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Reduction Formulas for Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$

Junesang Choi*
Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea
e-mail : junesang@mail.dongguk.ac.kr

Xiaoxia Wang
Department of Mathematics, Shanghai University, Shanghai, 200444, P. R. China
e-mail : xiaoxiawang@shu.edu.cn

Arjun K. Rathie
Department of Mathematics, School of Mathematical & Physical Sciences, Central University of Kerala, Riverside Transit Campus, Padennakad P.O. Nileshwar, Kasaragod-671 328, India
e-mail : akrathie@gmail.com

Abstract. Very recently the authors have obtained a very interesting reduction formula for the Srivastava's triple hypergeometric series $F^{(3)}(x, y, z)$ by applying the so-called Beta integral method to the Henrici's triple product formula for the hypergeometric series. In this sequel, we also present three more interesting reduction formulas for the function $F^{(3)}(x, y, z)$ by using the well known identities due to Bailey and Ramanujan. The results established here are simple, easily derived and (potentially) useful.

1. Introduction, Preliminaries and Main Result

The generalized hypergeometric series ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by (see, *e.g.*, [3, Chapter II]; see also [12, p.73]

* Corresponding Author.
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and [15, p.71–72]):

(1.1)

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\,\frac{z^n}{n!} = {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z). $$

Here $(\lambda)_\nu$ is the Pochhammer symbol defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

(1.2)

$$ (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} $$

it being understood conventionally that $(0)_0 := 1$, and $\mathbb{C}$ and $\mathbb{N}$ being the sets of complex numbers and positive integers, respectively. For convergence conditions and other details of ${}_pF_q$, we refer to the above-cited books.

It is noted here that whenever the hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ reduce to be expressed in terms of Gamma functions, the results are important in view of applications as well as themselves. Thus the well known classical summation theorems such as those of Gauss, Gauss second , Kummer and Bailey for the series ${}_2F_1$, and those of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, and others have played important roles.

Moreover, it is well known that, if the product of two generalized hypergeometric series can be expressed as a generalized hypergeometric series with argument x, the coefficient of x^n in the product should be expressed in terms of Gamma functions. Following this technique and using above mentioned classical summation theorems, in a well known, popular and very interesting paper [2], Bailey derived a large number of new as well as known results involving products of generalized hypergeometric series. Here, for our present investigation, we choose to recall some of those results in [2]:

$$ e^{-x}\,{}_1F_1(\alpha\,;\,\rho\,;x) = {}_1F_1(\rho-\alpha\,;\,\rho\,;-x). \tag{1.3} $$

$$ {}_1F_1(\alpha\,;\,2\alpha\,;x)\,{}_1F_1(\beta\,;\,2\beta\,;-x) = {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1)\,; \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta\,;\end{matrix}\; \frac{x^2}{4}\right]. \tag{1.4} $$

$$ {}_1F_1(\alpha\,;\,\rho\,;x)\,{}_1F_1(\alpha\,;\,\rho\,;-x) = {}_2F_3\left[\begin{matrix}\alpha,\ \rho-\alpha\,; \\ \rho,\ \frac{\rho}{2},\ \frac{\rho}{2}+\frac{1}{2}\,;\end{matrix}\; \frac{x^2}{4}\right]. \tag{1.5} $$

$$
\begin{aligned}
(1.6)\qquad & {}_1F_1\left[\begin{matrix}\alpha\,; \\ \rho\,;\end{matrix}\; x\right] {}_1F_1\left[\begin{matrix}\alpha-\rho+1\,; \\ 2-\rho\,;\end{matrix}\; -x\right] = {}_2F_3\left[\begin{matrix}\alpha-\frac{\rho}{2}+\frac{1}{2},\, \frac{\rho}{2}-\alpha+\frac{1}{2}\,; \\ \frac{1}{2},\, \frac{\rho}{2}+\frac{1}{2},\, \frac{3}{2}-\frac{\rho}{2}\,;\end{matrix}\; \frac{x^2}{4}\right] \\
& + \frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\, x\, {}_2F_3\left[\begin{matrix}\alpha-\frac{\rho}{2}+1,\, \frac{\rho}{2}-\alpha+1\,; \\ \frac{3}{2},\, \frac{\rho}{2}+1,\, 2-\frac{\rho}{2}\,;\end{matrix}\; \frac{x^2}{4}\right].
\end{aligned}
$$

It is interesting to note here that if we use the result (1.3) in (1.4), (1.5) and (1.6), we get, respectively, the following alternative forms:

$$
(1.7)\qquad e^{-x}\, {}_1F_1(\alpha\,;\, 2\alpha\,; x)\, {}_1F_1(\beta\,;\, 2\beta\,; x) = {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1)\,; \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta\,;\end{matrix}\; \frac{x^2}{4}\right].
$$

$$
(1.8)\qquad e^{-x}\, {}_1F_1(\alpha\,;\, \rho\,; x)\, {}_1F_1(\rho-\alpha\,;\, \rho\,; x) = {}_2F_3\left[\begin{matrix}\alpha,\, \rho-\alpha\,; \\ \rho,\, \frac{\rho}{2},\, \frac{\rho}{2}+\frac{1}{2}\,;\end{matrix}\; \frac{x^2}{4}\right].
$$

$$
\begin{aligned}
(1.9)\qquad & e^{-x}\, {}_1F_1\left[\begin{matrix}\alpha\,; \\ \rho\,;\end{matrix}\; x\right] {}_1F_1\left[\begin{matrix}1-\alpha\,; \\ 2-\rho\,;\end{matrix}\; x\right] = {}_2F_3\left[\begin{matrix}\alpha-\frac{\rho}{2}+\frac{1}{2},\, \frac{\rho}{2}-\alpha+\frac{1}{2}\,; \\ \frac{1}{2},\, \frac{\rho}{2}+\frac{1}{2},\, \frac{3}{2}-\frac{\rho}{2}\,;\end{matrix}\; \frac{x^2}{4}\right] \\
& + \frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\, x\, {}_2F_3\left[\begin{matrix}\alpha-\frac{\rho}{2}+1,\, \frac{\rho}{2}-\alpha+1\,; \\ \frac{3}{2},\, \frac{\rho}{2}+1,\, 2-\frac{\rho}{2}\,;\end{matrix}\; \frac{x^2}{4}\right].
\end{aligned}
$$

In 1987, Henrici [8] gave the following elegant result for a product of three generalized hypergeometric functions:

$$
\begin{aligned}
(1.10)\qquad & {}_0F_1\left[\begin{matrix}\text{—}; \\ 6c;\end{matrix}\; x\right] {}_0F_1\left[\begin{matrix}\text{—}; \\ 6c;\end{matrix}\; \omega x\right] {}_0F_1\left[\begin{matrix}\text{—}; \\ 6c;\end{matrix}\; \omega^2 x\right] \\
& = {}_2F_7\left[\begin{matrix}3c-\frac{1}{4},\; 3c+\frac{1}{4}; \\ 6c,\, 2c,\, 2c+\frac{1}{3},\, 2c+\frac{2}{3},\, 4c-\frac{1}{3},\, 4c,\, 4c+\frac{1}{3};\end{matrix}\; \left(\frac{4x}{9}\right)^3\right],
\end{aligned}
$$

where $\omega = \exp(\frac{2\pi i}{3})$.

It is interesting to mention here that by making use of certain known transformations in the theory of generalized hypergeometric functions, in 1990, Karlsson and Srivastava [9] established a general triple series identity which readily yields the Henrici's identity (1.10).

On the other hand, just as the Gauss function ${}_2F_1$ was extended to ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator, the four Appell functions were introduced and generalized by Appell and Kampé de Fériet [1] who defined a general hypergeometric function in two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [18, p. 423, Eq. (26)]. For this, let (H_h) denote the sequence of parameters $(H_1, H_2, \ldots, H_h)$ and for nonnegative integers define the Pochhammer symbols $((H_h))_n = (H_1)_n\,(H_2)_n \cdots (H_h)_n$, where, when $n = 0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as follows:

$$
\begin{aligned}
&F^{h:a;b}_{g:c;d}\left[\begin{matrix} (H_h): & (A_a); & (B_b); \\ (G_g): & (C_c); & (D_d); \end{matrix}\; x,\, y\right] \\
&\qquad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((H_h))_{m+n}\,((A_a))_m\,((B_b))_n}{((G_g))_{m+n}\,((C_c))_m\,((D_d))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}.
\end{aligned}
\tag{1.11}
$$

The symbol (H) is a convenient contraction for the sequence of the parameters $H_1, H_2, \ldots, H_h$ and the Pochhammer symbol $(H)_n$ is defined by in (1.2). For more details about the convergence for this function, we refer to [17].

Later on, a unification of Lauricella's 14 triple hypergeometric series $F_1, \ldots, F_{14}$ [16] and the additional three triple hypergeometric series H_A, H_B and H_C was introduced by Srivastava [14] who defined the following general triple hypergeometric series $F^{(3)}[x, y, z]$ (see, e.g., [16, p. 44, Eqs. (14) and (15)]):

$$
\begin{aligned}
&F^{(3)}[x,\, y,\, z] \equiv F^{(3)}\left[\begin{matrix} (a) :: & (b); & (b'); & (b''): & (c); & (c'); & (c''); \\ (e) :: & (g); & (g'); & (g''): & (h); & (h'); & (h''); \end{matrix}\; x,\, y,\, z\right] \\
&\qquad = \sum_{m,\,n,\,p=0}^{\infty} \Lambda(m,\, n,\, p)\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},
\end{aligned}
\tag{1.12}
$$

where, for convenience,

$$
\begin{aligned}
\Lambda(m,\, n,\, p) = &\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\,\prod_{j=1}^{B}(b_j)_{m+n}\,\prod_{j=1}^{B'}(b'_j)_{n+p}\,\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\,\prod_{j=1}^{G}(g_j)_{m+n}\,\prod_{j=1}^{G'}(g'_j)_{n+p}\,\prod_{j=1}^{G''}(g''_j)_{p+m}} \\
&\cdot \frac{\prod_{j=1}^{C}(c_j)_m\,\prod_{j=1}^{C'}(c'_j)_n\,\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\,\prod_{j=1}^{H'}(h'_j)_n\,\prod_{j=1}^{H''}(h''_j)_p},
\end{aligned}
\tag{1.13}
$$

and (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b''), and so on.

Very recently, Choi *et al.* [7] have obtained the following very interesting reduction formula for the Srivastava's triple hypergeometric series $F^{(3)}(x, y, z)$ by applying the so-called Beta integral method (see [10]; see also [6]) to the Henrici's triple product formula (1.10):

(1.14)

$$F^{(3)}\left[\begin{array}{l} e :: \ -;\ -;\ -:\ -;\ -;\ -; \\ d :: \ -;\ -;\ -:\ 6c;\ 6c;\ 6c; \end{array} 1,\, \omega,\, \omega^2\right]$$
$$= {}_5F_{10}\left[\begin{array}{r} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \end{array} \left(\frac{4}{9}\right)^3\right],$$

where $\omega = \exp(\frac{2\pi i}{3})$.

In this sequel, motivated essentially by (1.14), we establish three more reduction formulas for the function $F^{(3)}(x,\, y,\, z)$ by using the identities (1.7), (1.8) and (1.9) due to Bailey and Ramanujan asserted by the following theorem.

Theorem. *Each of the following reduction formulas holds true.*

(1.15)

$$F^{(3)}\left[\begin{array}{l} d :: \ -;\ -;\ -:\ -;\ \alpha;\ \beta; \\ e :: \ -;\ -;\ -:\ -;\ 2\alpha;\ 2\beta; \end{array} -1,\, 1,\, 1\right]$$
$$= {}_4F_5\left[\begin{array}{r} \frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1),\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{1}{4}\right].$$

(1.16)

$$F^{(3)}\left[\begin{array}{l} d :: \ -;\ -;\ -:\ -;\ \alpha;\ \rho-\alpha; \\ e :: \ -;\ -;\ -:\ -;\ \rho;\ \rho; \end{array} -1,\, 1,\, 1\right]$$
$$= {}_4F_5\left[\begin{array}{r} \alpha,\ \rho-\alpha,\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \rho,\ \frac{\rho}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{1}{4}\right].$$

(1.17)

$$F^{(3)}\left[\begin{array}{l} d :: \ -;\ -;\ -:\ -;\ \alpha;\ 1-\alpha; \\ e :: \ -;\ -;\ -:\ -;\ \rho;\ 2-\rho; \end{array} -1,\, 1,\, 1\right]$$
$$= {}_4F_5\left[\begin{array}{r} \alpha-\frac{\rho}{2}+\frac{1}{2},\ \frac{\rho}{2}-\alpha+\frac{1}{2},\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \frac{1}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{1}{4}\right]$$
$$+ \frac{d\,(2\alpha-\rho)(1-\rho)}{e\,\rho(2-\rho)}\,{}_4F_5\left[\begin{array}{r} \alpha-\frac{\rho}{2}+1,\ \frac{\rho}{2}-\alpha+1,\ \frac{d}{2}+\frac{1}{2},\ \frac{d}{2}+1; \\ \frac{3}{2},\ \frac{\rho}{2}+1,\ 2-\frac{\rho}{2},\ \frac{e}{2}+\frac{1}{2},\ \frac{e}{2}+1; \end{array} \frac{1}{4}\right].$$

2. Outline of Proof of Theorem

The proofs of the above results (1.15) to (1.17) are quite straightforward. In order to prove (1.15), we first replace e^{-x} by ${}_0F_0[\text{—};\text{—};\, -x]$ in (1.7). Then multiply each side of the resulting identity by $x^{d-1}\,(1-x)^{e-d-1}$ and expand the involved generalized hypergeometric functions as series. Now integrate both sides of the present resulting identity with respect to x between 0 to 1 and then change the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series. The integrations are easily evaluated to be expressed in terms of Gamma functions Γ by just recalling the well known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function (see, *e.g.*, [15, p. 8, Eq. (42)]). After some simplification, the left-hand side of the last resulting identity becomes

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)} \sum_{m,n,p=0}^{\infty} \frac{(d)_{m+n+p}}{(e)_{m+n+p}} \frac{(-1)^m}{m!} \frac{(\alpha)_n}{(2\alpha)_n\, n!} \frac{(\beta)_p}{(2\beta)_p\, p!},$$

which, except for the Gamma fraction in front of the triple summations, in view of (1.12), is easily seen to correspond with the left-hand side of (1.15).

On the other hand, applying the Legendre's duplication formula for the Gamma function (see, *e.g.*, [15, p. 6, Eq. (29)]):

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \quad \left(z \neq 0,\, -\frac{1}{2},\, -1,\, -\frac{3}{2},\, \cdots\right) \tag{2.1}$$

to the right-hand side of the above last resulting identity, we obtain

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}(\alpha+\beta))_n\,(\frac{1}{2}(\alpha+\beta+1))_n\,(\frac{d}{2})_n\,(\frac{d}{2}+\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n\,(\beta+\frac{1}{2})_n\,(\alpha+\beta)_n\,(\frac{e}{2})_n\,(\frac{e}{2}+\frac{1}{2})_n} \frac{1}{4^n\, n!},$$

which, except for the head-located Gamma fraction, is easily seen to correspond with the right-hand side of (1.15). This completes the proof of (1.15). A similar argument will establish the results (1.16) and (1.17).

We conclude this section by mentioning some special cases of our main results. The special case of (1.15) when $\beta = \alpha$ is equal to that of (1.16) when $\rho = 2\,\alpha$. The special case of (1.17) when $\rho = 2\alpha$ is also equal to that of (1.15) when $\beta = 1-\alpha$.

3. Further Observations

On the other hand, if we apply beta integral method to Equations (1.4) to (1.6), we, after some simplification, obtain the following transformation formulas between Kampé de Fériet functions and generalized hypergeometric functions:

$$\begin{aligned} &F^{1:1;1}_{1:1;1}\left[\begin{array}{lll} d: & \alpha; & \beta; \\ e: & 2\alpha; & 2\beta; \end{array} 1,\, -1\right] \\ &\quad = {}_4F_5\left[\begin{array}{c} \frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1),\, \frac{1}{2}d,\, \frac{1}{2}d+\frac{1}{2}\,; \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta,\, \frac{1}{2}e,\, \frac{1}{2}e+\frac{1}{2}\,; \end{array} \frac{1}{4}\right]. \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \alpha; \\ e: & \rho; & \rho; \end{matrix}\ 1,\,-1\right] \\
(3.2)\qquad &= {}_4F_5\left[\begin{matrix} \alpha,\ e-\alpha,\ \frac{1}{2}d,\ \frac{1}{2}d+\frac{1}{2}\,; \\ \rho,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}e,\ \frac{1}{2}e+\frac{1}{2}\,; \end{matrix}\ \frac{1}{4}\right].
\end{aligned}
$$

$$
\begin{aligned}
&F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \alpha-e+1\,; \\ e: & \rho; & 2-\rho\,; \end{matrix}\ 1,\,-1\right] \\
(3.3)\qquad &= {}_4F_5\left[\begin{matrix} \alpha-\frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho-\alpha+\frac{1}{2},\ \frac{1}{2}d,\ \frac{1}{2}d+\frac{1}{2}\,; \\ \frac{1}{2},\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{3}{2}-\frac{1}{2}\rho,\ \frac{1}{2}e,\ \frac{1}{2}e+\frac{1}{2}\,; \end{matrix}\ \frac{1}{4}\right] \\
&\quad + \frac{d(2\alpha-\rho)(1-\rho)}{e\rho(2-\rho)}\,F_5\left[\begin{matrix} \alpha-\frac{1}{2}\rho+1,\ \frac{1}{2}\rho-\alpha+1,\ \frac{1}{2}d+\frac{1}{2},\ \frac{1}{2}d+1\,; \\ \frac{3}{2},\ \frac{1}{2}\rho+1,\ 2-\frac{1}{2}\rho,\ \frac{1}{2}e+\frac{1}{2},\ \frac{1}{2}e+1\,; \end{matrix}\ \frac{1}{4}\right].
\end{aligned}
$$

It is noted that the results (3.1) and (3.2) are seen to be special cases when $p = q = 1$ of the more general results [17, p. 31, Eqs.(47) and (46)], while (3.3) is a special case of a more general result in [11, p. 21, Eq. (2.6)].

Finally, comparing Equations (1.15), (1.16) and (1.17) with Equations (3.1), (3.2) and (3.3), respectively, we get the following transformation formulas between Srivastava's triple hypergeometric series $F^{(3)}(x,\,y,\,z)$ and Kampé de Fériet double series:

$$
\begin{aligned}
(3.4)\qquad &F^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & \beta; \\ e:: & \text{—}; & \text{—}; & \text{—}; & 2\alpha; & 2\beta; \end{matrix}\ -1,\,1,\,1\right] \\
&= F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha\,; & \beta\,; \\ e: & 2\,\alpha\,; & 2\,\beta\,; \end{matrix}\ 1,\,-1\right].
\end{aligned}
$$

$$
\begin{aligned}
(3.5)\qquad &F^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & \rho-\alpha; \\ e:: & \text{—}; & \text{—}; & \text{—}; & \rho; & \rho; \end{matrix}\ -1,\,1,\,1\right] \\
&= F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha\,; & \alpha\,; \\ e: & \rho\,; & \rho\,; \end{matrix}\ 1,\,-1\right].
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\qquad &F^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & 1-\alpha; \\ e:: & \text{—}; & \text{—}; & \text{—}; & \rho; & 2-\rho; \end{matrix}\ -1,\,1,\,1\right] \\
&= F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha\,; & \alpha-\rho+1\,; \\ e: & \rho\,; & 2-\rho\,; \end{matrix}\ 1,\,-1\right].
\end{aligned}
$$

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References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergeometriques et Hyperspheriques; Polynomes d'Hermite*, Gauthier - Villars, Paris, 1926.

[2] W. N. Bailey, *Products of generalized hypergeometric series*, Proc. London Math. Soc., ser 2, **28**(1928), 242–254.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, **32**, Cambridge University Press, Cambridge, London and New York, 1935; Reprinted by Stechert-Hafner Service Agency, New York and London, 1964.

[4] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions II*, Quart. J. Math. (Oxford ser.), **11**(1940), 249–270.

[5] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math. (Oxford ser.), **12**(1941), 112–128.

[6] J. Choi, A. K. Rathie and H. M. Srivastava, *Certain hypergeometric identities deducible by using the Beta integral method*, Bull. Korean Math. Soc., **50**(2013), 1673–1681.

[7] J. Choi, X. Wang and A. K. Rathie, *A reducibility of Srivastava's triple hypergeometric series* $F^{(3)}[x, y, z]$, Commun. Korean Math. Soc., **28(2)**(2013), 297–301.

[8] P. Henrici, *A triple product theorem for hypergeometric series*, SIAM J. Math. Anal., **18**(1987), 1513–1518.

[9] P. W. Karlsson and H. M. Srivastava, *A note of Henrici's triple product theorem*, Proc. Amer. Math. Soc., **110**(1990), 85–88.

[10] C. Krattenthaler and K. S. Rao, *Automatic generation of hypergeometric identities by the beta integral method*, J. Comput. Appl. Math., **160**(2003), 159–173.

[11] M. A. Pathan, M. I. Qureshi and F. V. Khan, *Reducibility of Kampé de Fériet's hypergeometric series of higher order*, Serdica, **2**(1985), 20–24.

[12] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[13] S. Ramanujan, *Notebooks*, **1** and **2**, TIFR, Bombay, 1957.

[14] H. M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc., **63**(1967), 425–429.

[15] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[16] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[17] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Chichester, Brisbane, and Toronto, 1984.

[18] H. M. Srivastava and R. Panda, *An integral representation for the product of two Jacobi polynomials*, J. London Math. Soc., **12(2)**(1976), 419–425.