KYUNGPOOK Math. J. 55(2015), 395-410
http://dx.doi.org/10.5666/KMJ.2015.55.2.395
pISSN 1225-6951 eISSN 0454-8124

Radii of Starlikeness and Convexity for Analytic Functions with Fixed Second Coefficient Satisfying Certain Coefficient Inequalities

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Abstract. For functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ with $|a_2| = 2b$, $b \geq 0$, sharp radii of starlikeness of order α $(0 \leq \alpha < 1)$, convexity of order α $(0 \leq \alpha < 1)$, parabolic starlikeness and uniform convexity are derived when $|a_n| \leq M/n^2$ or $|a_n| \leq Mn^2$ $(M > 0)$. Radii constants in other instances are also obtained.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. It is well-known that the Taylor coefficients of functions $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n \geq 2$. The function $f(z) = 2z - z/(1-z)^2$ shows that the inequality $|a_n| \leq n$ $(n \geq 2)$ is not a sufficient criterion for univalence. Gavrilov [6] determined the radius of univalence of functions satisfying the inequality $|a_n| \leq n$ $(n \geq 2)$, which also turned out to be their radius of starlikeness, a result proved by Yamashita [25]. Ravichandran [20] and Ali *et al.* [4] worked in the similar direction and obtained several radii constants for functions with fixed second coefficient. In this paper, we continue to investigate the related radius problems for the following classes of analytic functions.

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Received October 25, 2013; accepted April 11, 2014.
2010 Mathematics Subject Classification: 30C45, 30C80.
Key words and phrases: starlike functions, convex functions, uniformly convex functions, parabolic starlike functions, radius problems, fixed second coefficient.
This work was supported in part by a grant from University of Delhi.

For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of starlike functions of order α and convex functions of order α, respectively, defined analytically by the equivalences

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad f \in \mathcal{C}(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha.$$

These classes were introduced by Robertson [22]. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the familiar classes of starlike and convex functions respectively. Let $\mathcal{S}^*_\alpha$ and $\mathcal{C}_\alpha$ be the subclasses of $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ defined as

$$\mathcal{S}^*_\alpha = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \right\}$$

and

$$\mathcal{C}_\alpha = \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < 1 - \alpha \right\}.$$

Goodman [8] introduced the class $\mathcal{UCV}$ of uniformly convex functions $f \in \mathcal{A}$, which map every circular arc γ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ onto a convex arc. For $f \in \mathcal{A}$, the equivalence

$$f \in \mathcal{UCV} \quad \Leftrightarrow \quad \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{D})$$

was independently proved by Rønning [23] and Ma and Minda [15]. Rønning [23] considered the class $\mathcal{S}_P$ of parabolic starlike functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|.$$

Note that the class $\mathcal{S}_P$ consists of functions $f = zF'$ where $F \in \mathcal{UCV}$. Moreover, $\mathcal{C}_{1/2} \subset \mathcal{UCV}$ and $\mathcal{S}^*_{1/2} \subset \mathcal{S}_P$ (see [1]).

Two more classes of analytic functions will be needed in our investigation. One of them is the class $\mathcal{L}(\alpha, \beta)$ defined as

$$\mathcal{L}(\alpha,\beta) = \left\{ f \in \mathcal{A} : \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1 + (1-2\beta)z}{1-z}, \beta \in \mathbb{R}\backslash\{1\}, \alpha \geq 0 \right\}$$

This class includes a variety of well known classes of analytic functions (see [11, 13, 14, 18, 19, 24]). Li and Owa [12] proved that $\mathcal{L}(\alpha, \beta) \subset \mathcal{S}^*$ for $-\alpha/2 \leq \beta < 1$. The other one is the class $\mathcal{ST}[A, B]$ of Janowski [9] starlike functions:

$$\mathcal{ST}[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1 \right\}.$$

The following two lemmas provide a sufficient condition for a function $f \in \mathcal{A}$ to be in the classes $\mathcal{L}(\alpha, \beta)$ and $\mathcal{ST}[A, B]$.

Lemma 1.1.([14],[24]) *Let* $\beta \in \mathbb{R}\backslash\{1\}$ *and* $\alpha \geq 0$. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *satisfies the inequality*

$$\sum_{n=2}^{\infty}(\alpha n^2 + (1-\alpha)n - \beta)|a_n| \leq |1-\beta|,$$

then $f \in \mathcal{L}(\alpha, \beta)$.

Lemma 1.2.([7]) *Let* $-1 \leq B < A \leq 1$. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *satisfies the inequality*

$$\sum_{n=2}^{\infty}((1-B)n - (1-A))|a_n| \leq A - B,$$

then $f \in \mathcal{ST}[A, B]$.

For $b \geq 0$, let $\mathcal{A}_b$ denote the class of functions f given by (1.1) with $|a_2| = 2b$. Recent work in the study of univalent functions in $\mathcal{A}_b$ $(0 \leq b \leq 1)$ include those of [2, 3, 10, 16]. In [20], Ravichandran obtained the sharp radii of starlikeness of order α, convexity of order α, uniform convexity and parabolic starlikeness for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$, $|a_n| \leq M$ or $|a_n| \leq M/n$ $(M > 0)$ for $n \geq 3$. Ali, Nargesi and Ravichandran [4] obtained sharp $\mathcal{L}(\alpha, \beta)$-radius and $\mathcal{ST}[A, B]$-radius for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq cn + d$ $(c, d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$ and deduced the results of Ravichandran [20] and Yamashita [25] as particular cases. Recently, Nagpal and Ravichandran [17] obtained radii of starlikeness and convexity of order α for harmonic functions satisfying similar coefficient inequalities.

If $f \in \mathcal{A}$ given by (1.1), satisfy $\operatorname{Re}(f'(z) + zf''(z)) > 0$ for $z \in \mathbb{D}$, then $|a_n| \leq 2/n^2$ (see [5]). But the converse does not hold. To see this, consider the function $\phi(z) = z + 2z^n/n^2$ and observe that the analytic function $z\phi''(z) + \phi'(z)$ vanishes inside $\mathbb{D}$. Reade [21] proved that a close-to-star function $f \in \mathcal{A}$ given by (1.1) satisfies $|a_n| \leq n^2$ for $n \geq 2$. However, the function $\psi(z) = z + n^2 z^n$ is not close-to-star since $\int_0^z (\psi(t)/t)\,dt$ is not even univalent (see [21, p. 61]). This paper determines sharp $\mathcal{S}^*_\alpha$, $\mathcal{C}_\alpha$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{UCV}$, $\mathcal{S}_P$, $\mathcal{L}(\alpha, \beta)$ and $\mathcal{ST}[A, B]$ radii for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying either $|a_n| \leq M/n^2$ or $|a_n| \leq Mn^2$ $(M > 0)$.

2. Radius Constants Concerning $|a_n| \leq M/n^2$

In this section, the sharp radius constants are obtained for functions $f \in \mathcal{A}_b$ satisfying the condition $|a_n| \leq M/n^2$ $(M > 0)$ for $n \geq 3$.

Theorem 2.1. *Let* $f \in \mathcal{A}_b$ *be given by* (1.1) *with* $|a_n| \leq M/n^2$ $(M > 0)$ *for* $n \geq 3$. *Then we have the following.*

(i) *f satisfies the inequality*

$$\operatorname{Re}(f'(z) + zf''(z)) > 0$$

in $|z| < r_0$*, where* r_0 *is the real root in* $(0,1)$ *of the equation*

$$(8b - M)r^2 - (8b+1)r + 1 = 0. \tag{2.1}$$

(ii) f *satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha$$

in $|z| < r_1$ *where* $r_1 = r_1(\alpha)$ *is the real root in* $(0,1)$ *of the equation*

$$\begin{aligned}(8b - M)(2-\alpha)r^2 - 4(1-\alpha)(1+M)r \\ -4M\log(1-r) - 4M\alpha Li_2(r) = 0.\end{aligned} \tag{2.2}$$

where $Li_2(z)$ *is the polylogarithm function of order* 2 *defined by the power series*

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots \quad (z \in \mathbb{D}).$$

The number $r_1(\alpha)$ *is also the radius of starlikeness of order* α*. The number* $r_1(1/2)$ *is the radius of parabolic starlikeness of the given functions.*

(iii) f *satisfies the inequality*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \alpha$$

in $|z| < r_2$ *where* $r_2 = r_2(\alpha)$ *is the real root in* $(0,1)$ *of the equation*

$$\begin{aligned}(M - 8b)(2-\alpha)r^3 + ((8b+1)(2-\alpha) - \alpha(1+M))r^2 \\ + 2(\alpha(1+M) - 1)r + 2\alpha M(1-r)\log(1-r) = 0.\end{aligned} \tag{2.3}$$

The number $r_2(\alpha)$ *is also the radius of convexity of order* α*. The number* $r_2(1/2)$ *is the radius of uniform convexity of the given functions.*

All these results are sharp.

Proof. (i) Using $|a_2| = 2b$ for the function $f \in \mathcal{A}_b$ and the inequality $|a_n| \leq M/n^2$ for $n \geq 3$, a calculation shows that, for $|z| < r_0$,

$$\begin{aligned} f'(z) + zf''(z) &= 1 + \sum_{n=2}^{\infty} na_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \\ &= 1 + 8bz + \sum_{n=3}^{\infty} n^2 a_n z^{n-1} \end{aligned}$$

so that

$$\begin{aligned}\operatorname{Re}(f'(z)+zf''(z)) &> 1-8br_0-\sum_{n=3}^{\infty} Mr_0^{n-1}\\ &= 1-8br_0-\frac{Mr_0^2}{1-r_0}=0,\end{aligned}$$

where r_0 is the real root of (2.1) in $(0,1)$. The result is sharp by considering the function f_0 given by

$$f_0(z)=z-2bz^2-\sum_{n=3}^{\infty}\frac{M}{n^2}z^n. \tag{2.4}$$

(ii) We need to show that the function $f(r_1z)/r_1\in \mathcal{S}^*_\alpha$ where r_1 is the real root of (2.2) in $(0,1)$. In view of Lemma 1.2, it is sufficient to verify the inequality

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|r_1^{n-1}\le 1-\alpha$$

(by setting $A=1-\alpha$, $B=0$). Using the identities

$$\sum_{n=3}^{\infty}\frac{r_1^{n-1}}{n}=-\frac{\log(1-r_1)}{r_1}-1-\frac{r_1}{2}\quad\text{and}\quad\sum_{n=3}^{\infty}\frac{r_1^{n-1}}{n^2}=\frac{Li_2(r_1)}{r_1}-1-\frac{r_1}{4}$$

it follows that

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha)|a_n|r_1^{n-1} &= 2(2-\alpha)br_1+\sum_{n=3}^{\infty}(n-\alpha)|a_n|r_1^{n-1}\\ &\le 2(2-\alpha)br_1+M\sum_{n=3}^{\infty}\frac{r_1^{n-1}}{n}-\alpha M\sum_{n=3}^{\infty}\frac{r_1^{n-1}}{n^2}\\ &= \frac{1}{4}(2-\alpha)(8b-M)r_1-(1-\alpha)M\\ &\quad -\frac{M}{r_1}\log(1-r_1)-\frac{\alpha M}{r_1}Li_2(r_1)\\ &= 1-\alpha.\end{aligned}$$

Thus $f\in\mathcal{S}^*_\alpha$ for $|z|<r_1$ where r_1 is the real root of (2.2) in $(0,1)$. For sharpness, the function f_0 defined by (2.4) satisfies

$$\frac{zf_0'(z)}{f_0(z)}-1=-\frac{(8b-M)z^2-4M\log(1-z)-4MLi_2(z)}{4(1+M)z-(8b-M)z^2-4MLi_2(z)}.$$

At the point $z=r_1$, where r_1 is the real root of (2.2) in $(0,1)$, we obtain

$$\operatorname{Re}\frac{zf_0'(z)}{f_0(z)}=1-\frac{(8b-M)r_1^2-4M\log(1-r_1)-4MLi_2(r_1)}{4(1+M)r_1-(8b-M)r_1^2-4MLi_2(r_1)}=\alpha. \tag{2.5}$$

Note that

$$\begin{aligned}
&(8b - M)r_1^2 - 4M\log(1-r_1) - 4MLi_2(r_1)\\
&= (8b-M)r_1^2 + 4M\sum_{n=2}^{\infty}\left(\frac{n-1}{n^2}\right)r_1^n\\
&> (8b-M)r_1^2 + Mr_1^2 = 8br_1^2 \geq 0,
\end{aligned}$$

and since $\alpha < 1$, (2.5) shows that the denominator of the rational expression in the middle is positive. Thus, it follows that, at the point $z = r_1$,

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{(8b-M)r_1^2 - 4M\log(1-r_1) - 4MLi_2(r_1)}{4(1+M)r_1 - (8b-M)r_1^2 - 4MLi_2(r_1)} = 1-\alpha. \tag{2.6}$$

This shows that the radius r_1 given by (2.2) is sharp.

Since $\mathcal{S}^*_\alpha \subset \mathcal{S}^*(\alpha)$, it is easily seen that the radius of starlikeness of order α is at least $r_1(\alpha)$. However, (2.5) shows that this radius is sharp for the same function f_0.

Also, since $\mathcal{S}^*_{1/2} \subset \mathcal{S}_P$, the radius of parabolic starlikeness is at least $r_1(1/2)$. But (2.5) and (2.6) with $\alpha = 1/2$ shows that, at the point $z = r_1(1/2)$,

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{1}{2} = \frac{zf_0'(z)}{f_0(z)}.$$

Hence the radius of parabolic starlikeness is also sharp.

(iii) If r_2 is the real root of (2.3) in $(0,1)$, then it suffices to show that $f(r_2 z/r_2) \in \mathcal{C}_\alpha$. By Lemma 1.2 and using the equivalence $f \in \mathcal{C}_\alpha$ if and only if $zf' \in \mathcal{S}^*_\alpha$, it is sufficient to establish the inequality

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n|r_2^{n-1} \leq 1-\alpha.$$

Observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-\alpha)|a_n|r_2^{n-1}\\
&= 4(2-\alpha)br_2 + \sum_{n=3}^{\infty} n(n-\alpha)|a_n|r_2^{n-1}\\
&\leq 4(2-\alpha)br_2 + M\sum_{n=3}^{\infty} r_2^{n-1} - \alpha M\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n}\\
&= 4(2-\alpha)br_2 + \frac{Mr_2^2}{1-r_2} - \alpha M\left(-\frac{\log(1-r_2)}{r_2} - 1 - \frac{r_2}{2}\right)\\
&= 1-\alpha.
\end{aligned}$$

To prove the sharpness, consider the function f_0 defined by (2.4). For this function, a calculation shows that

$$\frac{zf_0''(z)}{f_0'(z)} = \frac{4bz + M\sum_{n=3}^{\infty}\left(\frac{n-1}{n}\right)z^{n-1}}{4bz + M\sum_{n=3}^{\infty}\frac{z^{n-1}}{n} - 1}$$

In particular, at the point $z = r_2$, we have

$$\operatorname{Re}\frac{zf_0''(z)}{f_0'(z)} = \frac{4br_2 + M\sum_{n=3}^{\infty}\left(\frac{n-1}{n}\right)r_2^{n-1}}{4br_2 + M\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n} - 1} = \alpha - 1. \tag{2.7}$$

Since $\alpha < 1$ and the numerator of the rational function in the middle of (2.7) is positive, it follows that the denominator is negative. Hence, at $z = r_2$, where r_2 is the real root of (2.3) in $(0, 1)$,

$$\left|\frac{zf_0''(z)}{f_0'(z)}\right| = \frac{4br_2 + M\sum_{n=3}^{\infty}\left(\frac{n-1}{n}\right)r_2^{n-1}}{1 - 4br_2 - M\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n}} = 1 - \alpha. \tag{2.8}$$

Using the fact that $\mathcal{C}_\alpha \subset \mathcal{C}(\alpha)$, the radius of convexity of order α is at least $r_2(\alpha)$. But, (2.7) shows that this radius is sharp for the function f_0.

As $\mathcal{C}_{1/2} \subset \mathcal{UCV}$, the radius of uniform convexity is at least $r_2(1/2)$. With $\alpha = 1/2$, (2.7) and (2.8) shows that, at the point $z = r_2(1/2)$,

$$\left|\frac{zf_0''(z)}{f_0'(z)}\right| = \frac{1}{2} = \operatorname{Re}\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right).$$

Therefore, the radius of uniform convexity is also sharp. □

The logarithm in (2.2) and (2.3) is the branch that takes the value 1 at $z = 0$. Setting $M = 2$ and $b = 1/4$ in Theorem 2.1, we obtain the sharp radii constants for functions $f \in \mathcal{A}$ satisfying $|a_n| \le 2/n^2$ for $n \ge 2$.

Corollary 2.2. *Let $\mathcal{F}$ be the class of functions $f \in \mathcal{A}$ given by (1.1) with $|a_n| \le 2/n^2$ for $n \ge 2$. Then we have the following.*

(i) *A function $f \in \mathcal{F}$ satisfies the inequality $\operatorname{Re}(f'(z) + zf''(z)) > 0$ in $|z| < 1/3$.*

(ii) *The radius of starlikeness of order α of the class $\mathcal{F}$ is the real root $r_1 = r_1(\alpha)$ of the equation*

$$3(1-\alpha)r + 2\log(1-r) + 2\alpha Li_2(r) = 0$$

in $(0, 1)$. In particular, the radius of starlikeness is the root $r_1(0) \approx 0.582812$ of the equation $3r + 2\log(1-r) = 0$ and the radius of parabolic starlikeness is the root $r_1(1/2) \approx 0.442017$ of the equation $3r + 4\log(1-r) + 2Li_2(r) = 0$.

(iii) *The radius of convexity of order* α *of the class* $\mathcal{F}$ *is the real root* $r_2 = r_2(\alpha)$ *of the equation*

$$3(1-\alpha)r^2 + (3\alpha - 1)r + 2\alpha(1-r)\log(1-r) = 0$$

in $(0,1)$. *In particular, the radius of convexity is* $1/3$ *and the radius of uniform convexity is the root* $r_2(1/2) \approx 0.244312$ *of the equation* $3r^2 + r + 2(1-r)\log(1-r) = 0$.

The results are sharp.

If the second coefficient in the Taylor series expansion of $f \in \mathcal{A}$ is zero, then the results of Corollary 2.2 can be further improved, as seen by the following corollary.

Corollary 2.3. *Let* $\mathcal{F}_0$ *be the class of functions* $f \in \mathcal{A}$ *given by* (1.1) *with* $a_2 = 0$ *and* $|a_n| \leq 2/n^2$ *for* $n \geq 3$. *Then we have the following.*

(i) *A function* $f \in \mathcal{F}_0$ *satisfies the inequality* $\operatorname{Re}(f'(z) + zf''(z)) > 0$ *in* $|z| < 1/2$.

(ii) *The radius of starlikeness of order* α *of the class* $\mathcal{F}_0$ *is the real root* $r_1 = r_1(\alpha)$ *of the equation*

$$(2-\alpha)r^2 + 6(1-\alpha)r + 4\log(1-r) + 4\alpha Li_2(r) = 0$$

in $(0,1)$. *In particular, the radius of starlikeness is the root* $r_1(0) \approx 0.76088$ *of the equation* $r^2 + 3r + 2\log(1-r) = 0$ *and the radius of parabolic starlikeness is the root* $r_1(1/2) \approx 0.648957$ *of the equation* $3r^2 + 6r + 8\log(1-r) + 4Li_2(r) = 0$.

(iii) *The radius of convexity of order* α *of the class* $\mathcal{F}_0$ *is the real root* $r_2 = r_2(\alpha)$ *of the equation*

$$(2-\alpha)r^3 + (1-2\alpha)r^2 + (3\alpha - 1)r + 2\alpha(1-r)\log(1-r) = 0$$

in $(0,1)$. *In particular, the radius of convexity is* $1/2$ *and the radius of uniform convexity is the root* $r_2(1/2) \approx 0.41368$ *of the equation* $3r^3 + r + 2(1-r)\log(1-r) = 0$.

The results are sharp.

The sharp $\mathcal{L}(\alpha, \beta)$ and $\mathcal{ST}[A, B]$ radii of functions $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq M/n^2$ are obtained in the following theorem.

Theorem 2.4. *Let* $\beta \in \mathbb{R}\backslash\{1\}$, $\alpha \geq 0$ *and* $-1 \leq B < A \leq 1$. *Then*

(a) *The* $\mathcal{L}(\alpha, \beta)$*-radius of functions* $f \in \mathcal{A}_b$ *given by* (1.1) *satisfying the coefficient inequality* $|a_n| \leq M/n^2$, $n \geq 3$, *is the real root* $t_0 = t_0(\alpha, \beta)$ *in* $(0,1)$ *of the equation*

$$\begin{aligned} &(2\alpha + 2 - \beta)(M - 8b)t^3 \\ &+ (M(2 - 2\alpha - 3\beta) + 8b(2\alpha + 2 - \beta) + 4|1-\beta|)t^2 \\ &+ 4(M(\alpha + \beta - 1) - |1-\beta|)t \\ &\quad = 4M(1-t)((1-\alpha)\log(1-t) + \beta Li_2(t)). \end{aligned} \tag{2.9}$$

(b) *The* $\mathcal{ST}[A,B]$ *radius for functions* $f \in \mathcal{A}_b$ *given by* (1.1) *satisfying the coefficient inequality* $|a_n| \le M/n^2$ *for* $n \ge 3$, *is the real root* $t_1 = t_1(A,B)$ *in* $(0,1)$ *of the equation*

$$(8b - M)(1 + A - 2B)t^2 - 4(A - B)(M + 1)t = 4M((1 - B)\log(1 - t) + (1 - A)Li_2(t)). \tag{2.10}$$

The results are all sharp.

Proof. (a) By Lemma 1.1, it is sufficient to verify the inequality

$$\sum_{n=2}^{\infty}(\alpha n^2 + (1-\alpha)n - \beta)|a_n|t_0^{n-1} \le |1-\beta|$$

where t_0 is the real root of (2.9) in $(0,1)$. Using $|a_2| = 2b$ and the inequality $|a_n| \le M/n^2$ for $n \ge 3$, we deduce that

$$\begin{aligned}
&\sum_{n=2}^{\infty}(\alpha n^2 + (1-\alpha)n - \beta)|a_n|t_0^{n-1} \\
&\le 2(2\alpha + 2 - \beta)bt_0 + M\alpha\sum_{n=3}^{\infty}t_0^{n-1} \\
&\qquad + M(1-\alpha)\sum_{n=3}^{\infty}\frac{t_0^{n-1}}{n} - M\beta\sum_{n=3}^{\infty}\frac{t_0^{n-1}}{n^2} \\
&= 2(2\alpha + 2 - \beta)bt_0 + \frac{M\alpha t_0^2}{1 - t_0} \\
&\qquad + M(1-\alpha)\left(-\frac{\log(1-t_0)}{t_0} - 1 - \frac{t_0}{2}\right) \\
&\qquad - M\beta\left(\frac{Li_2(t_0)}{t_0} - 1 - \frac{t_0}{4}\right) \\
&= |1-\beta|.
\end{aligned}$$

For $\beta < 1$, the function f_0 defined by (2.4) satisfies

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\alpha z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)}\right) \\
&= \frac{1 - 4b(1+\alpha)t_0 - \frac{\alpha M t_0^2}{1-t_0} + (\alpha - 1)M\sum_{n=3}^{\infty}\frac{t_0^{n-1}}{n}}{1 - 2bt_0 - M\sum_{n=3}^{\infty}\frac{t_0^{n-1}}{n^2}} = \beta
\end{aligned}$$

at the point $z = t_0$, where t_0 is the real root of (2.9) in $(0,1)$. This shows that t_0 is the sharp $\mathcal{L}(\alpha,\beta)$-radius for $f \in \mathcal{A}_b$. For the case $\beta > 1$, the function

$$F_0(z) = z + 2bz^2 + \sum_{n=3}^{\infty}\frac{M}{n^2}z^n$$

verifies the sharpness of the result.

(b) If t_1 is the real root of (2.10) in $(0,1)$, then by Lemma 1.2, it suffices to show that

$$\sum_{n=2}^{\infty}((1-B)n-(1-A))|a_n|t_1^{n-1} \leq A-B.$$

A simple calculation shows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}((1-B)n-(1-A))|a_n|t_1^{n-1} \\
&\leq 2(2(1-B)-(1-A))bt_1 \\
&\quad + (1-B)M\sum_{n=3}^{\infty}\frac{t_1^{n-1}}{n} - (1-A)M\sum_{n=2}^{\infty}\frac{t_1^{n-1}}{n^2} \\
&= \frac{1}{4}(8b-M)(1+A-2B)t_1 + (B-A)M \\
&\qquad - \frac{M}{t_1}((1-B)\log(1-t_1) + (1-A)Li_2(t_1)) \\
&= A-B.
\end{aligned}$$

The function f_0 given by (2.4) shows that the result is sharp. Indeed,

$$\left|\frac{zf_0'(z)}{f_0(z)}-1\right| = \left|A - B\frac{zf_0'(z)}{f_0(z)}\right| \quad (-1 \leq B < A \leq 1; z = t_1). \qquad \square$$

Remark 2.5. If either $\alpha = 0$, $0 \leq \beta < 1$ in Theorem 2.4(a), or $A = 1-2\beta$, $B = -1$ in Theorem 2.4(b), we obtain the result of Theorem 2.1(ii).

3. Radius Constants Concerning $|a_n| \leq Mn^2$

In this section, we obtain the sharp radius constants for functions $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq Mn^2$ $(M > 0)$ for $n \geq 3$.

Theorem 3.1. *Let $f \in \mathcal{A}_b$ be given by (1.1) with $|a_n| \leq Mn^2$ $(M > 0)$ for $n \geq 3$. Then we have the following.*

(i) *f is close-to-star in $|z| < s_0$, where s_0 is the real root in $(0,1)$ of the equation*

$$(3.1)\quad 2(b-2M)s^4 + (11M-6b-1)s^3 + (6b+3-9M)s^2 - (2b+3)s + 1 = 0.$$

(ii) *f satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha$$

in $|z| < s_1$ *where* $s_1 = s_1(\alpha)$ *is the real root in* $(0,1)$ *of the equation*

$$(3.2)\qquad \begin{aligned}&(2(2-\alpha)(b-2M)s-(1-\alpha)(1+M))(1-s)^4\\&\qquad +M(s^2(1+\alpha)+4s+1-\alpha)=0.\end{aligned}$$

The number $s_1(\alpha)$ *is also the radius of starlikeness of order* α. *The number* $s_1(1/2)$ *is the radius of parabolic starlikeness of the given functions.*

(iii) f *satisfies the inequality*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\alpha$$

in $|z| < s_2$ *where* $s_2 = s_2(\alpha)$ *is the real root in* $(0,1)$ *of the equation*

$$(3.3)\qquad \begin{aligned}&(4(2-\alpha)(b-2M)s-(1+M)(1-\alpha))(1-s)^5\\&+M((1+\alpha)s^3+(11+3\alpha)s^2+(11-3\alpha)s+(1-\alpha))=0.\end{aligned}$$

The number $s_2(\alpha)$ *is also the radius of convexity of order* α. *The number* $s_2(1/2)$ *is the radius of uniform convexity of the given functions.*

All these results are sharp.

Proof. (i) Recall that [21] a function $f \in \mathcal{A}$ is close-to-star if and only if the function $\int_0^z (f(t)/t)\,dt$ is close-to-convex. Consequently, the coefficient inequality $\sum_{n=2}^{\infty} |a_n| \le 1$ is a sufficient condition for a function $f \in \mathcal{A}$ given by (1.1) to be close-to-star. Therefore, it suffices to show that

$$\sum_{n=2}^{\infty} |a_n| s_0^{n-1} \le 1$$

where s_0 is the real root of (3.1) in $(0,1)$. Using the coefficient bounds $|a_2| = 2b$ and $|a_n| \le Mn^2$ for $n \ge 3$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} |a_n| s_0^{n-1} &\le 2bs_0 + M\sum_{n=3}^{\infty} n^2 s_0^{n-1}\\&= 2(b-2M)s_0 + \frac{M(1+s_0)}{(1-s_0)^3} - M = 1\end{aligned}$$

provided s_0 is the real root of (3.1) in $(0,1)$. For sharpness, we consider the function

$$(3.4)\qquad g_0(z) = z - 2bz^2 - M\sum_{n=3}^{\infty} n^2 z^n = (1+M)z + 2(2M-b)z^2 - \frac{Mz(1+z)}{(1-z)^3},$$

which satisfies the hypothesis of the theorem. Observe that

$$h_0(z) = \int_0^z \frac{g_0(t)}{t}\,dt = (1+M)z + (2M-b)z^2 - \frac{Mz}{(1-z)^2}$$

and its derivative h_0' vanishes at $z = s_0$, where s_0 is the real root of (3.1) in $(0, 1)$. This shows that the function h_0 is not univalent in $|z| < r$ if $r > s_0$ and hence, g_0 is not close-to-star in $|z| < r$ if $r > s_0$. This establishes the sharpness of the result.

(ii) Following the method of the proof of Theorem 2.1(ii) and using the identities

$$\sum_{n=3}^{\infty} n^2 s_1^{n-1} = \frac{1+s_1}{(1-s_1)^3} - 1 - 4s_1 \quad \text{and} \quad \sum_{n=3}^{\infty} n^3 s_1^{n-1} = \frac{1+4s_1+s_1^2}{(1-s_1)^4} - 1 - 8s_1,$$

it is easy to deduce that

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha)|a_n|s_1^{n-1} &\leq 2(2-\alpha)bs_1 + M\sum_{n=3}^{\infty} n^3 s_1^{n-1} - \alpha M \sum_{n=3}^{\infty} n^2 s_1^{n-1} \\ &= 2(2-\alpha)(b-2M)s_1 + M\frac{1+4s_1+s_1^2}{(1-s_1)^4} \\ &\quad - \alpha M \frac{1+s_1}{(1-s_1)^3} - (1-\alpha)M \\ &= 1-\alpha\end{aligned}$$

where s_1 is the real root of (3.2) in $(0, 1)$. For sharpness, the function g_0 defined by (3.4) satisfies

$$\frac{zg_0'(z)}{g_0(z)} - 1 = -\frac{2(b-2M)z + \frac{2Mz(z+2)}{(1-z)^4}}{(1+M) + 2(2M-b)z - \frac{M(1+z)}{(1-z)^3}}.$$

In particular, at the point $z = s_1$, where s_1 is the real root of (3.2) in $(0, 1)$, we obtain

$$\operatorname{Re}\frac{zg_0'(z)}{g_0(z)} = 1 - \frac{2(b-2M)s_1 + \frac{2Ms_1(s_1+2)}{(1-s_1)^4}}{(1+M) + 2(2M-b)s_1 - \frac{M(1+s_1)}{(1-s_1)^3}} = \alpha. \tag{3.5}$$

Also, observe that the function $(2+r)/(1-r)^4$ is an increasing function of $r \in (0, 1)$, so that

$$2(b-2M)s_1 + \frac{2Ms_1(s_1+2)}{(1-s_1)^4} > 2(b-2M)s_1 + 4Ms_1 = 2bs_1 \geq 0.$$

Using this fact, (3.5) shows that the denominator of the rational expression in the middle is positive, as $\alpha < 1$. This leads to the following equality:

$$\left|\frac{zg_0'(z)}{g_0(z)} - 1\right| = \frac{2(b-2M)s_1 + \frac{2Ms_1(s_1+2)}{(1-s_1)^4}}{(1+M) + 2(2M-b)s_1 - \frac{M(1+s_1)}{(1-s_1)^3}} = 1-\alpha. \tag{3.6}$$

at the point $z = s_1$, where s_1 is the real root of (3.2) in $(0, 1)$.

Equations (3.5) and (3.6), together with inclusions $\mathcal{S}^*_\alpha \subset \mathcal{S}^*(\alpha)$ and $\mathcal{S}^*_{1/2} \subset \mathcal{S}_P$, show that the numbers $s_1(\alpha)$ and $s_1(1/2)$ are sharp radius of starlikeness of order α and parabolic starlikeness, respectively.

(iii) A straightforward calculation shows that

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-\alpha)|a_n|s_2^{n-1} &\leq 4(2-\alpha)bs_2 + M\sum_{n=3}^{\infty} n^4 s_2^{n-1} - \alpha M\sum_{n=3}^{\infty} n^3 s_2^{n-1}\\ &= 4(2-\alpha)(b-2M)s_2 - M(1-\alpha)\\ &\qquad + M\frac{(1+s_2)(1+10s_2+s_2^2)}{(1-s_2)^5} - M\alpha\frac{1+4s_2+s_2^2}{(1-s_2)^4}\\ &= 1-\alpha,\end{aligned}$$

where s_2 is the real root of (3.3) in $(0,1)$. Thus $f \in \mathcal{C}_\alpha$ for $|z| < s_2$. To prove the sharpness, consider the function g_0 defined by (3.4). For this function, a calculation shows that

$$\frac{zg_0''(z)}{g_0'(z)} = \frac{4(2M-b)z - \frac{2Mz(4+7z+z^2)}{(1-z)^5}}{1+M+4(2M-b)z - \frac{M(1+4z+z^2)}{(1-z)^4}}$$

Using the similar analysis, it is easy to see that, at the point $z = s_2$, where s_2 is the real root of (3.3) in $(0,1)$, we have

$$\left|\frac{zg_0''(z)}{g_0'(z)}\right| = 1-\alpha \quad \text{and} \quad \operatorname{Re}\left(1+\frac{zg_0''(z)}{g_0'(z)}\right) = \alpha.$$

Using these observations and the inclusions $\mathcal{C}_\alpha \subset \mathcal{C}(\alpha)$, $\mathcal{C}_{1/2} \subset \mathcal{UCV}$, it follows that $\mathcal{C}_\alpha$, $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$-radii are all sharp. □

The particular cases $b = 2$ and $b = 0$ (with $M = 1$) of Theorem 3.1 are stated in the following two corollaries.

Corollary 3.2. *Let $\mathcal{G}$ be the class of functions $f \in \mathcal{A}$ given by* (1.1) *with $|a_n| \leq n^2$ for $n \geq 2$. Then we have the following.*

(i) *A function $f \in \mathcal{G}$ is close-to-star in $|z| < s_0$, where s_0 is given by*

$$s_0 = 1 + \frac{1}{6^{2/3}}((\sqrt{300}-18)^{1/3} - (\sqrt{330}+18)^{1/3}) \approx 0.164878.$$

(ii) *The radius of starlikeness of order α of the class $\mathcal{G}$ is the real root $s_1 = s_1(\alpha)$ of the equation*

$$2(1-\alpha)s^4 - 8(1-\alpha)s^3 + (11-13\alpha)s^2 - 4(3-2\alpha)s + (1-\alpha) = 0$$

in $(0,1)$. In particular, the radius of starlikeness is the root $s_1(0) \approx 0.0903331$ of the equation $2s^4 - 8s^3 + 11s^2 - 12s + 1 = 0$ and the radius of parabolic starlikeness is the root $s_1(1/2) \approx 0.064723$ of the equation $2r^4 - 8s^3 + 9s^2 - 16s + 1 = 0$.

(iii) *The radius of convexity of order* α *of the class* $\mathcal{G}$ *is the real root* $s_2 = s_2(\alpha)$ *of the equation*

$$2(1-\alpha)s^5-10(1-\alpha)s^4+(21-19\alpha)s^3-(9-23\alpha)s^2+(21-13\alpha)s-(1-\alpha)=0$$

in $(0,1)$. *In particular, the radius of convexity is the root* $s_2(0) \approx 0.0485162$ *of the equation* $2r^5-10s^4+21s^3-9s^2+21s-1=0$ *and the radius of uniform convexity is the root* $s_2(1/2) \approx 0.0342491$ *of the equation* $2s^5-10s^4+23s^3+5s^2+29s-1=0$.

The results are sharp.

Corollary 3.3. *Let* $\mathcal{G}_0$ *be the class of functions* $f \in \mathcal{A}$ *given by* (1.1) *with* $a_2 = 0$ *and* $|a_n| \leq n^2$ *for* $n \geq 3$. *Then we have the following.*

(i) *A function* $f \in \mathcal{G}_0$ *is close-to-star in* $|z| < s_0$, *where* $s_0 \approx 0.253571$ *is the real root of the equation* $4s^4-10s^3+6s^2+3s-1=0$ *in* $(0,1)$.

(ii) *The radius of starlikeness of order* α *of the class* $\mathcal{G}_0$ *is the real root* $s_1 = s_1(\alpha)$ *of the equation*

$$4(2-\alpha)s^5-2(15-7\alpha)s^4+8(5-2\alpha)s^3-3(7-\alpha)s^2-4(1-\alpha)s+(1-\alpha)=0$$

in $(0,1)$. *In particular, the radius of starlikeness is the root* $s_1(0) \approx 0.155972$ *of the equation* $8s^5-30s^4+40s^3-21s^2-4s+1=0$ *and the radius of parabolic starlikeness is the root* $s_1(1/2) \approx 0.125429$ *of the equation* $12s^5-46s^4+64s^3-39s^2-4s+1=0$.

(iii) *The radius of convexity of order* α *of the class* $\mathcal{G}_0$ *is the real root* $s_2 = s_2(\alpha)$ *of the equation*

$$8(2-\alpha)s^6-2(39-19\alpha)s^5+10(15-7\alpha)s^4-(139-61\alpha)s^3 \\ +(71-17\alpha)s^2+5(1-\alpha)s-(1-\alpha)=0$$

in $(0,1)$. *In particular, the radius of convexity is the root* $s_2(0) \approx 0.0944584$ *of the equation* $16s^6-78s^5+150s^4-139s^3+71s^2+5s-1=0$ *and the radius of uniform convexity is the root* $s_2(1/2) \approx 0.0753134$ *of the equation* $24s^6-118s^5+230s^4-217s^3+125s^2+5s-1=0$.

The results are sharp.

From Corollaries 3.2 and 3.3, it is evident that the radii constants improve if the second coefficient in the Taylor series expansion of $f \in \mathcal{A}$ is zero. The next theorem determines the sharp $\mathcal{L}(\alpha,\beta)$ and $\mathcal{ST}[A,B]$ radii of functions $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq Mn^2$. Proof of this theorem is omitted as it is similar to Theorem 2.4.

Theorem 3.4. *Let* $\beta \in \mathbb{R}\backslash\{1\}$, $\alpha \geq 0$ *and* $-1 \leq B < A \leq 1$. *Then*

(a) *The* $\mathcal{L}(\alpha,\beta)$*-radius of functions* $f \in \mathcal{A}_b$ *given by* (1.1) *satisfying the coefficient inequality* $|a_n| \leq Mn^2$, $n \geq 3$, *is the real root* $t_0 = t_0(\alpha,\beta)$ *in* $(0,1)$ *of the equation*

$$\begin{aligned}&(2(2\alpha+2-\beta)(b-2M)t - M(1-\beta) - |1-\beta|)(1-t)^5\\&+ M((1-\beta)+(\beta+8\alpha+3)t+(14\alpha+\beta-3)t^2-(1-2\alpha+\beta)t^3) = 0.\end{aligned}$$

(b) *The* $\mathcal{ST}[A,B]$ *radius for functions* $f \in \mathcal{A}_b$ *given by* (1.1) *satisfying the coefficient inequality* $|a_n| \leq Mn^2$ *for* $n \geq 3$, *is the real root* $t_1 = t_1(A,B)$ *in* $(0,1)$ *of the equation*

$$\begin{aligned}&(2(b-2M)(1+A-2B)t-(A-B)(M+1))(1-t)^4\\&+M((A-B)+4(1-B)t+(2-A-B)t^2) = 0.\end{aligned}$$

The results are all sharp.

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