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# Uniqueness of Meromorphic Functions Concerning the Difference Polynomials

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Abstract. In this article, we main study the uniqueness problem of meromorphic function which difference polynomials sharing common values. We consider the entire function $(f^n(f^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and the meromorphic function $f^n(f^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ to get the main results which extend Theorem 1.1 in paper[5] and theorem 1.4 in paper[6].

## 1. Introduction

In this paper, we assume that the reader is familiar with the fundamental results. We adopt the standard notations of the Nevanlinna theory of meromorphic function such as $m(r,f), N(r,f), \overline{N}(r,f)$ and $T(r,f)$ as explained in [1]. In addition, We set $N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}_{(2}(r,\frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r,\frac{1}{f-a})$, where $N_{(k}(r,\frac{1}{f-a})$ be the counting function for the zeros of $f-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}(r,\frac{1}{f-a})$ be the corresponding one for which the multiplicity is not counted. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane and let $a$ be a value in the extended plane. We say that $f(z)$ and $g(z)$ share the value $a$ CM, provided that $f(z)$ and $g(z)$ have the same $a$-points with the same multiplicities. We say that two meromorphic functions $f(z)$ and $g(z)$ share a finite value $a$ IM(ignoring multiplicities) when $f(z)-a$ and $g(z)-a$ have the same zeros. By $S(r,f)$, we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ and $r \notin E$, where $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We say that $\alpha(z)$ is a small function of $f(z)$, if $\alpha(z)$ is a meromorphic function satisfying $T(r,\alpha(z)) = S(r,f)$.

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Recently,the topic of a difference equation and a difference product in the complex plane C has attracted many mathematicians, a number of papers have focused on value distribution and uniqueness of differences and differences operator analogues of Nevanlinna theory (including [2,3,4]). In this paper, our aim is to investigate the uniqueness problems of difference monomials of meromorphic functions. We recall the uniqueness theorems due to Keyu Zhang,Hongxun Yi in paper[5] and Xiaomin Li,Hongxun Yi in paper[6].

**Theorem A.**([5]) *Let $f(z), g(z)$ be two transcendental entire functions with finite order. $\alpha(z)$ is a small function with respect to $f(z)$ and $g(z)$. $c_j (j = 1, 2, \ldots, s)$ are distinct complex constants, $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are non-negative integers, $\sigma = \sum_{j=1}^{s} \mu_j$. If $(f^n(f^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)} = \alpha(z) \rightleftharpoons (g^n(g^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)} = \alpha(z)$, and $n \geq 2k + m + \sigma + 5$, Then $f(z) = tg(z)$, where $t^m = 1$.*

The above theorem is discussed for values shared CM, next we will consider this case: If we change the condition of theorem A "$\alpha(z)$ is shared by $(f^n(f^m - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(g^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ $CM$" to "$\alpha(z)$ is shared by $(f^n(f^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(g^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ $IM$", Can we get the same conclusion? That's true. Theorem is given below in detail.

**Theorem 1.** *Let $f(z), g(z)$ be transcendental entire functions with finite order. $\alpha(z)$ is a small function with respect to $f(z)$ and $g(z)$. $c_j (j = 1, 2, \ldots, s)$ are distinct complex constants,$m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are non-negative integers,$\sigma = \sum_{j=1}^{s} \mu_j$. If $(f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)} = \alpha(z) \Longleftrightarrow (g^n(g^m - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j})^{(k)} = \alpha(z)$, and $n \geq 5k + 4m + 4\sigma + 8$, We have $f(z) = tg(z)$, where $t^m = 1$.*

Considering the meromorphic functions,$X.M, Li$ and $H.X, Yi$ get the following results in paper[6].

**Theorem B.**([6]) *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite orders. $\infty$ is shared by $f(z)$ and $g(z)$ $IM$ . $\alpha(z)$ is a nonzero entire function,satisfying $\rho(\alpha) < \rho(f)$. $c$ is a nonzero complex number. $m(\geq 2), n$ are positive integers. Let $F(z) = f^n(f^m - 1)f(z + c), G(z) = g^n(g^m - 1)g(z + c)$. $0, \infty$ is shared by $F(z) - \alpha(z), G(z) - \alpha(z)$ $CM$. If $n \geq m + 12$ then $f(z) = tg(z)$, where $t$ is a constant satisfying $t^m = 1$.*

We improve the above theorem to get the following conclusion.

**Theorem 2.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying $\rho(f) < \infty, \rho(g) < \infty$. $f(z)$ and $g(z)$ share $\infty$ $IM$. $\alpha(z) \not\equiv 0$ is an entire function satisfying $\rho(\alpha) < \rho(f)$. $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are non-negative integers, $\sigma = \sum_{j=1}^{s} \mu_j$. $c_j (j = 1, 2, \ldots, s)$ are non-zero complex constants. $F(z) = f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}, G(z) = g^n(g^m - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$. $F(z) - \alpha(z), G(z) - \alpha(z)$ share $0, \infty$ $CM$. If $n \geq m + 2s + 3\sigma + 7$ we get $f(z) = tg(z)$, where $t$ is a constant satisfying $t^m = 1$.*

## 2. Preliminary Lemmas

**Lemma 1.**([7]) *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying $f(z) = 1 \Longleftrightarrow g(z) = 1$. We set $h = \frac{f''}{f'} - 2\frac{f'}{f-1} - (\frac{g''}{g'} - 2\frac{g'}{g-1})$. If $h \neq 0$, we have*

$$\begin{aligned} T(r,f) + T(r,g) \leq & 2[N_2(r,\frac{1}{f}) + N_2(r,f) + N_2(r,\frac{1}{g}) + N_2(r,g)] \\ & + 3[\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{g})] + S(r,f) + S(r,g) \end{aligned}$$

**Lemma 2.**([8]) *Let $f(z)$ be nonconstant meromorphic function and $p, k$ be positive integers. Then*

$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f).$$

**Lemma 3.**([9]) *Let $f(z)$ be nonconstant meromorphic function and $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ be non-negative integers, $\sigma = \sum_{j=1}^{s} \mu_j$. $c_j (j = 1, 2, \ldots, s)$ are nonzero complex constants. Set $F = f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}$. We have $T(r,F) = (m + n + \sigma)T(r,f) + S(r,f)$.*

**Lemma 4.**([5]) *Let $f(z)$ be nonconstant meromorphic function and $k, m, n, s, \mu_j (j = 1, 2, \ldots, s)$ be non-negative integers, $\sigma = \sum_{j=1}^{s} \mu_j$. $c_j (j = 1, 2, \ldots, s)$ are nonzero distinct complex constants. Set $F = (f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)}$. Then we have $T(r,F) = (m + n + \sigma)T(r,f) + S(r,f)$.*

**Lemma 5.**([5]) *Let $f(z), g(z)$ be nonconstant entire functions and $k, m, n$ be positive integers. $F(z) = (f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)}, G(z) = (g^n(g^m - 1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$. If there are nonzero constants $a_1, a_2$ such that $\overline{N}(r,\frac{1}{F-a_1}) = \overline{N}(r,\frac{1}{G})$, $\overline{N}(r,\frac{1}{G-a_2}) = \overline{N}(r,\frac{1}{F})$, then we get $n \leq 2k + m + \sigma + 2$.*

**Lemma 6.**([9]) *Let $f(z), g(z)$ be nonconstant entire functions with finite order and $c_j (j = 1, 2, \ldots, s)$ be finite complex constants. $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are integers. If $n \geq m + 5\sigma$ and $f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(g^m - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$. then $f(z) = tg(z)$, where $t^m = t^{n+\sigma} = 1$.*

**Lemma 7.** *Let $f(z), g(z)$ be nonconstant transcendental meromorphic functions satisfying $\rho(f) < \infty, \rho(g) < \infty$. $c_j (j = 1, 2, \ldots, s)$ are nonzero distinct complex constants. $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are positive integers. $\sigma = \sum_{j=1}^{s} \mu_j$. If $n \geq m + 2s + 3\sigma + 7$, $f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(g^m - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$. We get $f(z) = tg(z)$, where $t^m = t^{n+\sigma} = 1$.*

**Lemma 8.** *Let $f(z)$ be nonconstant meromorphic function satisfying $\rho(f) < \infty$ and $c$ be nonzero complex constant. $m, n, s, \mu_j (j = 1, 2, \ldots, s), \sigma = \sum_{j=1}^{s} \mu_j$ are non-negative integers. $c_j (j = 1, 2, \ldots, s)$ are nonzero complex constants. We have*

$$m(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) = (m + n + \sigma)m(r,f) + S(r,f)$$

## 3. Proof of the Lemmas and Theorems

In this section, we will prove the main lemmas and theorems.

### 3.1 The proof of lemmas

**Lemma 7.** *Let $f(z), g(z)$ be nonconstant transcendental meromorphic functions satisfying $\rho(f) < \infty, \rho(g) < \infty$. $c_j (j = 1, 2, \ldots, s)$ are nonzero distinct complex constants. $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are positive integers. $\sigma = \sum_{j=1}^{s} \mu_j$. If $n \geq m + 2s + 3\sigma + 7$, $f^n(f^m - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(g^m - 1) \prod_{j=1}^{s} g(z + c_j)^{\mu_j}$. We get $f(z) = tg(z)$, where $t^m = t^{n+\sigma} = 1$.*

*Proof.* Let $h(z) = \frac{f(z)}{g(z)}$.

Without loss of generality, we assume that $h(z)$ is a nonconstant meromorphic function.

From

$$f^n(f^m - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(g^m - 1) \prod_{j=1}^{s} g(z + c_j)^{\mu_j}.$$

We get

$$g^m = \frac{h^n \prod_{j=1}^{s} h(z + c_j)^{\mu_j} - 1}{h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j} - 1}.$$

In the next, wo will prove that 1 is not the Picard exceptional value of $H_1 = h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j}$.

If 1 is the Picard exceptional value of $H_1 = h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j}$ .

By the Second Nevalinna theory, we obtain the following inequality:

$$\begin{aligned}
&T(r, h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j}) \\
&\leq \overline{N}(r, h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j}) + \overline{N}(r, \frac{1}{h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j}}) \\
&\quad + \overline{N}(r, \frac{1}{h^{m+n} \prod_{j=1}^{s} h(z + c_j)^{\mu_j} - 1}) + S(r, h) \\
&\leq \overline{N}(r, h) + \overline{N}(r, \prod_{j=1}^{s} h(z + c_j)^{\mu_j}) + \overline{N}(r, \frac{1}{h}) \\
&\quad + \overline{N}(r, \frac{1}{\prod_{j=1}^{s} h(z + c_j)^{\mu_j}}) + S(r, h) \\
&\leq \overline{N}(r, h) + \sum_{j=1}^{s} \overline{N}(r, h(z + c_j)) + \sum_{j=1}^{s} \overline{N}(r, \frac{1}{h(z + c_j)}) + \overline{N}(r, \frac{1}{h}) + S(r, h) \\
&\leq (2 + 2s)T(r, h) + S(r, h)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(m+n)T(r,h(z))\\
&\quad = T(r,h^{m+n}) + S(r,h(z))\\
&\quad \le T(r,h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j}) + T(r,\frac{1}{\prod_{j=1}^{s} h(z+c_j)^{\mu_j}}) + S(r,h)\\
&\quad = T(r,h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j}) + T(r,\prod_{j=1}^{s} h(z+c_j)^{\mu_j}) + S(r,h)\\
&\quad \le (2+2s+\sigma)T(r,h(z)) + S(r,h)
\end{aligned}
$$

Since $n > m + 2s + 3\sigma + 7$,
Then $m + n > 2m + 2s + 3\sigma + 7 > 2 + 2s + \sigma$. This is a contradiction.
So 1 is not the Picard exceptional value of $H_1 = h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j}$.
There exists $z_0$ such that

$$H_1(z_0) = h(z_0)^{m+n}\prod_{j=1}^{s} h(z_0+c_j)^{\mu_j} = 1$$

we will discussed in two cases as following:
Case I): $H_1 \not\equiv 1$.
If $h(z)$ is a rational function.
From the equation

$$g^m = \frac{h^n\prod_{j=1}^{s} h(z+c_j)^{\mu_j} - 1}{h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j} - 1}.$$

we can obtain that:
$g(z)$ is also a rational function. This is contradict to the condition $g(z)$ is a nonconstant transcendental meromorphic functions.
Hence $h(z)$ is a transcendental meromorphic functions.
Since $f(z), g(z)$ are transcendental meromorphic functions with finite order,
We get $h(z)$ is also a transcendental meromorphic function with finite order.
Let $H_2 = h^n\prod_{j=1}^{s} h(z+c_j)^{\mu_j}$.
We assume that $z_0 \in C$ is the 1-point of $H_1$ ,but not the 1-point of $H_2$.
From

$$g^m = \frac{h^n\prod_{j=1}^{s} h(z+c_j)^{\mu_j} - 1}{h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j} - 1}.$$

We can get that $z_0$ is the zero of $H_1 - 1$ with the multiplicities at least m.
Assume that $z_1 \in C$ is the common zero of $H_1 - 1$ and $H_2 - 1$.
Since

$$(h^{m+n}(z_1)\prod_{j=1}^{s} h(z_1+c_j)^{\mu_j} - 1)g^m(z_1) = h^n(z_1)\prod_{j=1}^{s} h(z_1+c_j)^{\mu_j} - 1,$$

$$h^{m+n}(z_1)\prod_{j=1}^{s}h(z_1+c_j)^{\mu_j}=1,$$

$$h^{n}(z_1)\prod_{j=1}^{s}h(z_1+c_j)^{\mu_j}=1.$$

Then $h^m(z_1)=1$.
Hence

$$\begin{aligned}\overline{N}(r,\frac{1}{H_1-1})&\le \overline{N}(r,|H_1-1=0,H_2-1\ne 0)+\overline{N}(r,\frac{1}{h^m-1})\\&\le \frac{1}{m}N(r,\frac{1}{h^{m+n}\prod_{j=1}^{s}h(z+c_j)^{\mu_j}})+mT(r,h)+O(1)\\&\le \frac{m+n+\sigma}{m}T(r,h)+mT(r,h)+S(r,f)\\&\le \frac{m^2+m+n+\sigma}{m}T(r,h)+S(r,f)\end{aligned}$$

Where $\overline{N}(r,|H_1-1=0,H_2-1\ne 0)$ denote the counting function of zero of $H_1(z)-1$ in $|z|<r$ where each such point is not a zero of $H_2(z)-1$,and each such point is counted ignore its multiplicity.

Since $h$ is a meromorphic function with finite order.

So $H_1$ is also a meromorphic function with finite order.

And

$$\begin{aligned}(m+n+\sigma)T(r,h)&=T(r,h^{m+n+\sigma})+S(r,h)\\&\le T(r,H_1)+T(r,\frac{h^{m+n+\sigma}}{H_1})+S(r,h)\\&=T(r,H_1)+T(r,\frac{h^{\sigma}}{\prod_{j=1}^{s}h(z+c_j)^{\mu_j}})+S(r,h)\\&\le T(r,H_1)+2\sigma T(r,h)+S(r,h)\\&\le (\frac{m^2+m+n+\sigma}{m}+2s+2+2\sigma)T(r,h)+S(r,h)\end{aligned}$$

By the second fundamental theorem:

$$\begin{aligned}T(r,H_1)&\le \overline{N}(r,H_1)+\overline{N}(r,\frac{1}{H_1})+\overline{N}(r,\frac{1}{H_1-1})+S(r,H_1)\\&\le \overline{N}(r,h)+\sum_{j=1}^{s}\overline{N}(r,h(z+c_j))+\overline{N}(r,\frac{1}{h})+\sum_{j=1}^{s}\overline{N}(r,\frac{1}{h(z+c_j)})\\&\quad+\frac{m^2+m+n+\sigma}{m}T(r,h)+S(r,h)\\&\le (\frac{m^2+m+n+\sigma}{m}+2s+2)T(r,h)+S(r,h)\end{aligned}$$

Since $n > m + 2s + 3\sigma + 7$
Then $m + n + \sigma > \frac{m^2+m+n+\sigma}{m} + 2s + 2 + 2\sigma$ This is a contradiction.
Case II): $H_1 \equiv 1$
Then

$$\begin{aligned}(m+n)T(r,h) &= T(r,h^{m+n}) \\ &\le T(r, h^{m+n}\prod_{j=1}^{s} h(z+c_j)^{\mu_j}) + T(r, \frac{1}{\prod_{j=1}^{s} h(z+c_j)^{\mu_j}}) + S(r,h) \\ &\le T(r, \prod_{j=1}^{s} h(z+c_j)^{\mu_j}) + S(r,h) \\ &\le \sigma T(r,h) + S(r,h)\end{aligned}$$

This is a contradiction with $n > m + 2s + 3\sigma + 7$.
Hence $h(z)$ is a constant.
Let $h(z) = t$ is a nonzero constant,then $f(z) = tg(z)$
Hence

$$f^n(f^m - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} = t^{n+\sigma} g^n(t^m g^m - 1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$$

And

$$f^n(f^m - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} = g^n(g^m - 1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}.$$

We get

$$g^n(g^m - 1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j} = t^{n+\sigma} g^n(t^m g^m - 1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$$

Therefore $t^{n+\sigma} = t^m = 1$ □

**Lemma 8.** *Let $f(z)$ be nonconstant meromorphic function satisfying $\rho(f) < \infty$ and $c$ be nonzero complex constant. $m, n, s, \mu_j (j = 1, 2, \ldots, s), \sigma = \sum_{j=1}^{s} \mu_j$ are non-negative integers. $c_j (j = 1, 2, \ldots, s)$ are nonzero complex constants. We have*

$$m(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}) = (m+n+\sigma)m(r,f) + S(r,f)$$

*Proof.* Let

$$F(z) = f^n(f^m - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$$

Then

$$\begin{aligned}
m(r,F) &\leq m(r,\frac{F}{f^{n+\sigma}(f^m-1)})+m(r,f^{n+\sigma}(f^m-1))\\
&\leq m(r,\frac{\prod_{j=1}^{s}f(z+c_j)^{\mu_j}}{f^{\sigma}})+(m+n+\sigma)m(r,f)+S(r,f)\\
&= m(r,\prod_{j=1}^{s}\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}})+(m+n+\sigma)m(r,f)+S(r,f)\\
&\leq (m+n+\sigma)m(r,f)+\sum_{j=1}^{s}m(r,\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}})+S(r,f)\\
&\leq (m+n+\sigma)m(r,f)+S(r,f)
\end{aligned}$$

On the other hand

$$\begin{aligned}
(m+n+\sigma)m(r,f) &= m(r,f^n(f^m-1)f(z)^{\sigma})+S(r,f)\\
&\leq m(r,\frac{f^n(f^m-1)f(z)^{\sigma}}{F})+m(r,F)+S(r,f)\\
&= m(r,\prod_{j=1}^{s}\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}})+m(r,F)+S(r,f)\\
&\leq \sum_{j=1}^{s}m(r,\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}})+m(r,F)+S(r,f)\\
&\leq m(r,F)+S(r,f).
\end{aligned}$$

Hence

$$m(r,F)=(m+n+\sigma)m(r,f)+S(r,f).$$

□

### 3.2. The proof of theorems

**Theorem 1.** *Let $f(z),g(z)$ be transcendental entire functions with finite order. $\alpha(z)$ is a small function with respect to $f(z)$ and $g(z)$. $c_j(j=1,2,\ldots,s)$ are distinct complex constants,$m,n,s,\mu_j(j=1,2,\ldots,s)$ are non-negative integers,$\sigma=\sum_{j=1}^{s}\mu_j$. If $(f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k)}=\alpha(z)\Longleftrightarrow(g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)}=\alpha(z)$, and $n\geq 5k+4m+4\sigma+8$, We have $f(z)=tg(z)$, where $t^m=1$.*

*Proof.* Let

$$F(z)=\frac{(f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k)}}{\alpha(z)},$$

$$G(z)=\frac{(g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)}}{\alpha(z)}.$$

From the condition of the theorem,we know that 1 is shared by $F(z)$ and $G(z)$ $CM$.

Let
$$H(z)=\frac{F''}{F'}-2\frac{F'}{F-1}-(\frac{G''}{G'}-2\frac{G'}{G-1})$$
If $H(z)\not\equiv 0$ by the lemma 1 and lemma 2, we obtain:

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\le 2[N_2(r,\frac{1}{F})+N_2(r,F)+N_2(r,\frac{1}{G})+N_2(r,G)]\\
&\quad+3[\overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})]+S(r,F)+S(r,G)\\
&\quad\le 2[N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})]+3[\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})]+S(r,F)+S(r,G)\\
&\quad= 2[N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})]+3[N_1(r,\frac{1}{F})+N_1(r,\frac{1}{G})]+S(r,F)+S(r,G)\\
&\quad\le 2[T(r,(f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k)})-T(r,f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})\\
&\qquad+N_{k+2}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+T(r,(g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)})\\
&\qquad-T(r,g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})+N_{k+2}(r,\frac{1}{f^n(f(z)^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]\\
&\qquad+3[T(r,(f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k)})-T(r,f^n(f^m-1)\prod_{j=1}^{s}(z+c_j)^{\mu_j})\\
&\qquad+N_{k+1}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+T(r,(g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)})\\
&\qquad-T(r,g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})+N_{k+1}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]\\
&\qquad+S(r,f)+S(r,g).
\end{aligned}$$

Together with lemma 3 and lemma 4:

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\le 2[N_{k+2}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+N_{k+2}(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]\\
&+3[N_{k+1}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+N_{k+1}(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]\\
&+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\leq 2[(k+2)\overline{N}(r,\frac{1}{f})+N_{k+2}(r,\frac{1}{(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+(k+2)\overline{N}(r,\frac{1}{g})\\
&\quad+N_{k+2}(r,\frac{1}{(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]+3[(k+1)\overline{N}(r,\frac{1}{f})\\
&\quad+N_{k+1}(r,\frac{1}{(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})+(k+1)\overline{N}(r,\frac{1}{g})\\
&\quad+N_{k+1}(r,\frac{1}{(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]+S(r,f)+S(r,g)\\
&\leq (5k+7)(\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g}))+(5m+5\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g)\\
&\leq (5k+5m+5\sigma+7)(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned}$$

Since $T(r,F)+T(r,G)=(m+n+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g)$.
We get $n\leq 5k+4m+4\sigma+7$, which contradicts the assumption that $n\geq 5k+4m+4\sigma+8$ .
Hence $H(z)\equiv 0$.
Integrating twice,we can get $\frac{1}{F-1}=\frac{a}{G-1}+b$, where $a(\neq 0),b$ are constants.
In another form $F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}, G=\frac{(a-b-1)-(a-b)F}{bF-(b+1)}$.
We will discuss the details in three cases:
Case I): $b=-1$
We can assert $a=-1$.
If $a\neq -1$, then:

$$F=\frac{a}{(a+1)-G}$$

therefore

$$\overline{N}(r,\frac{1}{G-(a+1)})=\overline{N}(r,F)=S(r,f).$$

By the second fundamental theorem,

$$\begin{aligned}
T(r,G)&\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-(a+1)})+S(r,G)\\
&\leq N_1(r,\frac{1}{G})+S(r,f)+S(r,g)\\
&\leq T(r,(g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)})-T(r,g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})\\
&\quad+N_{k+1}(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})+S(r,f)+S(r,g)\\
&\leq (k+1)\overline{N}(r,\frac{1}{g})+N_{k+1}(r,\frac{1}{(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}})]+S(r,f)+S(r,g)\\
&\leq (k+1+m+\sigma)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

This is a contradiction with the assumption that $n \geq 5k + 4m + 4\sigma + 8$.
Hence $a = -1$. Then $F = \frac{1}{G}$.
That is

$$(f^n(f^m - 1\prod_{j=1}^{s}(z+c_j)^{\mu_j})^{(k)}(g^n(g^m - 1\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)} \equiv \alpha(z)^2.$$

Since $n \geq 5k + 4m + 4\sigma + 8$.
We get

$$N(r,\frac{1}{f}) = S(r,f), N(r,\frac{1}{f-1}) = S(r,f).$$

Then

$$\delta(0,f) + \delta(1,f) + \delta(\infty,f) = 3$$

Which contradicts the conclusion $\sum_a \delta(a,f) \leq 2$.
Case II): $b = 0$
We assert $a = 1$.
If $a \neq 1$, then $F = \frac{G+(a-1)}{a}$.
Further

$$\overline{N}(r,\frac{1}{F}) = \overline{N}(r,\frac{1}{G+(a-1)}).$$

By the second fundamental theorem,

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G+(a-1)}) + S(r,G) \\
&\leq N_1(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F}) + S(r,f) + S(r,g) \\
&= N_1(r,\frac{1}{G}) + N_1(r,\frac{1}{F}) + S(r,g) \\
&\leq T(r,G) - T(r,g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}) \\
&\quad + N_{k+1}(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}}) + T(r,F) \\
&\quad - T(r,f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}) \\
&\quad + N_{k+1}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}}) + S(r,f) + S(r,g) \\
&\leq (k+1)(\overline{N}(r,\frac{1}{f}) + (m+\sigma)T(r,f) + (k+1)\overline{N}(r,\frac{1}{g}) \\
&\quad + (m+\sigma)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Similarly:

$$T(r,G) \le (k+1)(\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{f}))+(m+\sigma)(T(r,g)+T(r,f))+S(r,f)+S(r,g).$$

From the two inequalities above, we get:

$$(m+n+\sigma)(T(r,f)+T(r,g)) \le (2k+2m+2\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

This is a contradiction with the assumption that $n \ge 5k+4m+4\sigma+8$.
Hence $a = 1$.
Then $F(z) \equiv G(z)$.
By integration, we get

$$(f^n(f^m-1)\prod_{j=1}^{s}(z+c_j)^{\mu_j})^{(k-1)} = (g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k-1)} + a_{k-1}.$$

Where $a_{k-1}$ is a constant.
If $a_{k-1} \ne 0$ then by lemma 5, we obtain:
$n \le 2k+m+\sigma+2$, which contradicts the assumption $n \ge 5k+4m+4\sigma+8$.
Therefore $a_{k-1} = 0$. Repeating the same process $k-1$ times, we get:

$$(f^n(f^m-1)\prod_{j=1}^{s}(z+c_j)^{\mu_j}) = (g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}).$$

By lemma 6, we obtain that $f(z) = tg(z)$, where $t^m = 1$.
Case III) : $b \ne 0, -1$
If $a-b-1 = 0$ then

$$F = \frac{(b+1)G}{bG+1}.$$

Further

$$\overline{N}(r, G+\frac{1}{b}) = \overline{N}(r,F) = S(r,f).$$

With the similar proof of Case I) we can get a contradiction.
Hence $a-b-1 \ne 0$.
Then $\overline{N}(r,\frac{1}{F}) = \overline{N}(r,\frac{1}{G+\frac{a-b-1}{b=1}})$.
With the similar proof of CaseII) we can get a contradiction.
The proof is completed. □

**Theorem 2.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying $\rho(f) < \infty, \rho(g) < \infty$. $f(z)$ and $g(z)$ share $\infty$ $IM$. $\alpha(z) \not\equiv 0$ is an entire function satisfying $\rho(\alpha) < \rho(f)$. $m, n, s, \mu_j (j = 1, 2, \ldots, s)$ are non-negative integers, $\sigma = \sum_{j=1}^{s}\mu_j$. $c_j (j = 1, 2, \ldots, s)$ are non-zero complex constants. $F(z) = f^n(f^m-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}, G(z) = g^n(g^m-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$. $F(z)-\alpha(z), G(z)-\alpha(z)$ share $0,\infty$ $CM$. If $n \ge m+2s+3\sigma+7$ we get $f(z) =$*

*$tg(z)$, where $t$ is a constant satisfying $t^m = 1$.*

*Proof.* Let $F_1(z) = \frac{F(z)}{\alpha(z)}, G_1(z) = \frac{G(z)}{\alpha(z)}$.

Suppose that $z_0 \in C$ is a zero of $F_1(z) - 1$ of multiplicity $p$.

Since $\alpha(z) \not\equiv 0$, we can see that $z_0$ is a zero of $F(z) - \alpha(z)$ of multiplicity $p + q$, where $q(\geq 0)$ is the multiplicity of $z_0$ as a zero of $\alpha(z)$.

Since$\alpha(z)$ is shared by $F(z)$ and $G(z)$ $CM$.

So $z_0$ is a zero of $G(z) - \alpha(z)$ of multiplicity $p + q$.

Hence $z_0$ is a zero of $G_1 - 1$ of multiplicity $p$.

Therefore $F_1$ and $G_1$ share 1 $CM$.

By lemma 8

$$m(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) = (m + n + \sigma)m(r, f) + o(\frac{T(r, f)}{r^{1-\varepsilon}}) + O(1).$$

By the lemma 2.4 in the paper[12]

$$\begin{aligned}
N(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) &\geq (m + n)N(r, f) - N(r, \frac{1}{\prod_{j=1}^{s} f(z + c_j)^{\mu_j}}) \\
&\geq (m + n)N(r, f) - \sum_{j=1}^{s} N(r, \frac{1}{f(z + c_j)^{\mu_j}}) \\
&\geq (m + n)N(r, f) - \sigma N(r, \frac{1}{f}) + S(r, f).
\end{aligned}$$

Therefore:

$$\begin{aligned}
&T(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) \\
&= m(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) + N(r, f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}) \\
&\geq (m + n + \sigma)m(r, f) + (m + n)N(r, f) - \sigma N(r, \frac{1}{f}) + S(r, f) \\
&= (m + n - \sigma)T(r, f) + \sigma m(r, f) + S(r, f)
\end{aligned}$$

the generalized second fundamental theorem:

$$\begin{aligned}T(r,F_1) &\le \overline{N}(r,F_1)+\overline{N}(r,\frac{1}{F_1})+\overline{N}(r,\frac{1}{F_1-1})+o(1)+S(r,F_1)\\
&\le \overline{N}(r,f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})+\overline{N}(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}})\\
&\quad +\overline{N}(r,\frac{1}{\alpha(z)})+\overline{N}(r,\frac{1}{G_1-1})+S(r,f)\\
&\le \overline{N}(r,f)+\sum_{j=1}^{s}\overline{N}(r,f(z+c_j))+\overline{N}(r,\frac{1}{\alpha(z)})+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^m-1})\\
&\quad +\sum_{j=1}^{s}\overline{N}(r,\frac{1}{f(z+c_j)})+\overline{N}(r,\frac{1}{G_1-1})+S(r,f)\\
&\le (m+2s+2)T(r,f)+T(r,G_1)+S(r,f)\\
&\le (m+2s+2)T(r,f)+(m+n+\sigma)T(r,g)+S(r,f)\end{aligned}$$

and

$$\begin{aligned}T(r,f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}) &\le T(r,F_1)+T(r,\alpha(z))\\
&\le T(r,F_1)+S(r,f).\end{aligned}$$

We have

$$\begin{aligned}(m+n-\sigma)T(r,f)+\sigma m(r,f) &\le (m+2s+2)T(r,f)+T(r,G_1)+S(r,f)\\
&\le (m+2s)T(r,f)+(m+n+\sigma)T(r,g)+S(r,f).\end{aligned}$$

Hence $\rho(f)\le\rho(G_1)\le\rho(g)$
in the same way, we can prove that $\rho(g)\le\rho(F_1)\le\rho(f)$.
So $\rho(f)=\rho(g)=\rho(F_1)=\rho(G_1)$.
Since

$$\begin{aligned}(m+n-\sigma)T(r,f)+\sigma m(r,f)+O(1) &\le T(r,f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j})\\
&\le T(r,F_1)+T(r,\alpha(z)).\end{aligned}$$

So

$$(m+n-\sigma)T(r,f)\le T(r,F_1)+O(1)$$

$$\begin{aligned} N_2(r,\frac{1}{F_1})+2\overline{N}(r,F_1) &\leq 2\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f(z)^m-1})+N(r,\frac{1}{\prod_{j=1}^{s}f(z+c_j)^{\mu_j}}) \\ &\quad +2\overline{N}(r,f)+2\overline{N}(r,\prod_{j=1}^{s}f(z+c_j)^{\mu_j})+2\overline{N}(r,\frac{1}{\alpha(z)}) \\ &\leq (m+\sigma+2s+4)T(r,f)+S(r,f) \\ &\leq \frac{m+\sigma+2s+4}{m+n-\sigma}T(r,F_1)+O(T(r,F_1)) \end{aligned}$$

$$\begin{aligned} N_2(r,\frac{1}{G_1}) &\leq 2\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g(z)^m-1})+N(r,\frac{1}{\prod_{j=1}^{s}g(z+c_j)^{\mu_j}}) \\ &\leq (m+\sigma+2)T(r,g)+S(r,g) \\ &\leq \frac{m+\sigma+2}{m+n-\sigma}T(r,G_1)+O(T(r,G_1)). \end{aligned}$$

Hence

$$N_2(r,\frac{1}{F_1})+2\overline{N}(r,F_1)+N_2(r,\frac{1}{G_1}) \leq \frac{2m+2\sigma+2s+6}{m+n-\sigma}T(r)+o(T(r))$$

Where $T(r)=\max\{T(r,F_1),T(r,G_1)\}$.
Since

$$n \geq m+2s+3\sigma+7.$$

Therefore

$$\frac{2m+2s+2\sigma+6}{m+n-\sigma}<1.$$

By the lemma 2.7 in the paper[12], we get $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$.
If $F_1 \equiv G_1$, then by lemma 7, we have $f=tg$.
If $F_1G_1 \equiv 1$, then
$f^n(f^m-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}g^n(g^m-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}=\alpha(z)^2$.
That is

$$f^n(f^m-1)g^n(g^m-1)=\frac{\alpha(z)^2}{\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\prod_{j=1}^{s}g(z+c_j)^{\mu_j}}.$$

By the previous proof,we obtain

$$T(r,g)=O(T(r,f)),T(r,f)=O(T(r,g)),T(r,\alpha)=O(T(r,f)).$$

Since $f(z)$ and $g(z)$ share $\infty IM$, and

$$N(r,f^n(f^m-1)g^n(g^m-1))=N(r,\frac{\alpha(z)^2}{\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\prod_{j=1}^{s}g(z+c_j)^{\mu_j}}).$$

We get that:

$$\begin{aligned}(m+n)(N(r,f)+N(r,g)) &\le N(r,\frac{1}{\prod_{j=1}^{s} f(z+c_j)^{\mu_j}})+N(r,\frac{1}{\prod_{j=1}^{s} g(z+c_j)^{\mu_j}})\\ &\le \sigma(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+S(r,f)+S(r,g).\end{aligned}$$

Since

$$\frac{1}{f^n(f^m-1)g^n(g^m-1)}=\frac{\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\prod_{j=1}^{s} g(z+c_j)^{\mu_j}}{\alpha(z)^2}.$$

We obtain

$$\begin{aligned}&n(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+N(r,\frac{1}{f^m-1})+N(r,\frac{1}{g^m-1})\\ &\le N(r,\prod_{j=1}^{s} f(z+c_j)^{\mu_j})+N(r,\prod_{j=1}^{s} g(z+c_j)^{\mu_j})+2N(r,\frac{1}{\alpha(z)})\\ &\le \sigma(N(r,f)+N(r,g))+2T(r,\alpha(z))+S(r,f)+S(r,g)\\ &\le \sigma(N(r,f)+N(r,g))+O(T(r,f))+S(r,f)+S(r,g).\end{aligned}$$

Hence

$$N(r,f)+N(r,g)+N(r,\frac{1}{f})+N(r,\frac{1}{g})+N(r,\frac{1}{f^m-1})+N(r,\frac{1}{g^m-1})=O(T(r,f)).$$

By the second fundamental theorem,

$$mT(r,f)\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\sum_{j=1}^{m}\overline{N}(r,\frac{1}{f-w_j})+O(\log r)=O\{T(r,f)\}.$$

Where $w_j(j=1,2,\ldots,m)$ are the roots of $w^m=1$.

This is a contradiction.

The proof is completed. □

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