

Extended by Balk Metrics

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ABSTRACT. Let X be a nonempty set and $\mathcal{F}(X)$ be the set of nonempty finite subsets of X . The paper deals with the extended metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ recently introduced by Peter Balk. Balk's metrics and their restriction to the family of sets A with $|A| \leq n$ make possible to consider "distance functions" with n variables and related them quantities. In particular, we study such type generalized diameters diam_{τ^n} and find conditions under which $B \mapsto \text{diam}_{\tau^n} B$ is a Balk's metric. We prove the necessary and sufficient conditions under which the restriction τ to the set of $A \in \mathcal{F}(X)$ with $|A| \leq 3$ is a symmetric G -metric. An infinitesimal analog for extended by Balk metrics is constructed.

1. Introduction

The following generalized metrics were introduced by P. Balk in 2009 for applications to some inverse geophysical problems ([2]).

Let X be a nonempty set and $\mathcal{F}(X)$ be the set of all nonempty finite subsets of X .

Definition 1.1.([3]) A function $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is an extended (by Balk) metric on X if the equivalence

$$(1.1) \quad (\tau(A) = 0) \Leftrightarrow (|A| = 1),$$

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and the equality

$$(1.2) \quad \tau(A \cup B) \leq \tau(A \cup C) + \tau(C \cup B)$$

hold for all $A, B, C \in \mathcal{F}(X)$.

Example 1.2.([3]) If ρ is a metric on X , then the function $\tau(A) = \text{diam}_\rho(A)$, with $\text{diam}_\rho(A) = \sup\{\rho(x, y) : x, y \in A\}$, is an extended by Balk metric.

If τ is an extended by Balk metric on X then, as shown in Proposition 2.1, the function $\tau^2 : X^2 \rightarrow \mathbb{R}$, with

$$(1.3) \quad \tau^2(x, y) = \tau(\text{Im}(x, y)), \quad \text{Im}(x, y) = \begin{cases} \{x\} & \text{if } x = y \\ \{x, y\} & \text{if } x \neq y, \end{cases}$$

is a metric on X . Analogously, for all integer numbers $k \geq 1$ we can define the functions $\tau^k : X^k \rightarrow \mathbb{R}$ as

$$(1.4) \quad \tau^k(x_1, \dots, x_k) = \tau(\text{Im}(x_1, \dots, x_k)),$$

where $\text{Im}(x_1, \dots, x_k)$ is the image of the set $\{1, \dots, k\}$ under the map $i \mapsto x_i$,

$$(1.5) \quad (x \in \text{Im}(x_1, \dots, x_k)) \Leftrightarrow (\exists i \in \{1, \dots, k\} : x = x_i).$$

Formula (1.4) turns to formula (1.3) when $k = 2$, thus we obtain a "generalized metric" which is a function of k variables (while the usual metric is a function of two variables).

In what follows the important role will play some "generalized diameters" generated by τ^k .

Definition 1.3. Let $X \neq \emptyset$, k be an integer positive number and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric. For every nonempty $A \subseteq X$ we set

$$\text{diam}_{\tau^k} A = \sup\{\tau^k(x_1, \dots, x_k) : x_1, \dots, x_k \in A\}$$

that is equivalent to

$$(1.6) \quad \text{diam}_{\tau^k} A = \sup\{\tau(B) : B \subseteq A, |B| \leq k\}.$$

Remark 1.4. It is clear that $\text{diam}_{\tau^k} A$ is the usual diameter of A if $k = 2$. Definition 1.1 implies $\text{diam}_{\tau^1} A = 0$ for every $A \in \mathcal{F}(X)$.

In Theorem 2.12 of the second section of the paper we obtain a structural characteristic of extended by Balk metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ for which $\tau(A) = \text{diam}_{\tau^k} A$ holds with all $A \in \mathcal{F}(X)$ and $k \geq 2$.

In the third section we study the relationship between τ^3 and the so-called G -metrics which were introduced by Zead Mustafa and Brailey Sims in 2006.

Definition 1.5.([20]) Let X be a nonempty set. A function $G : X^3 \rightarrow \mathbb{R}$ is called a G -metric if the following properties hold.

- (i) $G(x, y, z) = 0$ for $x = y = z$.
- (ii) $0 < G(x, x, y)$ for $x \neq y$.
- (iii) $G(x, x, y) \leq G(x, y, z)$ for $z \neq y$.
- (iv) $G(x_1, x_2, x_3) = G(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3})$ for every permutation σ of the set $\{1, 2, 3\}$ and every $(x_1, x_2, x_3) \in X^3$.
- (v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $a, x, y, z \in X$.

Definition 1.6. A G -metric is called symmetric if the equality $G(x, y, y) = G(y, x, x)$ holds for all $x, y \in X$.

Remark 1.7. In [20] G -metrics were defined as some functions G with the codomain $[0, \infty)$, which is slightly different from Definition 1.5. In this connection it should be pointed out that conditions (i) – (iv) of Definition 1.5 imply the nonnegativity of G . Indeed, it is sufficient to prove $G(y, x, x) > 0$ for $x \neq y$, that follows from $0 < G(x, x, y) = G(x, y, x) = G(y, x, x)$.

We shall prove that for every symmetric G -metric on X there is an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ such that $\tau^3 = G$. Conversely, an arbitrary τ^3 is a G -metric if the corresponding extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is increasing. (See Theorem 3.7).

The infinitesimal structure of spaces (X, τ) with extended by Balk metrics τ is investigated in the fourth section. In particular, we transfer the extended by Balk metrics τ from X to spaces which are pretangent to (X, τ^2) . The pretangent spaces to the general metric spaces were introduced in [12] (see also [13]). For convenience, we recall some related definitions.

Let (X, d) be a metric space and let $p \in X$. Fix a sequence \tilde{r} of positive real numbers r_n which tend to zero. The sequence \tilde{r} will be called a *normalizing sequence*. Let us denote by \tilde{X}_p the set of all sequences of points from X which tend to p .

Definition 1.8. Two sequences $\tilde{x}, \tilde{y} \in \tilde{X}_p$, $\tilde{x} = (x_n)_{n \in \mathbb{N}}$ and $\tilde{y} = (y_n)_{n \in \mathbb{N}}$ are mutually stable with respect to a normalizing sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$, if there is a finite limit

$$(1.7) \quad \lim_{n \rightarrow \infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}).$$

The family $\tilde{F} \subseteq \tilde{X}_p$ is *self-stable* with respect to \tilde{r} , if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable, \tilde{F} is *maximal self-stable* if \tilde{F} is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}_p \setminus \tilde{F}$ there is $\tilde{x} \in \tilde{F}$ such that \tilde{x} and \tilde{z} are not mutually stable. Zorn’s lemma leads to the following

Proposition 1.9. Let (X, d) be a metric space and let $p \in X$. Then for every normalizing sequence $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ there exists a maximal self-stable family $\tilde{X}_{p, \tilde{r}}$ such that $\tilde{p} = \{p, p, \dots\} \in \tilde{X}_{p, \tilde{r}}$.

Let us consider a function $\tilde{d} : \tilde{X}_{p,\tilde{r}} \times \tilde{X}_{p,\tilde{r}} \rightarrow \mathbb{R}$, where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.7). Obviously, \tilde{d} is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_{p,\tilde{r}}$. Hence $(\tilde{X}_{p,\tilde{r}}, \tilde{d})$ is a pseudometric space.

Define a relation \sim on $\tilde{X}_{p,\tilde{r}}$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0$. Let us denote by $\Omega_{p,\tilde{r}}^X$ the set of equivalence classes in $\tilde{X}_{p,\tilde{r}}$ under the equivalence relation \sim . For $\alpha, \beta \in \Omega_{p,\tilde{r}}^X$ set

$$(1.8) \quad \rho(\alpha, \beta) = \tilde{d}(\tilde{x}, \tilde{y}),$$

where $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is a metric on $\Omega_{p,\tilde{r}}^X$ (see, for example, [16, Ch. 4, Theorem 15]).

Definition 1.10. The space $(\Omega_{p,\tilde{r}}^X, \rho)$ is pretangent to the space X at the point p with respect to a normalizing sequence \tilde{r} .

Let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric, let $p \in X$ and let $(\Omega_{p,\tilde{r}}^X, \rho)$ be a pretangent space to the metric space (X, τ^2) . Now the "lifting" of τ on $(\Omega_{p,\tilde{r}}^X, \rho)$ is defined as follows. Let \mathfrak{U} be a nontrivial ultrafilter on \mathbb{N} . For $\{\alpha_1, \dots, \alpha_n\} \in \mathcal{F}(\Omega_{p,\tilde{r}}^X)$, $(x_m^1)_{m \in \mathbb{N}} \in \alpha_1, \dots, (x_m^n)_{m \in \mathbb{N}} \in \alpha_n$ set

$$\mathfrak{X}_\tau(\{\alpha_1, \dots, \alpha_n\}) = \mathfrak{U} - \lim \frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m}.$$

In Theorem 4.3 it is proved that \mathfrak{X}_τ is an extended by Balk metric on $(\Omega_{p,\tilde{r}}^X, \rho)$ and $\mathfrak{X}_\tau^2 = \rho$. Theorem 4.8 provides a characteristic of extended metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ for which the equality

$$\mathfrak{X}_\tau(A) = \text{diam}_\rho A$$

holds for every $A \in \mathcal{F}(\Omega_{p,\tilde{r}}^X)$. This result is used in Corollary 4.10 for characterization of τ for which \mathfrak{X}_τ are the extended "ultrametrics", i.e. satisfy the inequality

$$\mathfrak{X}_\tau(A \cup B) \leq \max\{\mathfrak{X}_\tau(A \cup C), \mathfrak{X}_\tau(B \cup C)\}$$

instead of inequality (1.2).

2. Extended by Balk Metrics and Generalized Diameters

Let X be a nonempty set and $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric on X . Set

$$(2.1) \quad \tau^2(x, y) := \begin{cases} \tau(\{x, y\}), & \text{if } x \neq y \\ 0, & \text{if } x=y \end{cases}$$

for every ordered pair $(x, y) \in X \times X$, where $\{x, y\}$ is the set whose elements are the points x and y .

Proposition 2.1. *The function $\tau^2 : X^2 \rightarrow \mathbb{R}$ is a metric for every nonempty set X and extended metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$.*

Proof. Obviously, the function τ^2 is symmetric and by (1.1) $\tau^2(x, y) = 0$ if and only if $x = y$. Putting in (1.2) $A = B = C$ we obtain $\tau(A) \leq 2\tau(A)$ for every $A \in \mathcal{F}(X)$ that is an equivalent to $\tau(A) \geq 0$. The last inequality implies the nonnegativity of the function τ^2 . It remains to prove the triangle inequality for τ^2 . Let x, y, z be arbitrary points from X . Putting $A = \{x\}$, $B = \{y\}$ and $C = \{z\}$ into inequality (1.2) we obtain

$$\begin{aligned} \tau^2(x, y) &= \tau(\{x\} \cup \{y\}) \leq \tau(\{x\} \cup \{z\}) + \tau(\{z\} \cup \{y\}) \\ &\leq \tau(\{x, z\}) + \tau(\{z, y\}) = \tau^2(x, z) + \tau^2(z, y). \end{aligned}$$

Thus the triangle inequality is satisfied. □

If d is a metric and τ is an extended by Balk metric on the same set X and the equality $d(x, y) = \tau^2(x, y)$ holds for all $x, y \in X$, we say that τ is *compatible* with d .

Remark 2.2. The nonnegativity of τ was earlier proved in [3].

Recall that a mapping $f : X \rightarrow Y$ from a partially ordered set (X, \leq_X) to a partially ordered set (Y, \leq_Y) is called *increasing* if the implication

$$(x \leq_X y) \Rightarrow (f(x) \leq_Y f(y))$$

holds for all $x, y \in X$.

Let us put in order the set $\mathcal{F}(X)$ by the set-theoretic inclusion \subseteq and consider \mathbb{R} with the standard order \leq . If ρ is a metric on X , then the mapping

$$\mathcal{F}(X) \ni A \mapsto \text{diam}_\rho(A) \in \mathbb{R}$$

is increasing.

Definition 2.3. Let $X \neq \emptyset$ and k be an integer number greater or equal two. A mapping $f : \mathcal{F}(X) \rightarrow \mathbb{R}$ is called *k-increasing* if the implication

$$(B \subseteq A) \Rightarrow (f(B) \leq f(A))$$

holds for $A, B \in \mathcal{F}(X)$ with $|B| \leq k$.

Remark 2.4. It is clear that every increasing mapping $f : \mathcal{F}(X) \rightarrow \mathbb{R}$ is *k-increasing* for every $k \geq 2$. It is not hard to check that, if $|X| \leq k + 1$, then all *k-increasing* mappings are increasing.

The next example shows that for $|X| \geq k + 2$ there are extended by Balk metrics on X which are *k-increasing* but not *k + 1-increasing* mappings.

Example 2.5. Let $|X| \geq k + 2$ and $t_i, i = 2, \dots, k + 2$ be some numbers from the interval $(1, 2)$ such that $t_k < t_{k+2} < t_{k+1}$ and $t_i < t_{i+1}$ for $i = 2, \dots, k$. For $A \in \mathcal{F}(X)$ set

$$(2.2) \quad \tau(A) = \begin{cases} 0, & \text{for } |A| = 1 \\ t_n, & \text{for } |A| = n, \text{ if } 2 \leq n \leq k + 1 \\ t_{k+2}, & \text{for } |A| \geq k + 2. \end{cases}$$

It follows directly from (2.2) and the restrictions to the numbers t_n that τ is k -increasing but not $k + 1$ -increasing. If $|A \cup C| \neq 1 \neq |B \cup C|$ and $\tau(A \cup B) = t_i, \tau(A \cup C) = t_j, \tau(B \cup C) = t_l$ then $t_i, t_j, t_l \in (1, 2)$. Hence $t_i \leq t_j + t_l$ that implies (1.2). Assuming, for example, the equality $1 = |B \cup C|$ we obtain the existence of $x \in X$ such that $B = C = \{x\}$. Then inequality (1.2) turns into an equality. Case $|A \cup C| = 1$ is similar.

Lemma 2.6. *The following conditions are equivalent for all $X \neq \emptyset, \tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ and integer numbers $k \geq 2$.*

- (i) *The mapping τ is a k -increasing function from $(\mathcal{F}(X), \subseteq)$ to (\mathbb{R}, \leq) .*
- (ii) *The inequality*

$$(2.3) \quad \tau(A) \geq \max\{\tau(B) : B \subseteq A, |B| \leq k\}$$

holds for every $A \in \mathcal{F}(X)$.

The proof can be obtained directly from definitions and we omit it here.

Corollary 2.7. *Let $X \neq \emptyset$ and k be an integer number greater or equal two. An extended metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is a k -increasing mapping from $(\mathcal{F}(X), \subseteq)$ to (\mathbb{R}, \leq) if and only if the inequality $\tau(A) \geq \text{diam}_{\tau^k} A$ holds for every $A \in \mathcal{F}(X)$ where $\text{diam}_{\tau^k} A$ is defined by relation (1.6).*

Let (X, \leq_X) and (Y, \leq_Y) be partially ordered sets. A mapping $f : X \rightarrow Y$ is called *decreasing* if the implication $(z \leq_X y) \Rightarrow (f(z) \geq_Y f(y))$ holds for all $z, y \in X$.

In the following definition the relation $B \subset A$ means that we have $B \subseteq A$ and $B \neq A$.

Definition 2.8. Let $X \neq \emptyset$ and $k \geq 2$ be an integer number. A mapping $f : \mathcal{F}(X) \rightarrow \mathbb{R}$ is k -weakly decreasing if for every $A \in \mathcal{F}(X)$ with $|A| > k$ there is a finite nonempty set $B \subset A$ such that $f(B) \geq f(A)$.

Lemma 2.9. *The following conditions are equivalent for all $X \neq \emptyset, k \geq 2$ and mappings $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$.*

- (i) *The mapping τ is k -weakly decreasing.*

(ii) *The inequality*

$$(2.4) \quad \tau(A) \leq \max\{\tau(B) : B \subseteq A, |B| \leq k\}$$

holds for every $A \in \mathcal{F}(X)$.

Proof. The implication (ii) \Rightarrow (i) follows directly from Definition 2.8. Let us check the implication (i) \Rightarrow (ii). Assume that condition (i) is true. Let us prove inequality (2.4) using induction by $|A|$. If $|A| = 1, \dots, k$ inequality (2.4) is obvious. Suppose that (2.4) is proved for $|A| \leq n, n \in \mathbb{N}$. Assume $|A| = n + 1 \geq k + 1$. By (i) the mapping τ is k -weakly decreasing. Therefore there is $B \subset A$ such that $\tau(A) \leq \tau(B)$. From the inclusion $B \subset A$ follows the inequality $|B| \leq n$. Using the induction hypothesis we get

$$(2.5) \quad \tau(A) \leq \tau(B) \leq \max\{\tau(C) : C \subseteq B, |C| \leq k\}.$$

Since $(C \subseteq B)$ implies $(C \subseteq A)$, we obtain

$$\max\{\tau(C) : C \subseteq B, |C| \leq k\} \leq \max\{\tau(C) : C \subseteq A, |C| \leq k\}.$$

The last inequality and (2.5) give (2.4). □

The next corollary directly follows from Definition 1.3 and Lemma 2.9.

Corollary 2.10. *Let $X \neq \emptyset$ and $k \geq 2$ be an integer number. An extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is a k -weakly decreasing mapping from $(\mathcal{F}(X), \subseteq)$ into (\mathbb{R}, \leq) if and only if the inequality $\tau(A) \leq \text{diam}_{\tau^k}(A)$ holds for every $A \in \mathcal{F}(X)$.*

Lemmas 2.6 and 2.9 give the following.

Corollary 2.11. *Let $X \neq \emptyset$, let $k \geq 2$ be an integer number and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be a k -weakly decreasing mapping. Then τ is increasing if and only if it is k -increasing.*

Proof. It is sufficient to verify that if τ is k -increasing then τ is increasing. Indeed, if τ is k -increasing, then inequalities (2.3) and (2.4) imply

$$\tau(A) = \max\{\tau(B) : B \subseteq A, |B| \leq k\}, \quad A \in \mathcal{F}(A).$$

The increase of τ follows. □

Combining corollaries 2.7, 2.10 and 2.11 we get

Theorem 2.12. *The following statements are equivalent for all nonempty X , integer $k \geq 2$ and extended metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$.*

- (i) *The equality $\tau(A) = \text{diam}_{\tau^k} A$ holds for every $A \in \mathcal{F}(X)$, where $\text{diam}_{\tau^k} A$ is determined by Definition 1.3.*
- (ii) *τ is k -increasing and k -weakly decreasing.*

(iii) τ is increasing and k -weakly decreasing.

Definition 2.13. Let ρ be a metric on X and $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric on X . We say that τ is generated by ρ if $\tau(A) = \text{diam}_\rho A$ for any $A \in \mathcal{F}(A)$.

Theorem 2.14. Let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric on a nonempty set X . The following statements are equivalent.

- (i) There is a mapping $\mu : X^2 \rightarrow \mathbb{R}$ such that $\mathcal{F}(A) = \max\{\mu(x, y) : x, y \in A\}$ for every $A \in \mathcal{F}(X)$.
- (ii) There is a metric on X which generates τ .
- (iii) τ is generated by τ^2 .
- (iv) τ is 2-increasing and 2-weakly decreasing.
- (v) τ is increasing and 2-weakly decreasing.

Proof. The implications (iii) \Rightarrow (ii) and (ii) \Rightarrow (i) are obvious. The equivalences (v) \Leftrightarrow (iv) and (iv) \Leftrightarrow (iii) follow from Theorem 2.12. It remains to note that (i) \Leftrightarrow (v) follows immediately from the definitions of increasing mapping and 2-weakly decreasing one. \square

In the next section we will prove an analog of Theorem 2.14 for the symmetric G -metrics.

3. Extended by Balk Metrics and G -metrics

The domain of τ^3 (see formula (1.4)) is the set $X^3 = X \times X \times X$. Different generalized metrics with this domain were considered at least since 60s of the last century [14, 15, 6]. The so-called G -metric (see Definition 1.5) is among the most important from these generalizations. The G -metric was introduced by Mustafa and Sims [18, 20] and has applications in the fixed point theory.

In the current section we, in particular, show that the functions $\tau^3 : X^3 \rightarrow \mathbb{R}$ generated by increasing extended by Balk metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ are symmetric (in the sense of Definition 1.6 G -metrics on X).

Lemma 3.1. Let $X \neq \emptyset$ and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an increasing extended by Balk metric. Then τ^3 is a symmetric G -metric on X .

Proof. By Definition (1.3) τ^3 is a symmetric G -metric if and only if the equality $\tau^3(x, y, y) = \tau^3(y, x, x)$ holds for all $x, y \in X$. This equality immediately follows from (1.4) and (1.5).

Let us check conditions (i)-(v) of Definition 1.5.

- (i) For every x the equality $\tau^3(x, x, x) = 0$ follows from (1.1).

- (ii) The inequality $\tau^3(x, x, y) > 0$ for $x \neq y$ follows from the equality $\tau^3(x, x, y) = \tau^2(x, y)$ and the fact that τ^2 is a metric on X (see Proposition 2.1).
- (iii) The inequality $\tau^3(x, x, y) \leq \tau^3(x, y, z)$ follows because τ is increasing.
- (iv) The arguments of the function τ on the right-hand side of equality (1.4) are sets, that automatically gives the invariance of τ^3 with respect to the permutations of arguments.
- (v) We must prove the inequality

$$(3.1) \quad \tau^3(x, y, z) \leq \tau^3(x, a, a) + \tau^3(a, y, z)$$

for all $x, y, z, a \in X$. The inequality holds if $x = y = z$ since τ^3 is a non-negative function and $\tau(x, x, x) = 0$. Now let $x \neq y \neq z \neq x$. Substituting $A = \{x\}$, $B = \{y, z\}$, $C = \{a\}$ in (1.2) we obtain

$$\tau^3(x, y, z) = \tau(A \cup B) \leq \tau(A \cup C) + \tau(B \cup C) = \tau^3(x, a, a) + \tau^3(a, y, z).$$

If $y = z$, inequality (3.1) is equivalent to the triangle inequality

$$\tau^2(x, y) \leq \tau^2(x, a) + \tau^2(a, y),$$

that was proved in Proposition 2.1. Let $x = z$. Then (3.1) can be written as

$$(3.2) \quad \tau^3(x, x, y) \leq \tau^3(x, a, a) + \tau^3(a, y, x).$$

Since τ is increasing, the inequality $\tau^3(a, y, x) \geq \tau^3(a, y, y)$ holds. Therefore, it is sufficient to check $\tau^3(x, x, y) \leq \tau^3(x, a, a) + \tau^3(a, y, y)$ which again reduces to the triangle inequality for τ^2 . It remains to consider the case where $x = y$. With this assumption (3.1) we get

$$(3.3) \quad \tau^3(x, x, z) \leq \tau^3(x, a, a) + \tau^3(a, x, z).$$

Again from the increase of τ we obtain $\tau^3(a, x, z) \geq \tau^3(a, z, z)$. Hence it suffices to prove the inequality $\tau^3(x, x, z) \leq \tau^3(x, a, a) + \tau^3(a, z, z)$, which also follows from the triangle inequality.

□

Now we want to prove the converse of Lemma 3.1. To do this it suffices for given symmetric G -metric $\Phi : X^3 \rightarrow \mathbb{R}$ to construct an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ such that

$$(3.4) \quad \Phi(x_1, x_2, x_3) = \tau(Im(x_1, x_2, x_3)), \quad (x_1, x_2, x_3) \in X^3,$$

where $Im(x_1, x_2, x_3)$ was defined by relation (1.5). We will carry out this construction in two steps.

- For given G -metric $\Phi : X^3 \rightarrow \mathbb{R}$ we first find an increasing mapping $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$, $\mathcal{F}^3(X) = \{A \in \mathcal{F}(X) : |A| \leq 3\}$, which satisfies (3.4) and (1.1), (1.2) with $\tau = \tilde{\tau}$. (This is almost what we need but the domain of $\tilde{\tau}$ is $\mathcal{F}^3(X)$).
- Second, we expand $\tilde{\tau}$ to an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$.

Lemma 3.2. *Let $X \neq \emptyset$. The following statements are equivalent for every function $G : X^3 \rightarrow \mathbb{R}$.*

- (i) *G satisfies condition (iv) of Definition 1.5 and is symmetric in the sense that*

$$(3.5) \quad G(x, y, y) = G(y, x, x)$$

for all $x, y \in X$.

- (ii) *There is a mapping $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ such that equality (3.4) holds for every $(x_1, x_2, x_3) \in X^3$ with $\Phi = G$ and $\tau = \tilde{\tau}$.*

Proof. The implication (ii) \Rightarrow (i) has already been proved in the proof of Lemma 3.1.

Let us verify the implication (i) \Rightarrow (ii). Suppose (i) holds. It is sufficient to check that the equality

$$(3.6) \quad Im(x_1, x_2, x_3) = Im(y_1, y_2, y_3)$$

implies

$$(3.7) \quad G(x_1, x_2, x_3) = G(y_1, y_2, y_3).$$

Let (3.6) hold. If $|Im(x_1, x_2, x_3)| = 1$, then there is $x \in X$ such that $x_i = x = y_i$ for every $i \in \{1, 2, 3\}$. In this case (3.7) transforms to the trivial equality $G(x, x, x) = G(x, x, x)$. If $|Im(x_1, x_2, x_3)| = 3$, then (3.7) follows from the invariance of G with respect to the permutations of arguments. For the case $|Im(x_1, x_2, x_3)| = 2$ there are $x, y \in X$ for which the triple (x_1, x_2, x_3) coincides with one of the triples $(x, x, y), (x, y, x), (y, x, x), (y, y, x), (y, x, y), (x, y, y)$. The same holds for (y_1, y_2, y_3) also. Now to prove (3.7) we can use (3.5) and the invariance of G with respect to the permutations of arguments. \square

Remark 3.3. Since the mapping $X^3 \ni (x_1, x_2, x_3) \mapsto Im(x_1, x_2, x_3) \in \mathcal{F}^3(X)$ is surjective, the existence of $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ for which the equality

$$(3.8) \quad G(x_1, x_2, x_3) = \tilde{\tau}(Im(x_1, x_2, x_3))$$

holds for every $(x_1, x_2, x_3) \in X^3$ implies the uniqueness of $\tilde{\tau}$.

Lemma 3.4. *Let $X \neq \emptyset$ and let $G : X^3 \rightarrow \mathbb{R}$ be a symmetric G -metric. Then there is an increasing mapping $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ such that: equality (3.8) holds for every $(x_1, x_2, x_3) \in X^3$; equivalence (1.1) holds with $\tilde{\tau} = \tau$ for every $A \in \mathcal{F}^3(X)$; inequality (1.2) holds with $\tilde{\tau} = \tau$ for all $A \cup B, A \cup C, B \cup C \in \mathcal{F}^3(X)$.*

Proof. The existence of $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ which satisfies (3.8) for $(x_1, x_2, x_3) \in X^3$ has already proved in Lemma 3.2. The increase of $\tilde{\tau}$ and the equivalence

$$(\tilde{\tau}(A) = 0) \Leftrightarrow (|A| = 1)$$

follow from conditions (i) – (iii) of Definition 1.5 and equality (3.8). We must prove only the inequality

$$(3.9) \quad \tilde{\tau}(A \cup B) \leq \tilde{\tau}(A \cup C) + \tilde{\tau}(B \cup C)$$

for $A \cup B, A \cup C, B \cup C \in \mathcal{F}^3(X)$.

Note that (3.9) is trivial if $|A \cup B| = 1$ because in this case $\tilde{\tau}(A \cup B) = 0$ holds. So we can suppose $|A \cup B| = 2$ or $|A \cup B| = 3$. Since $\tilde{\tau}$ is increasing, it suffices to prove (3.9) for $C = \{a\}$ where a is an arbitrary point of X .

Let $|A \cup B| = 2$. If, in addition, we have $|A| = 2$, then

$$(3.10) \quad A \cup B = A \subseteq A \cup C.$$

Hence using the nonnegativity of G (see Remark 1.7) and the increase of $\tilde{\tau}$ we obtain (3.9). If $|B| = 2$, then the proof is similar. Now let $A = \{x\}$, $B = \{y\}$ and $x \neq y$. Then

$$(3.11) \quad \begin{aligned} \tilde{\tau}(A \cup B) &= G(x, y, y), & \tilde{\tau}(A \cup C) &= G(x, a, a), \\ \tilde{\tau}(B \cup C) &= G(y, a, a) = G(a, y, y). \end{aligned}$$

Putting $z = y$ in condition (v) of Definition 1.5 we find

$$G(x, y, y) \leq G(x, a, a) + G(a, y, y).$$

This inequality and (3.11) give (3.9).

Suppose $|A \cup B| = 3$. If $\max(|A|, |B|) = 3$, then we have (3.10) or $A \cup B \subseteq B \cup C$. Hence using the increase of τ^* we get (3.9). If $|A| = 2$ and $|B| = 2$, then there are some distinct $x, y, z \in X$ for which $A = \{x, y\}$ and $B = \{y, z\}$. Consequently $\tilde{\tau}(A \cup B) = G(x, y, z)$, $\tilde{\tau}(A \cup C) = G(x, y, a)$ and $\tilde{\tau}(B \cup C) = G(y, z, a)$. Inequality (3.9) can be rewritten as

$$(3.12) \quad G(x, y, z) \leq G(x, y, a) + G(y, z, a).$$

Using condition (iii) of Definition 1.5 and the symmetry of G we obtain the inequality

$$(3.13) \quad G(x, a, a) \leq G(x, y, a)$$

for all $x, y, a \in X$. Now (3.13) and condition (v) of Definition 1.5 imply (3.12). To complete the proof it remains to consider the next alternative

$$\text{either } |A| = 1 \text{ and } |B| = 2 \quad \text{or} \quad |B| = 1 \text{ and } |A| = 2.$$

By the symmetry of the occurrences of A and B in (3.9) it suffices to consider the first case. Putting $A = \{x\}$ and $B = \{y, z\}$ and expressing $\tilde{\tau}$ via G , we get from (3.9) to the inequality $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$. Condition (v) of Definition 1.5 claims the validity of the last inequality. \square

In accordance with our plan it remains expand the function $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ to an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$. It is easy enough to do for all increasing $\tilde{\tau} : \mathcal{F}^k(X) \rightarrow \mathbb{R}$, $\mathcal{F}^k(X) = \{A \in \mathcal{F}(X) : |A| \leq k\}$ with an arbitrary integer $k \geq 2$.

For $A \in \mathcal{F}(X)$ and $\tilde{\tau} : \mathcal{F}^k(X) \rightarrow \mathbb{R}$, $k \geq 1$ we set

$$(3.14) \quad \text{diam}_{\tilde{\tau}}(A) := \max\{\tilde{\tau}(B) : B \subseteq A, B \in \mathcal{F}^k(X)\},$$

c.f. formula (1.6).

Proposition 3.5. *Let $X \neq \emptyset$, let $k \geq 2$ be an integer number and let $\tilde{\tau} : \mathcal{F}^k(X) \rightarrow \mathbb{R}$ be an increasing mapping such that the equivalence*

$$(3.15) \quad (\tilde{\tau}(A) = 0) \Leftrightarrow (|A| = 1)$$

holds for each $A \in \mathcal{F}^k(X)$ and the inequality

$$(3.16) \quad \tilde{\tau}(A \cup B) \leq \tilde{\tau}(A \cup C) + \tilde{\tau}(B \cup C)$$

holds as soon as $A \cup B, A \cup C, B \cup C \in \mathcal{F}^k(X)$. Then the function

$$(3.17) \quad \tau : \mathcal{F}(X) \rightarrow \mathbb{R}, \quad \tau(A) = \text{diam}_{\tilde{\tau}}(A), \quad A \in \mathcal{F}(X)$$

is an increasing extended by Balk metric such that

$$(3.18) \quad \tau^k(x_1, \dots, x_k) = \tilde{\tau}(\text{Im}(x_1, \dots, x_k)) \quad \text{for } (x_1, \dots, x_k) \in X^k.$$

Proof. The increase of τ follows directly from equality (3.14). This equality and the increase of $\tilde{\tau}$ give also equality (3.18). Using (3.14) it is easy to prove (3.15) for every $A \in \mathcal{F}(X)$. It remains to show that the inequality

$$(3.19) \quad \tau(A \cup B) \leq \tau(A \cup C) + \tau(B \cup C),$$

holds for all $A, B, C \in \mathcal{F}(X)$.

Let A, B and C be arbitrary elements of $\mathcal{F}(X)$. Let us choose an element $D \in \mathcal{F}^k(X)$ such that

$$(3.20) \quad D \subseteq A \cup B \text{ and } \tau(A \cup B) = \tilde{\tau}(D).$$

If $D \subseteq A$ or $D \subseteq B$, then by increase of τ we have $\tau(D) \leq \tau(A) \leq \tau(A \cup C)$ or, respectively, $\tau(D) \leq \tau(B) \leq \tau(B \cup C)$. These inequalities together with (3.20) give (3.19). Thus we can assume that

$$(3.21) \quad D \setminus A \neq \emptyset \neq D \setminus B.$$

Set $A' := A \cap D$, $B' := B \cap D$. Then we have

$$(3.22) \quad D = D \cap (A \cup B) = A' \cup B'.$$

Condition (3.21) and the inequality $|A| \leq k$ give the inequalities

$$(3.23) \quad |A'| \leq k - 1, \quad \text{and} \quad |B'| \leq k - 1.$$

Using (3.20) and (3.22) we write (3.19) in the form

$$\tau(A' \cup B') \leq \tau(A \cup C) + \tau(B \cup C).$$

Since τ is increasing, $A' \subseteq A$ and $B' \subseteq B$, it suffices to check the inequality

$$\tau(A' \cup B') \leq \tau(A' \cup C) + \tau(B' \cup C).$$

Let $C' = \{c\}$ where $c \in C$. Since τ is increasing, we have

$$\tau(A' \cup C) + \tau(B' \cup C) \geq \tau(A' \cup C') + \tau(B' \cup C').$$

Therefore it is sufficient to show that

$$(3.24) \quad \tau(A' \cup B') \leq \tau(A' \cup C') + \tau(B' \cup C').$$

To prove (3.24) note that (3.23) implies that

$$|A' \cup C'| \leq |A'| + |C'| \leq (k - 1) + 1 = k,$$

and, similarly that $|B' \cup C'| \leq k$. Thus $A' \cup C', B' \cup C' \in \mathcal{F}^k(X)$. In addition we have $A' \cup B' = D \in \mathcal{F}^k(X)$. Now using (3.18) we can rewrite (3.24) in the form $\tilde{\tau}(A' \cup B') \leq \tilde{\tau}(A' \cup C') + \tilde{\tau}(B' \cup C')$ that holds by (3.16).

Thus the function τ defined on $\mathcal{F}(X)$ by formula (3.17) has all properties of extended by Balk metric. \square

Remark 3.6. Proposition 3.5 is false for $k = 1$. In this case we have $\text{diam}_{\tilde{\tau}}(A) = 0$ for every $A \in \mathcal{F}(X)$.

Theorem 3.7. *Let $X \neq \emptyset$. The following statements are equivalent for every function $G : X^3 \rightarrow \mathbb{R}$.*

- (i) G is a symmetric G -metric in the sense of Definition 1.5.
- (ii) There is an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ such that $\tau^3 = G$.

Proof. The implication (ii) \Rightarrow (i) was obtained in Lemma 3.1. Let us prove the implication (i) \Rightarrow (ii). Suppose (i) holds. By Lemma 3.2 there is $\tilde{\tau} : \mathcal{F}^3(X) \rightarrow \mathbb{R}$ such that $G(x_1, x_2, x_3) = \tilde{\tau}(Im(x_1, x_2, x_3))$ for every $(x_1, x_2, x_3) \in X^3$. Using Lemma 3.4 we get the equivalence $(\tilde{\tau}(A) = 0) \Leftrightarrow (|A| = 1)$ for every $A \in \mathcal{F}^3(X)$

and the inequality $\tilde{\tau}(A \cup B) \leq \tilde{\tau}(A \cup C) + \tilde{\tau}(B \cup C)$ for $A \cup B, A \cup C, B \cup C \in \mathcal{F}^3(X)$. By Proposition 3.5 there is an increasing extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ such that $\tau|_{\mathcal{F}^3(X)} = \tilde{\tau}$. The implication (i) \Rightarrow (ii) is proved. \square

The next theorem is an analog of Theorem 2.14.

Theorem 3.8. *Let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric on $X \neq \emptyset$. The following statements are equivalent.*

(i) *There is a function $G : X^3 \rightarrow \mathbb{R}$ such that the equality*

$$(3.25) \quad \tau(A) = \max\{G(x, y, z) : x, y, z \in A\}$$

holds for every $A \in \mathcal{F}(X)$.

(ii) *There is a symmetric G -metric $G : X^3 \rightarrow \mathbb{R}$ such that equality (3.25) holds for every $A \in \mathcal{F}(X)$.*

(iii) *For every $A \in \mathcal{F}(X)$ the equality (3.25) holds with $G = \tau^3$.*

(iv) *τ is 3-increasing and 3-weakly decreasing.*

(v) *τ is increasing and 3-weakly decreasing.*

Proof. If (iii) holds, then τ is increasing. Then, by Lemma 3.1, τ^3 is symmetric G -metric. Therefore the implication (iii) \Rightarrow (ii) is true. The implication (ii) \Rightarrow (i) is obvious. The implication (i) \Rightarrow (v) follows directly from definitions. To complete the proof it remains to note that Theorem 2.12 gives (v) \Leftrightarrow (iv) and (iv) \Leftrightarrow (iii). \square

4. Extended Metrics on Pretangent Spaces

Let (X, d) be a metric space with a metric d . The infinitesimal geometry of the space X can be investigated by constructing of metric spaces that, in some sense, are tangent to X . If X is equipped with an additional structure, then the question arises of the lifting this structure on the tangent spaces. More specifically, let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric and let (Ω, ρ) be a tangent space to a metric space (X, d) . Suppose τ is compatible with d .

How to build an extended by Balk metric which is compatible with the metric ρ ?

The answer to this question depends on the construction of the tangent space (Ω, ρ) . Today there are several approaches to construct tangent spaces to metric spaces. Probably, the most famous of these are the Gromov-Hausdorff convergence and the ultra-convergence. The sequential approach to the construction of "pretangent" and "tangent" spaces was proposed in [12] and developed in [10, 1, 5, 4, 7, 9].

To construct Balk's extended metrics on pretangent spaces we will use ultrafilters on \mathbb{N} . Recall the necessary definitions.

Let X be a nonempty set and let $\mathfrak{B}(X)$ be the set of all its subsets. A set $\mathfrak{U} \subseteq \mathfrak{B}(X)$ is called a *filter* on X if $\emptyset \notin \mathfrak{U} \neq \emptyset$ and the implication

$$(A \in \mathfrak{U} \& B \in \mathfrak{U}) \Rightarrow (A \cap B \in \mathfrak{U}) \quad \text{and} \quad (A \supseteq B \& B \in \mathfrak{U}) \Rightarrow (A \in \mathfrak{U})$$

hold for all $A, B \subseteq X$.

A filter \mathfrak{U} on X is called an *ultrafilter* if the implication $(\mathcal{P} \supseteq \mathfrak{U}) \Rightarrow (\mathfrak{U} = \mathcal{P})$ holds for every filter \mathcal{P} on X . An ultrafilter \mathfrak{U} on X is called *trivial* if there is a point $x_0 \in X$ such that $(A \in \mathfrak{U}) \Leftrightarrow (x_0 \in A)$ for $A \in \mathfrak{B}(X)$. Otherwise \mathfrak{U} is a nontrivial ultrafilter. Let \mathfrak{U} be a filter on X . A mapping $\Phi : X \rightarrow \mathbb{R}$ converges to a point $t \in \mathbb{R}$ by the filter \mathfrak{U} , symbolically $\mathfrak{U} - \lim \Phi(x) = t$, if

$$(4.1) \quad \{x \in X : |\Phi(x) - t| < \varepsilon\} \in \mathfrak{U}$$

for every $\varepsilon > 0$.

Example 4.1. If \mathcal{M} is the Frechet filter on \mathbb{N} , (is the family of subsets of \mathbb{N} with finite complements) and $(x_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, then the limit $\lim_{n \rightarrow \infty} x_n$ exists if and only if there is $\mathcal{M} - \lim x_n$. In this case $\lim_{n \rightarrow \infty} x_n = \mathcal{M} - \lim x_n$.

We shall use the following properties of the nontrivial ultrafilters \mathfrak{U} on \mathbb{N} .

- (i₁) Every bounded sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in \mathbb{R}$, has $\mathfrak{U} - \lim x_n$;
- (i₂) If $(x_n)_{n \in \mathbb{N}}$ converges in the usual sense, then $\lim_{n \rightarrow \infty} x_n = \mathfrak{U} - \lim x_n$;
- (i₃) The relations $\mathfrak{U} - \lim cx_n = c(\mathfrak{U} - \lim x_n)$ and

$$\mathfrak{U} - \lim(x_n + y_n) = (\mathfrak{U} - \lim x_n) + (\mathfrak{U} - \lim y_n)$$

hold for every $c \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ which have \mathfrak{U} -limits;

- (i₄) If $(x_n)_{n \in \mathbb{N}}$ is \mathfrak{U} -convergent and $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$, then $(y_n)_{n \in \mathbb{N}}$ is \mathfrak{U} -convergent and $\mathfrak{U} - \lim x_n = \mathfrak{U} - \lim y_n$.

The above is a trivial modification of Problem 19 from Chapter 17 [17].

Lemma 4.2. *Let \mathfrak{U} be a nontrivial ultrafilter on \mathbb{N} . Then for every bounded sequence $(x_m)_{m \in \mathbb{N}}$, $x_m \in \mathbb{R}$ its \mathfrak{U} -limit coincides with a limit point of this sequence. Conversely, if t is a limit point of $(x_m)_{m \in \mathbb{N}}$, then there is a nontrivial ultrafilter \mathfrak{U} on \mathbb{N} such that $\mathfrak{U} - \lim x_m = t$.*

Proof. The first statement of the lemma follows from the definition of the limit points and formula (4.1) if put $X = \mathbb{N}$ and $\Phi(n) = x_n$ in this formula and take into account that all elements of nontrivial ultrafilter on \mathbb{N} are infinite subsets of \mathbb{N} .

To prove the second statement, note that for every limit point a of the sequence $(x_m)_{m \in \mathbb{N}}$ there is an infinite $A \subseteq \mathbb{N}$ such that $\lim_{\substack{m \rightarrow \infty \\ m \in A}} x_m = a$. Choose an ultrafilter \mathfrak{U} on \mathbb{N} for which $A \in \mathfrak{U}$. Now using property (i₁) we obtain $a = \mathfrak{U} - \lim x_m$. \square

Let (X, d) be a metric space, $X \neq \emptyset$, and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be a compatible with d extended by Balk metric. Let $\{\alpha_1, \dots, \alpha_n\}$ be a finite nonempty subset of pretangent space $\Omega_{p, \tilde{r}}^X$ and $\tilde{X}_{p, \tilde{r}}$ be a maximal self-stable subset of \tilde{X}_p which corresponds $\Omega_{p, \tilde{r}}^X$. Denote by π the projection $\tilde{X}_{p, \tilde{r}}$ on $\Omega_{p, \tilde{r}}^X$, i.e. if $\tilde{x} \in \tilde{X}_{p, \tilde{r}}$ then

$\pi(\tilde{x}) = \{\tilde{y} \in \tilde{X}_{p,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$. (See formula (1.7)). Choose $\tilde{x}^i = (x_m^i)_{m \in \mathbb{N}}$, $i = 1, \dots, n$ from $\tilde{X}_{p,\tilde{r}}$ such that $\pi(\tilde{x}^i) = \alpha_i$, $i = 1, \dots, n$. Put

$$(4.2) \quad \mathcal{X}_\tau(\{\alpha_1, \dots, \alpha_n\}) := \mathfrak{U} - \lim \frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m},$$

where $Im(x_m^1, \dots, x_m^n)$ is the set whose elements are the m -th terms of the sequences $(x_m^i)_{m \in \mathbb{N}}$, $i = 1, \dots, n$, r_m is m -th term of normalizing sequence \tilde{r} and \mathfrak{U} is a nontrivial ultrafilter on \mathbb{N} .

Theorem 4.3. *Let (X, d, p) be a metric space with a marked point p and $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric. If τ is compatible with d , then for every pretangent space $(\Omega_{p,\tilde{r}}^X, \rho)$ and every nontrivial ultrafilter \mathfrak{U} on \mathbb{N} the mapping*

$$\mathcal{F}(\Omega_{p,\tilde{r}}^X) \ni A \mapsto \mathcal{X}_\tau(A)$$

is correctly defined extended by Balk metric which is compatible with the metric ρ .

To prove this theorem we need the next lemma.

Lemma 4.4. *Let (X, d) be a nonempty metric space and $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric. If τ is compatible with d , then the inequalities*

$$(4.3) \quad \tau(\{x_1, x_2, \dots, x_n\}) \leq d(x_1, x_2) + \dots + d(x_{n-1}, x_n),$$

and

$$(4.4) \quad |\tau(\{x_1, \dots, x_n\}) - \tau(\{x'_1, \dots, x'_n\})| \leq \sum_{i=1}^n d(x_i, x'_i),$$

hold for every integer $n \geq 1$. Here $\{x_1, \dots, x_n\}$ and $\{x'_1, \dots, x'_n\}$ are arbitrary n -elements subsets of the set X .

Proof. Without loss of generality we can suppose that $n \geq 2$. Let $\{x_1, \dots, x_n\} \in \mathcal{F}(X)$. Using (2.1) with $B = \{x_n\}$, $C = \{x_{n-1}\}$ and $A = \{x_1, \dots, x_{n-1}\}$ we find

$$\begin{aligned} \tau(\{x_1, \dots, x_n\}) &\leq \tau(\{x_1, \dots, x_{n-1}\}) + \tau(\{x_{n-1}, x_n\}) \\ &\leq \tau(\{x_1, \dots, x_{n-1}\}) + d(x_{n-1}, x_n). \end{aligned}$$

Repeating this procedure we obtain inequality (4.3).

Let us check (4.4). To this end note that

$$\begin{aligned} &|\tau(\{x_1, \dots, x_n\}) - \tau(\{x'_1, \dots, x'_n\})| \\ &\leq |\tau(Im(x_1, x_2, \dots, x_n)) - \tau(Im(x'_1, x_2, \dots, x_n))| \\ &\quad + |\tau(Im(x'_1, x_2, \dots, x_n)) - \tau(Im(x'_1, x'_2, x_3, \dots, x_n))| \dots \\ &\quad + |\tau(Im(x'_1, x'_2, \dots, x'_{n-1}, x_n)) - \tau(Im(x'_1, \dots, x'_{n-1}, x'_n))|. \end{aligned}$$

The sets, which are the arguments of the function τ under the signs of the absolute value on the right-hand side of the last inequality, differ from each other by no more than one element. Therefore, it suffices to verify the inequality

$$(4.5) \quad |\tau(Im(x'_1, \dots, x'_{n-1}, x_n)) - \tau(Im(x'_1, \dots, x'_{n-1}, x'_n))| \leq d(x_n, x'_n).$$

Without loss of generality we can suppose that

$$(4.6) \quad \tau(Im(x'_1, \dots, x'_{n-1}, x_n)) \geq \tau(Im(x'_1, \dots, x'_{n-1}, x'_n)).$$

Using inequality (1.2) with $A = \{x'_1, \dots, x'_{n-1}\}$, $B = \{x_n\}$, $C = \{x'_n\}$ we get

$$A \cup B = Im(x'_1, \dots, x'_{n-1}, x_n), \quad A \cup C = Im(x'_1, \dots, x'_{n-1}, x'_n), \quad B \cup C = Im(x_n, x'_n)$$

and

$$\tau(Im(x'_1, \dots, x'_{n-1}, x_n)) - \tau(Im(x'_1, \dots, x'_{n-1}, x'_n)) \leq \tau(\{x_n, x'_n\}) = d(x_n, x'_n).$$

The last inequality together with (4.6) gives (4.5). □

Lemma 4.5. *Let (X, d) be a nonempty metric space, let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric and let $K \in \mathcal{F}(X)$. If τ is compatible with d , then the inequality*

$$(4.7) \quad \tau(K) \geq \frac{1}{2}d(x, y)$$

holds for all $x, y \in K$.

Proof. Let $A = \{x\}$, $B = \{y\}$ and $C = K$. Then by inequality (1.2) we have

$$d(x, y) = \tau(A \cup B) \leq \tau(A \cup K) + \tau(B \cup K) = 2\tau(K). \quad \square$$

The proof of the Theorem 4.3. Let us check the existence of the finite \mathfrak{U} -limit on the right-hand side of (4.2). According to property (i₁) of the ultrafilters it suffices to prove the inequality

$$(4.8) \quad \limsup_{m \rightarrow \infty} \frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m} < \infty.$$

Using (1.7), (1.8) and (4.3) we find

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m} &\leq \lim_{m \rightarrow \infty} \frac{d(x_m^1, x_m^2)}{r_m} \\ &+ \lim_{m \rightarrow \infty} \frac{d(x_m^2, x_m^3)}{r_m} + \dots + \lim_{m \rightarrow \infty} \frac{d(x_m^{n-1}, x_m^n)}{r_m} = \sum_{i=1}^{n-1} \rho(\alpha_i, \alpha_{i+1}). \end{aligned}$$

Inequality (4.8) follows.

Let us make sure that the value of $\mathcal{X}_\tau(\{\alpha_1, \dots, \alpha_n\})$ given in formula (4.2) does not depend of the choice of $\tilde{x}^i, i = 1, \dots, n$. Let $\tilde{y}^i = (y_m^i)$ be some elements of the set $\tilde{X}_{p, \tilde{r}}$ such that $\pi(\tilde{y}^i) = \pi(\tilde{x}^i) = \alpha_i, i = 1, \dots, n$.

By (4.4) we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{\tau(Im(x_m^1, \dots, x_m^n)) - \tau(Im(y_m^1, \dots, y_m^n))}{r_m} \\ & \leq \sum_{i=1}^n \left(\lim_{m \rightarrow \infty} \frac{d(x_m^i, y_m^i)}{r_m} \right) = \sum_{i=1}^n \rho(\alpha_i, \alpha_i) = 0. \end{aligned}$$

The wanted independence follows from (i₄).

Let us verify that $\mathcal{X}_\tau : \mathcal{F}(\Omega_{p, \tilde{r}}^X) \rightarrow \mathbb{R}$ has the characteristic properties of extended metric i.e.,

$$(4.9) \quad (\mathcal{X}_\tau(A) = 0) \Leftrightarrow (|A| = 1)$$

and

$$(4.10) \quad \mathcal{X}_\tau(A \cup B) \leq \mathcal{X}_\tau(A \cup C) + \mathcal{X}_\tau(B \cup C)$$

hold for all $A, B, C \in \mathcal{F}(\Omega_{p, \tilde{r}}^X)$.

Let $|A| = 1$. Then we have $A = \{\alpha\}$ for some $\alpha \in \Omega_{p, \tilde{r}}^X$. If $\tilde{x} = (x_m)_{m \in \mathbb{N}} \in \tilde{X}_{p, \tilde{r}}$ and $\pi(\tilde{x}) = \alpha$, then (4.2) and property (i₂) of the ultrafilters imply

$$\mathcal{X}_\tau(A) = \mathfrak{U} - \lim \frac{\tau(\{x_m\})}{r_m} = \mathfrak{U} - \lim 0 = 0.$$

Suppose now that A has at least two distinct points $\alpha_1 = \pi((x_m^1)_{m \in \mathbb{N}})$ and $\alpha_2 = \pi((x_m^2)_{m \in \mathbb{N}})$ where $(x_m^1)_{m \in \mathbb{N}}, (x_m^2)_{m \in \mathbb{N}} \in \tilde{X}_{p, \tilde{r}}$. Then inequality (4.7) implies

$$\frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m} \geq \frac{1}{2} \frac{d(x_m^1, x_m^2)}{r_m}$$

for every $m \in \mathbb{N}$. By the definition we have

$$\lim_{m \rightarrow \infty} \frac{1}{2} \frac{d(x_m^1, x_m^2)}{r_m} = \frac{1}{2} \rho(\alpha_1, \alpha_2) > 0.$$

Hence all limit points of the sequence with the common term

$$\frac{\tau(Im(x_m^1, \dots, x_m^n))}{r_m}$$

are positive. Therefore, by Lemma 4.2, we obtain the strict inequality $\mathcal{X}_\tau(A) > 0$. Equivalence (4.9) is proved.

Similarly, considering the limit points of the sequence that defines the value

$$(\mathcal{X}_\tau(A \cup B) - \mathcal{X}_\tau(A \cup C) - \mathcal{X}_\tau(B \cup C))$$

and using (1.2) and (i₃) we obtain (4.10). Thus \mathcal{X}_τ is an extended by Balk metric on $\Omega_{p,\tilde{r}}^X$.

To complete the proof it remains to check that \mathcal{X}_τ is compatible with ρ . Let $\alpha_1, \alpha_2 \in \Omega_{p,\tilde{r}}^X$ and $\alpha_1 = \pi((x_m^1)_{m \in \mathbb{N}}), \alpha_2 = \pi((x_m^2)_{m \in \mathbb{N}})$ where $(x_m^1)_{m \in \mathbb{N}}, (x_m^2)_{m \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$. Then from (1.7), (1.8), (4.2) and the fact that τ is compatible with d we find

$$\begin{aligned} \mathcal{X}_\tau(\{\alpha_1, \alpha_2\}) &= \mathfrak{U} - \lim \frac{\tau(Im(x_m^1, x_m^2))}{r_m} \\ &= \mathfrak{U} - \lim \frac{d(x_m^1, x_m^2)}{r_m} = \lim_{m \rightarrow \infty} \frac{d(x_m^1, x_m^2)}{r_m} = \rho(\alpha_1, \alpha_2), \end{aligned}$$

which is what had to be proved. □

It is rather easy to show if an extended by Balk metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is generated by a metric $d : X^2 \rightarrow \mathbb{R}$, then for all pretangent spaces $(\Omega_{p,\tilde{r}}^X, \rho)$ and nontrivial ultrafilters \mathfrak{U} the extended metrics \mathcal{X}_τ are generated by ρ . On the other hand if the space (X, d) is discrete, then every pretangent space $\Omega_{p,\tilde{r}}^X$ is single-point. Consequently \mathcal{X}_τ is generated by the metric ρ as the extended by Balk metric on the single-point space, irrespective of whether τ is generated by the metric d or not.

To describe the class of extended metrics $\tau : X \times X \rightarrow \mathbb{R}$ for which \mathcal{X}_τ is generated by ρ , we need some "infinitesimal" variant of Definition 2.13.

Let (X, d) be a metric space, $p \in X$ and $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended metric for which $\tau^2 = d$, i.e. τ is compatible with d .

Definition 4.6. The extended metric τ is generated by the metric d at the point p if for every $n \in \mathbb{N}$ and every finite set of sequences $(x_m^1)_{m \in \mathbb{N}}, \dots, (x_m^n)_{m \in \mathbb{N}}$ which converge to p with $m \rightarrow \infty$, the relation

$$(4.11) \quad |\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))| = o(\max\{d(x_m^1, p), \dots, d(x_m^n, p)\})$$

holds, where $Im(x_m^1, \dots, x_m^n)$ is defined by (1.5).

Remark 4.7. Relation (4.11) means that

$$(4.12) \quad \lim_{m \rightarrow \infty} \frac{|\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))|}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} = 0,$$

with $\lim_{m \rightarrow \infty} \frac{|\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))|}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} := 0$ for $x_m^1 = \dots = x_m^n = p$.

Theorem 4.8. Let $X \neq \emptyset, p \in X$ and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric. Then the following statements are equivalent.

- (i) For every nontrivial ultrafilter \mathfrak{U} on \mathbb{N} and every space $(\Omega_{p,\tilde{r}}^X, \rho)$ which is pretangent to the metric space (X, τ^2) at the point p , the extended metric $\mathfrak{X}_\tau : \mathcal{F}(\Omega_{p,\tilde{r}}^X) \rightarrow \mathbb{R}$ is generated by ρ .
- (ii) The extended metric τ is generated by the metric τ^2 at the point p .

Proof. For convenience write $d := \tau^2$.

First consider the case when p is an isolated point of the space (X, d) . In this case the equality $x_m = p$ holds for every sequence $(x_m)_{m \in \mathbb{N}} \in \tilde{X}_p$ if $m \in \mathbb{N}$ is sufficiently large. Using Remark 4.7 we see that τ is generated by d at the point p . For isolated p any $\Omega_{p,\tilde{r}}^X$ is a single-point space. Property (ii) holds. To prove (i) observe, that for the single-point space there is the unique extended metric which is generated by the unique metric on such space.

Let us turn now to less trivial case when p is a limit point of X . Suppose (ii) holds. Consider an arbitrary pretangent space $(\Omega_{p,\tilde{r}}^X, \rho)$, a nontrivial ultrafilter \mathfrak{U} on \mathbb{N} and the extended metric $\mathfrak{X}_\tau : \mathcal{F}(\Omega_{p,\tilde{r}}^X) \rightarrow \mathbb{R}$ which is defined by \mathfrak{U} according to (4.2).

We must prove the equality

$$(4.13) \quad \mathfrak{X}_\tau(A) = \text{diam}_\rho A$$

for arbitrary $A = \{\alpha_1, \dots, \alpha_n\} \in \mathcal{F}(\Omega_{p,\tilde{r}}^X)$.

By Theorem 4.3, \mathfrak{X}_τ is compatible with ρ . Therefore (4.13) holds for $n \leq 2$. So we can assume $n \geq 3$. Let $\tilde{X}_{p,\tilde{r}}$ be a maximal self-stable family in \tilde{X}_p corresponding to $\Omega_{p,\tilde{r}}^X$ and let $\pi : \tilde{X}_{p,\tilde{r}} \rightarrow \Omega_{p,\tilde{r}}^X$ be the projection that maps the sequences $(x_m)_{m \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ to their equivalence classes (see (1.8)). Relation (4.13) can be rewritten as

$$(4.14) \quad \mathfrak{U} - \lim \frac{\tau(\text{Im}(x_m^1, \dots, x_m^n))}{r_m} = \text{diam}_\rho A,$$

where r_m and x_m^1, \dots, x_m^n are the m -th elements of the normalizing sequence $\tilde{r} = (r_m)_{m \in \mathbb{N}}$ and, respectively, of the sequences $(x_m^1)_{m \in \mathbb{N}}, \dots, (x_m^n)_{m \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ for which $\pi((x_m^i)) = \alpha_i, i = 1, \dots, n$. The following limit relations directly follow from the definition of the metric ρ on $\Omega_{p,\tilde{r}}^X$,

$$(4.15) \quad \begin{aligned} \text{diam}_\rho A &= \lim_{m \rightarrow \infty} \frac{\text{diam}_d(\text{Im}(x_m^1, \dots, x_m^n))}{r_m}, \\ \max(\{\rho(\alpha, \alpha_1), \dots, \rho(\alpha, \alpha_n)\}) &= \lim_{m \rightarrow \infty} \frac{\max(\{d(p, x_m^1), \dots, d(p, x_m^n)\})}{r_m}, \end{aligned}$$

where $\alpha = \pi(\tilde{p}), \tilde{p} = (p, p, p \dots)$. Using properties (i₂), (i₃) of the ultrafilters, equalities (1.8) and (4.2) and the first equality from (4.15), we can rewrite (4.14) in the form

$$(4.16) \quad \mathfrak{U} - \lim \frac{\tau(\text{Im}(x_m^1, \dots, x_m^n)) - \text{diam}_d(\text{Im}(x_m^1, \dots, x_m^n))}{r_m} = 0.$$

Since $n \geq 3$, then the strict inequality

$$\max\{d(x_m^1, p), \dots, d(x_m^n, p)\} > 0$$

holds for sufficient large m . Using this inequality, (4.12) and the second equality from (4.15) we find

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))}{r_m} = \\ \lim_{m \rightarrow \infty} & \left(\frac{\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \frac{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}}{r_m} \right) \\ & = 0 \cdot \max\{\rho(\alpha, \alpha_1), \dots, \rho(\alpha, \alpha_n)\} = 0. \end{aligned}$$

Now property (i₂) of the ultrafilters implies (4.16). The implication (ii) \Rightarrow (i) is proved.

To complete the proof it remains to establish the converse implication (i) \Rightarrow (ii). Suppose that (i) is true but (ii) is false. Then there are an integer number $n \geq 3$ and sequences $(x_m^i)_{m \in \mathbb{N}} \in \tilde{X}_p, i = 1, \dots, n$ such that a limit point b of the sequence $(y_m)_{m \in \mathbb{N}}$,

$$y_m = \frac{\tau(Im(x_m^1, \dots, x_m^n)) - \text{diam}_d(Im(x_m^1, \dots, x_m^n))}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}},$$

is nonzero, $b \neq 0$. The sequence $(y_m)_{m \in \mathbb{N}}$ is bounded. Indeed, if $\text{diam}_d(Im(x_m^1, \dots, x_m^n)) = d(x_m^{i_1}, x_m^{i_2}), 1 \leq i_1, i_2 \leq n$, then

$$(4.17) \quad 0 \leq \frac{\text{diam}_d(Im(x_m^1, \dots, x_m^n))}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \leq \frac{d(x_m^{i_1}, p) + d(x_m^{i_2}, p)}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \leq 2.$$

Similarly, using (4.3) we find

$$\tau(Im(x_m^1, \dots, x_m^n)) \leq \sum_{i=1}^{n-1} \tau(Im(x_m^i, x_m^{i+1})) = \sum_{i=1}^{n-1} d(x_m^i, x_m^{i+1}),$$

that gives

$$(4.18) \quad \begin{aligned} 0 \leq & \frac{\tau(Im(x_m^1, \dots, x_m^n))}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \leq \sum_{i=1}^{n-1} \frac{d(x_m^i, x_m^{i+1})}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \\ & \leq \sum_{i=1}^{n-1} \frac{d(x_m^i, p) + d(x_m^{i+1}, p)}{\max\{d(x_m^1, p), \dots, d(x_m^n, p)\}} \leq 2(n-1). \end{aligned}$$

Inequalities (4.17) and (4.18) imply the desirable boundedness. Passing from $(y_m)_{m \in \mathbb{N}}$ to a suitable subsequence of $(y_m)_{m \in \mathbb{N}}$ it can be assumed that

$$(4.19) \quad \lim_{m \rightarrow \infty} y_m = b \quad \text{and} \quad b \notin \{0, +\infty, -\infty\}.$$

Moreover, using the conditions $(x_m^i)_{m \in \mathbb{N}} \in \tilde{X}_p$, $i = 1, \dots, n$ and passing to a subsequence again we can assume $\lim_{m \rightarrow \infty} \max\{d(x_m^1, p), \dots, d(x_m^n, p)\} = 0$ and $\max\{d(x_m^1, p), \dots, d(x_m^n, p)\} > 0$ for $m \in \mathbb{N}$.

Thus the sequence $(r_m)_{m \in \mathbb{N}}$ with $r_m = \max\{d(x_m^1, p), \dots, d(x_m^n, p)\}$ can be selected as normalizing. Using the obvious inequalities

$$\frac{d(x_m^i, x_m^j)}{r_m} \leq 2 \quad \text{and} \quad \frac{d(x_m^i, p)}{r_m} \leq 1$$

and passing to a subsequence again we can assume that the sequences $\tilde{p}, \tilde{x}_1 = (x_m^1)_{m \in \mathbb{N}}, \dots, \tilde{x}_n = (x_m^n)_{m \in \mathbb{N}}$ are mutually stable. Let $\tilde{X}_{p, \tilde{r}}$ be a maximal self-stable family for which $\tilde{x}_i \in \tilde{X}_{p, \tilde{r}}$, $i = 1, \dots, n$ and $\Omega_{p, \tilde{r}}^{\tilde{X}}$ be the corresponding pretangent space. Let \mathfrak{U} be a nontrivial ultrafilter on \mathbb{N} . Denote by α_i the image of subsequence $\tilde{x}_i = (x_m^i)_{m \in \mathbb{N}}$ under the projection of $\tilde{X}_{p, \tilde{r}}$ on $\Omega_{p, \tilde{r}}^{\tilde{X}}$, $\alpha_i = \pi(\tilde{x}_i)$. Now using properties (i₂)–(i₃) and equality (4.19) we obtain

$$\begin{aligned} b &= \mathfrak{U} - \lim \frac{\tau(\text{Im}(x_m^1, \dots, x_m^n)) - \text{diam}_d(\text{Im}(x_m^1, \dots, x_m^n))}{r_m} \\ &= \left(\mathfrak{U} - \lim \frac{\tau(\text{Im}(x_m^1, \dots, x_m^n))}{r_m} \right) - \left(\mathfrak{U} - \lim \frac{\text{diam}_d(\text{Im}(x_m^1, \dots, x_m^n))}{r_m} \right) \\ &= \mathcal{X}_\tau(\{\alpha_1, \dots, \alpha_n\}) - \text{diam}_\rho(\{\alpha_1, \dots, \alpha_n\}). \end{aligned}$$

Since $b \neq 0$ it implies the relation

$$\mathcal{X}_\tau(\{\alpha_1, \dots, \alpha_n\}) \neq \text{diam}_\rho(\{\alpha_1, \dots, \alpha_n\}),$$

contrary to (i).

The implication (i) \Rightarrow (ii) follows. □

Theorem 4.8 and some known results about pretangent spaces allow, in some cases, to get the relatively simple answer to the question about infinitesimal structure of extended metrics $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ for which the corresponding extended metrics \mathcal{X}_τ on pretangent spaces are generated by metrics with some special properties.

Recall that a metric space (X, d) is called *ultrametric* if the inequality

$$d(x, y) \leq d(x, z) \vee d(z, y)$$

holds for all $x, y, z \in X$. Here and in the sequel we set $p \vee q = \max\{p, q\}$ and $p \wedge q = \min\{p, q\}$ for all $p, q \in \mathbb{R}$.

Let (X, d) be a metric spaces with a marked point p . Let us define a function $F_d : X^3 \rightarrow \mathbb{R}$ as

$$F_d(x, y) := \begin{cases} \frac{d(x, y)(d(x, p) \wedge d(y, p))}{(d(x, p) \vee d(y, p))^2} & \text{if } (x, y) \neq (p, p) \\ 0 & \text{if } (x, y) = (p, p) \end{cases}$$

and a function $\Phi_d : X^3 \rightarrow \mathbb{R}$ as $\Phi_d(x, y, z) := F_d(x, y) \vee F_d(x, z) \vee F_d(y, z)$ for every $(x, y, z) \in X^3$. For convenience we introduce the notations: $d_1(x, y, z)$ is length of greatest side of the triangle with the sides $d(x, y)$, $d(x, z)$ and $d(y, z)$ and $d_2(x, y, z)$ is length of greatest of the two remained sides of this triangle.

Lemma 4.9.([11]) *Let (X, d) be a metric space with a marked point p . All pretangent spaces $\Omega_{p, \bar{r}}^X$ are ultrametric if and only if*

$$(4.20) \quad \lim_{x, y, z \rightarrow p} \Phi_d(x, y, z) \left(\frac{d_1(x, y, z)}{d_2(x, y, z)} - 1 \right) = 0,$$

where $\frac{d_1(x, y, z)}{d_2(x, y, z)} := 1$ for $d_2(x, y, z) = 0$.

Using Theorem 4.8 and Lemma 4.9 we get

Corollary 4.10. *Let $X \neq \emptyset$, $p \in X$ and let $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ be an extended by Balk metric. The following statements are equivalent.*

- (i) *All extended metrics $\mathcal{X}_\tau : \mathcal{F}(\Omega_{p, \bar{r}}^X) \rightarrow \mathbb{R}$ are generated by ultrametrics.*
- (ii) *The extended metric $\tau : \mathcal{F}(X) \rightarrow \mathbb{R}$ is generated by the metric τ^2 at the point p and the equality*

$$\lim_{x, y, z \rightarrow p} \Phi_{\tau^2}(x, y, z) \left(\frac{\tau_1^2(x, y, z)}{\tau_2^2(x, y, z)} - 1 \right) = 0,$$

holds with $\frac{\tau_1^2(x, y, z)}{\tau_2^2(x, y, z)} := 1$ for $x = y = z = p$.

Remark 4.11. An extended metric $\mathcal{X}_\tau : \mathcal{F}(\Omega_{p, \bar{r}}^X) \rightarrow \mathbb{R}$ is generated by an ultrametric if and only if the inequality

$$\mathcal{X}_\tau(A \cup B) \leq \mathcal{X}_\tau(A \cup C) \vee \mathcal{X}_\tau(B \cup C)$$

holds for all $A, B, C \in \mathcal{F}(\Omega_{p, \bar{r}}^X)$. (See Theorem 2.1 in [8]).

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