}(x)$ and the line $y = k$. Then (3.8) shows that

$$2\phi(h) = \bar{S}_P(k). \tag{3.9}$$

It follows from (2.2) that

$$\frac{d}{dk}\bar{S}_P(k) = \bar{L}_P(k), \tag{3.10}$$

where $\bar{L}_P(k)$ denotes the length of the interval $\bar{I}_P(k) = \{x \in \mathbb{R} | \bar{f}(x) < k\}$.

Since $k = h^2$, $\bar{I}_P(k)$ coincides with the interval $I_P(h) = \{x \in \mathbb{R} | f(x) < h\}$. Hence we get $\bar{L}_P(k) = L_P(h)$. This, together with (3.9) and (3.10) shows that

$$\phi'(h) = hL_P(h), \tag{3.11}$$

which completes the proof of (3.6).

Finally, note that $\psi(h)$ is also given by

$$\psi(h) = \frac{1}{2}\int_{y=0}^{h} \{x_2(y)^2 - x_1(y)^2\}dy, \tag{3.12}$$

which shows that (3.7) holds.

This completes the proof of Lemma 7. □

We, now, suppose that X satisfies Condition (D). Then for each sufficiently small $h > 0$, $V = (\alpha h, h)$. Hence, we obtain $G = \frac{3}{5}(\alpha h, h)$. Therefore we get from (3.3) that

$$\frac{3}{5}hS_P(h) = \phi(h) \tag{3.13}$$

and

$$\frac{3}{5}\alpha hS_P(h) = \psi(h). \tag{3.14}$$

It also follows from (3.13) and (3.14) that

$$\psi(h) = \alpha\phi(h). \tag{3.15}$$

By differentiating (3.13) with respect to h, (3.6) shows that

$$S_P(h) = \frac{2}{3}hL_P(h). \tag{3.16}$$

Differentiating (3.16) with respect to h yields

$$2hL'_P(h) = L_P(h). \tag{3.17}$$

Integrating (3.17) shows that

$$L_P(h) = C(P)\sqrt{h}, \tag{3.18}$$

where $C(P)$ is a constant. Hence, it follows from Lemma 5 that

$$L_P(h) = \frac{2}{\sqrt{a}}\sqrt{h}, \tag{3.19}$$

from which we get

$$x_2(h) - x_1(h) = \frac{2}{\sqrt{a}}\sqrt{h}. \tag{3.20}$$

Now, differentiating (3.15) and applying Lemma 7 show that

$$x_2(h) + x_1(h) = 2\alpha h. \tag{3.21}$$

Hence, we get from (3.20) and (3.21) that

$$x_1(h) = \alpha h - \frac{1}{\sqrt{a}}\sqrt{h}, \quad x_2(h) = \alpha h + \frac{1}{\sqrt{a}}\sqrt{h}. \tag{3.22}$$

Since $I_P(h) = (x_1(h), x_2(h))$, we obtain from (3.22) that the graph $X : y = f(x)$ is given by

$$f(x) = \begin{cases} \frac{1}{2a\alpha^2}\{2a\alpha x + 1 - \sqrt{4a\alpha x + 1}\}, & \text{if } \alpha \neq 0, \\ ax^2, & \text{if } \alpha = 0. \end{cases} \tag{3.23}$$

It follows from (3.23) that X is an open arc of the parabola defined by

$$ax^2 - 2a\alpha xy + a\alpha^2y^2 - y = 0. \tag{3.24}$$

Note that if $\alpha \neq 0$, the function $f(x)$ in (3.23) is defined on an interval J such that $J \subset (-\infty, -1/(4a\alpha))$ or $J \subset (-1/(4a\alpha), \infty)$ according to the sign of α.

Finally, we use the following coordinate change from (3.1):

$$x = u\cos\theta + v\sin\theta - b\cos\theta - c\sin\theta,$$
$$y = -u\sin\theta + v\cos\theta + b\sin\theta - c\cos\theta. \tag{3.25}$$

Then, after a long calculation we see that the curve $X : v = g(u)$ is an open arc of the parabola determined by the following quadratic polynomial

$$g(u) = \begin{cases} aw^3(u-b)^2 + \alpha(u-b) + c, & \text{if } \alpha \neq 0, \\ a(u-b)^2 + c, & \text{if } \alpha = 0. \end{cases} \tag{3.26}$$

Note that $g(b) = c, g'(b) = \alpha$ and $\kappa(P) = 2a$. This completes the proof of the if part of Theorem 3.

It is elementary to show the only if part of Theorem 3, or see Chapter 7 of [25], which is originally due to Archimedes. This completes the proof of Theorem 3.

4. Proof of Theorem 4

In this section, using Proposition 1, we give the proof of Theorem 4.

Let X be a strictly locally convex plane curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point P on X and a sufficiently small $h > 0$, we denote by l the parallel line through $P + hN(P)$ to the tangent t of the curve X at P. We let $d_P(h)$ the distance from the center G of gravity of the section of X cut off by l to the tangent t of the curve X at P.

First, suppose that X satisfies Condition (E).

For a fixed point $P \in X$, we adopt a coordinate system (x, y) of $\mathbb{R}^2$ as in the beginning of Section 2. Then the curve X is locally the graph of a non-negative strictly convex $C^{(3)}$ function $f : \mathbb{R} \to \mathbb{R}$. Hence, the Taylor's formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x), \tag{4.1}$$

where $a = f''(0)/2$, and $f_3(x)$ is an $O(|x|^3)$ function. Since $\kappa(P) = 2a > 0$, we see that a is positive.

It follows from (E) and the definition of $d_P(h)$ that

$$\frac{3}{5}hS_P(h) = \phi(h), \tag{4.2}$$

where we put

$$\phi(h) = \frac{1}{2}\int_{f(x)<h} \{h^2 - f(x)^2\}dx. \tag{4.3}$$

If we differentiate $\phi(h)$ with respect to h, then Lemma 7 shows that

$$\phi'(h) = hL_P(h). \tag{4.4}$$

By differentiating both sides of (4.2) with respect to h, we get from (4.4) and (2.2) that

$$S_P(h) = \frac{2}{3} h L_P(h), \tag{4.5}$$

which shows that the curve X satisfies Condition (C) in Proposition 1. Note that the argument in the proof of Proposition 1 given by [18] can be applied even if the curve X is a strictly locally convex plane curve. This completes the proof of the if part of Theorem 4.

For a proof of the only if part of Theorem 4, see Chapter 7 of [25], which is originally due to Archimedes. This completes the proof of Theorem 4.

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