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Certain Subclasses of $k-$Uniformly Starlike and Convex Functions of Order α and Type β with Varying Argument Coefficients

Mohamed Kamal Aouf
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
e-mail : mkaouf127@yahoo.com

Nanjundan Magesh*
P.G. and Research Department of Mathematics, Govt Arts College for Men, Krishnagiri - 635001, India
e-mail : nmagi_2000@yahoo.co.in

Jagadesan Yamini
Department of Mathematics, Govt First Grade College, Vijayanagar, Bangalore-560104, Karnataka, India
e-mail : yaminibalaji@gmail.com

Abstract. In this paper, we define two new subclass of $k-$uniformly starlike and convex functions of order α type β with varying argument of coefficients. Further, we obtain coefficient estimates, extreme points, growth and distortion bounds, radii of starlikeness, convexity and results on modified Hadamard products.

1. Introduction

Denoted by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

* Corresponding Author.
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that are analytic and univalent in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and by S^* and $\mathcal{K}$ the subclasses of S that are respectively, starlike and convex. Goodman [6, 7] introduced and defined the following subclasses of $\mathcal{K}$ and S^*.

A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathcal{U}$ if $f(z)$ is in $\mathcal{K}$ (S^*) and has the property that for every circular arc γ contained in $\mathcal{U}$, with center ξ also in $\mathcal{U}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV and the class of uniformly starlike functions by UST (for details see [6]). It is well known from [12, 14] that

$$f \in UCV \Leftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|.$$

In [14], Rønning introduced a new class of starlike functions related to UCV and defined as

$$f \in S_p \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further Rønning generalized the class S_p by introducing a parameter α, $\; -1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

In 1997, Bharati et al. [2] introduced the following classes of $k-$starlike functions of order α $(k-ST(\alpha))$ and $k-$uniformly convex functions of order α $(k-UCV(\alpha))$.

$$f \in k-ST(\alpha) \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \geq 0,\ 0 \leq \alpha < 1.$$

and

$$f \in k-UCV(\alpha) \Leftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right|, \quad k \geq 0,\ 0 \leq \alpha < 1.$$

It follows that $f \in k-UCV(\alpha) \Leftrightarrow zf' \in k-ST(\alpha)$. Further, we note that, for $\alpha = 0$ the classes $k-UCV(\alpha)$ and $k-ST(\alpha)$ reduce to $k-$uniformly convex $(k-UCV)$ and $k-$uniformly starlike $(k-ST)$ functions respectively. The classes $k-UCV$ and $k-ST$ were introduced and studied by Kanas and Wisniowska [9, 10]. Latter Kanas and Srivastava [8] extended the study to find the connections between the classes $k-UCV$ and $k-ST$ considering the Hohlov linear operator which is a special case of the Dziok-Srivastava linear operator [3].

Recently, Sim et al. [18] introduced the subclasses $k-UCV(\alpha,\beta)$ and $k-ST(\alpha,\beta)$ of the univalent function class S as follows (see El-Ashwah et al.[5]):

$$f \in k-UCV(\alpha,\beta) \Leftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq k\left|1 + \frac{zf''(z)}{f'(z)} - \beta\right|,$$

where $0 \le \alpha < \beta \le 1$ and $k(1-\beta) < (1-\alpha)$ and

$$f \in k - ST(\alpha,\beta) \Leftrightarrow \Re\left\{\frac{zf^{'}(z)}{f(z)} - \alpha\right\} \ge k\left|\frac{zf^{'}(z)}{f(z)} - \beta\right|,$$

where $0 \le \alpha < \beta \le 1$ and $k(1-\beta) < (1-\alpha)$.

Notice that $f \in k - UCV(\alpha,\beta) \Leftrightarrow zf^{'} \in k - ST(\alpha,\beta)$.

Motivated by the above said classes and the work of second author [11], we define the unified subclasses of univalent function class S as follows:

For $k \ge 0$, $-1 \le \alpha < \beta \le 1$, $0 \le \lambda < 1$, $k(1-\beta) < 1-\alpha$, we let $\mathcal{S}(\lambda,\alpha,\beta,k)$ be the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - \alpha\right\} > k\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - \beta\right|, \quad z \in \mathcal{U}. \tag{1.2}$$

and also, let $k - UCV(\lambda,\alpha,\beta)$ be the subclasses of S consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{zf'(z)+z^2f''(z)}{zf'(z)+\lambda z^2 f''(z)} - \alpha\right\} > k\left|\frac{zf'(z)+z^2f''(z)}{zf'(z)+\lambda z^2 f''(z)} - \beta\right|, \quad z \in \mathcal{U}. \tag{1.3}$$

We also let $\mathcal{V}S_\eta(\lambda,\alpha,\beta,k) = \mathcal{S}(\lambda,\alpha,\beta,k) \bigcap V_\eta$ and $k - \mathcal{V}UCV_\eta(\lambda,\alpha,\beta,k) = k - UCV(\lambda,\alpha,\beta,k) \bigcap V_\eta$, where $\mathcal{V}_\eta$ the class of functions $f \in S$ of the form (1.1) for which $arg(a_n) = \pi + (n-1)\eta$, $n \ge 2$. For $\eta = 0$, we obtain the familiar class T of functions with negative coefficients [16]. Moreover, we define $\mathcal{V} := \cup_{\eta\in\mathbb{R}}\mathcal{V}_\eta$. The class $\mathcal{V}$ was introduced by Sliverman [17] (see also [4]). It is called the class of functions with varying argument of coefficients.

We note that, by specializing the parameters λ, α, and k we obtain the following subclasses studied by various authors.

1. $\mathcal{VS}_\eta(0,\alpha,1,0) = \mathcal{V}S^*(\alpha)$ and $0 - \mathcal{V}UCV_\eta(0,\alpha,0) = \mathcal{VK}(\alpha)$ (Silverman [17])
2. $\mathcal{VS}_0(0,\alpha,1,0) = S^*(\alpha)$ and $0 - \mathcal{V}UCV_0(0,\alpha,0) = \mathcal{K}(\alpha)$ (Silverman [16])
3. $\mathcal{VS}_0(\lambda,\alpha,1,0;1) = \mathcal{S}^*(\lambda,\alpha)$ and $0 - \mathcal{V}UCV_0(\lambda,\alpha,1) = \mathcal{K}(\lambda,\alpha)$ (Altintas and Owa [1]).
4. $\mathcal{VS}_0(0,\alpha,1,k) = k - ST(\alpha)$ and $k - \mathcal{V}UCV_0(0,\alpha,1) = k - UCV(\alpha)$ (Bharati et al. [2] and Shams et al [15])
5. $\mathcal{S}(0,\alpha,1,1) = S_p(\alpha)$ and $1 - \mathcal{U}(0,\alpha,1) = UCV(\alpha)$ (Rønning [14])
6. $\mathcal{S}(0,0,1,k) = k - St$ and $k - \mathcal{U}(0,1,1) = k - UCV$ (Kanas and Wisnowska [9, 10] and Subramanian et al. [19])
7. $\mathcal{VS}_0(0,\alpha,\beta,k) = k - ST(\alpha,\beta)$ and $k - UCV(0,\alpha,\beta) = k - UCV(\alpha,\beta)$ (Sim et al [18] and El-Ashwah et al. [5]).

8. $\mathcal{VS}_0(\lambda,\alpha,1,k) = k - ST(\lambda,\alpha)$ and $k - \mathcal{V}UCV_0(\lambda,\alpha,1) = UCV(\lambda,\alpha,k)$ (Murugusundaramoorthy and Magesh [13]).

The main object of this paper is to obtain the sufficient coefficient conditions for functions f of the form (1.1) to be in the classes $\mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$ and $k - \mathcal{V}UCV_\eta(\lambda,\alpha,\beta,k)$. We show that they are also the necessary condition for functions belong to those classes. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and results on modified Hadamard products for the class aforementioned classes.

2. Main Results

In the first theorem of this section, we obtain sufficient condition for functions $f(z)$ in the class $\mathcal{S}(\lambda,\alpha,\beta,k)$.

Theorem 2.1 *A function $f(z)$ of the form* (1.1) *is in* $\mathcal{S}(\lambda,\alpha,\beta,k)$ *if*

$$\sum_{n=2}^{\infty}[\phi_n(1+k) - (k\beta+\alpha)\psi_n]|a_n| \leq 1-\alpha-k(1-\beta), \tag{2.1}$$

where

$$\phi_n = n, \quad \psi_n = [1+\lambda(n-1)] \tag{2.2}$$

and $-1 \leq \alpha < \beta \leq 1$, $0 \leq \lambda < 1$, $k(1-\beta) < 1-\alpha$ *and* $z \in \mathcal{U}$.

Proof. It suffices to show that the inequality (1.2) holds true. Upon using the fact that

$$\Re(w) > k|w-\beta| + \alpha \quad \text{iff} \quad \Re((1+ke^{i\theta})w - \beta ke^{i\theta}) > \alpha, \tag{2.3}$$

then the inequality (1.2) may be written as

$$\Re\left((1+ke^{i\theta})\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - \beta ke^{i\theta}\right) \geq \alpha.$$

That is,

$$\Re\left(\frac{A(z)}{B(z)}\right) > \alpha,$$

where

$$A(z) = (1+ke^{i\theta})zf'(z) - \beta ke^{i\theta}[(1-\lambda)f(z)+\lambda zf'(z)]$$

and

$$B(z) = (1-\lambda)f(z)+\lambda zf'(z)$$

then we have

$$|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \geq 0. \tag{2.4}$$

Now,

$$\begin{aligned}|A(z)+(1-\alpha)B(z)| &= \Big|((1-\beta)ke^{i\theta}+2-\alpha)z \\ &\qquad -\sum_{n=2}^{\infty}[(\beta\psi_n-\phi_n)ke^{i\theta}-(1-\alpha)\psi_n-\phi_n]a_nz^n\Big| \\ &\geq (-(1-\beta)k+2-\alpha)|z| \\ &\qquad -\sum_{n=2}^{\infty}[(\beta\psi_n-\phi_n)k+(1-\alpha)\psi_n+\phi_n]|a_n||z|^n \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}|A(z)-(1+\alpha)B(z)| &= \Big|((1-\beta)ke^{i\theta}-\alpha)z \\ &\qquad +\sum_{n=2}^{\infty}[(\phi_n-\beta\psi_n)ke^{i\theta}+\phi_n-(1+\alpha)\psi_n]a_nz^n\Big| \\ &\leq ((1-\beta)k+\alpha)|z| \\ &\qquad +\sum_{n=2}^{\infty}[(\phi_n-\beta\psi_n)k-(1+\alpha)\psi_n+\phi_n]|a_n||z|^n. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\ &\geq [2(1-\alpha)-2k(1-\beta)]|z|-2\sum_{n=2}^{\infty}[(\phi_n-\beta\psi_n)k+(\phi_n-\alpha\psi_n)]|a_n||z|^n \\ &=2\left[[(1-\alpha)-k(1-\beta)]|z|-\sum_{n=2}^{\infty}[\phi_n(1+k)-\psi_n(k\beta+\alpha)]|a_n||z|^n\right].\end{aligned}$$

The last expression is bounded below by 0 if

$$\sum_{n=2}^{\infty}[\phi_n(1+k)-(k\beta+\alpha)\psi_n]|a_n|\leq 1-\alpha-k(1-\beta),$$

and hence the proof is complete. □

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f\in\mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$.

Theorem 2.2. *Let $f(z)$ of the form* (1.1) *and in $\mathcal{V}_\eta$, then $f\in\mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$ if and only if*

$$\sum_{n=2}^{\infty}[\phi_n(1+k)-(k\beta+\alpha)\psi_n]\ |a_n|\leq 1-\alpha-k(1-\beta). \tag{2.7}$$

Proof. In view of Theorem 2.1, we need only to show that $f(z) \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ satisfies the coefficient inequality (2.7). If $f(z) \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ then by definition, we have

$$\Re\left(\frac{(1-\alpha)+\sum\limits_{n=2}^{\infty}(\phi_n-\alpha\psi_n)\ a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\psi_n\ a_n z^{n-1}}\right) \geq k\left|\frac{(1-\beta)+\sum\limits_{n=2}^{\infty}(\phi_n-\beta\psi_n)\ a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\psi_n\ a_n z^{n-1}}\right|.$$

Since f is a function of the form (1.1) with the argument property given in the class $\mathcal{V}_\eta$ and setting $z = re^{i\eta}$ in the above inequality, we have

$$\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}(\phi_n-\alpha\psi_n)\ |a_n| r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\psi_n\ |a_n| r^{n-1}} \geq k\frac{(1-\beta)+\sum\limits_{n=2}^{\infty}(\phi_n-\beta\psi_n)\ |a_n| r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\psi_n\ |a_n| r^{n-1}}. \tag{2.8}$$

Letting $r \to 1$, (2.8) leads the desired inequality

$$\sum_{n=2}^{\infty}[\phi_n(1+k)-(k\beta+\alpha)\psi_n]\ |a_n| \leq 1-\alpha-k(1-\beta),\ -1 \leq \alpha < 1,\ k \geq 0.$$

Finally, the function $f(z)$ given by

$$f_{n,\eta}(z) = z - \frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}z^n,\ 0 \leq \eta < 2\pi,\ n \geq 2, \tag{2.9}$$

where ϕ_2 and ψ_2 as written in (2.2), is extremal for the function. □

Corollary 2.1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$. Then*

$$|a_n| \leq \frac{1-\alpha-k(1-\beta)}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]},\ \ n \geq 2. \tag{2.10}$$

The equality in (2.10) *is attained for the function $f(z)$ given by* (2.9).

Using the same technique of Theorem 2.1, we state the following theorem without proof.

Theorem 2.3. *A function $f(z)$ of the form* (1.1) *is in $k-UCV(\lambda, \alpha, \beta, k)$ if*

$$\sum_{n=2}^{\infty} n[\phi_n(1+k)-(k\beta+\alpha)\psi_n]|a_n| \leq 1-\alpha-k(1-\beta), \tag{2.11}$$

where ϕ_n and ψ_n are given by (2.2).

In the following theorem, it is shown that the condition (2.11) is also necessary for functions $f \in k - \mathcal{V}UCV_\eta(\lambda, \alpha, \beta, k)$. The proof is lines similar to the proof of Theorem 2.2, so we skip the details.

Theorem 2.4. *Let $f(z)$ of the form* (1.1) *and in $\mathcal{V}_\eta$, then $f \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ if and only if*

$$\sum_{n=2}^{\infty} n[\phi_n(1+k) - (k\beta+\alpha)\psi_n]\ |a_n| \leq 1 - \alpha - k(1-\beta), \tag{2.12}$$

where ϕ_n and ψ_n are given by (2.2).

Next, we obtain growth and distortion bounds for functions in the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$.

Theorem 2.5. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$. Then for $|z| < r = 1$*

$$r - \frac{1-\alpha-k(1-\beta)}{[\phi_2(1+k) - (k\beta+\alpha)\psi_2]} r^2 \leq |f(z)| \leq r + \frac{1-\alpha-k(1-\beta)}{[\phi_2(1+k) - (k\beta+\alpha)\psi_2]} r^2. \tag{2.13}$$

and

$$1 - \frac{2(1-\alpha-k(1-\beta))}{[\phi_2(1+k) - (k\beta+\alpha)\psi_2]} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha-k(1-\beta))}{[\phi_2(1+k) - (k\beta+\alpha)\psi_2]} r. \tag{2.14}$$

The result (2.13) *is attained for the function $f(z)$ given by* (2.9) *for $z = \pm r$.*

Proof. The proof of the Theorem 2.5, follows on line similar to the proof of the theorem on distortion bounds given in [5]. □

Theorem 2.6. *Let $f \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ with argument property as in the class $\mathcal{V}_\eta$. Define $f_j(z) = z$, and*

$$f_{n,\eta}(z) = z - \frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\phi_n(1+k) - (k\beta+\alpha)\psi_n]} z^n, \quad 0 \leq \eta < 2\pi,\ n \geq 2. \tag{2.15}$$

Then $f(z)$ is in the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_{n,\eta}(z), \tag{2.16}$$

where $\mu_n \geq 0$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. The proof of the Theorem 2.6, follows on lines similar to the proof of the theorem on extreme points given in [5]. □

Next, we obtain the radius of close-to-convexity for the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$.

Theorem 2.7. *Let $f \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$. Then $f(z)$ is close-to-convex of order σ $(0 \leq \sigma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \inf \left[\frac{(1-\sigma)[\phi_n(1+k) - (k\beta + \alpha)\psi_n]}{n(1-\alpha-k(1-\beta))} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \tag{2.17}$$

The result is sharp, with extremal function $f(z)$ given by (2.9).

Proof. Given $f \in \mathcal{V}_\eta$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma. \tag{2.18}$$

For the left hand side of (2.18) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n|a_n||z|^{n-1}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\sigma}|a_n||z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$, if and only if

$$\sum_{n=2}^{\infty} \frac{[\phi_n(1+k) - (k\beta + \alpha)\psi_n]}{(1-\alpha-k(1-\beta))}|a_n| \leq 1.$$

We can say (2.18) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{[\phi_n(1+k) - (k\beta + \alpha)\psi_n]}{(1-\alpha-k(1-\beta))}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[\phi_n(1+k) - (k\beta + \alpha)\psi_n]}{n(1-\alpha-k(1-\beta))} \right],$$

which completes the proof. □

Employing the technique as in Theorem 2.7, we state , radii of starlikeness and convexity for the class $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$ in the following theorem with out proof.

Theorem 2.8. *Let $f \in \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$. Then*

(i) *f is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_2$; where*

$$r_2 = \inf \left[\left(\frac{1-\sigma}{n-\sigma} \right) \frac{[\phi_n(1+k) - (k\beta + \alpha)\psi_n]}{(1-\alpha-k(1-\beta))} \right]^{\frac{1}{n-1}}, \; n \geq 2, \tag{2.19}$$

(ii) *f is convex of order σ $(0 \le \sigma < 1)$ in the unit disc $|z| < r_3$, where*

$$r_3 = \inf\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}\right]^{\frac{1}{n-1}},\ n \ge 2. \tag{2.20}$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.9).

3. Results on Modified Hadamard Product

Let the functions $f_j(z)(j = 1,2)$ be defined by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,i}z^n, \quad a_{n,i} \ge 0; i = 1,2, \tag{3.1}$$

then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{3.2}$$

Now, we prove the following.

Theorem 3.1. *Let each of the functions $f_j(z)(j = 1,2)$ defined by* (3.1) *be in the class $\mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$. Then $(f_1 * f_2) \in \mathcal{VS}_\eta(\lambda,\delta_1,k)$, for*

$$\delta_1 = \frac{[\phi_2(1+k)-(k\beta+\alpha)\psi_2]^2 - [\phi_2(1+k)-k\beta\psi_2](1-\alpha-k(1-\beta))^2}{[\phi_2(1+k)-(k\beta+\alpha)\psi_2]^2 - \psi_2(1-\alpha-k(1-\beta))^2}. \tag{3.3}$$

The result is sharp.

Proof. We need to prove the largest δ_1 such that

$$\sum_{n=2}^{\infty} \frac{[\phi_n(1+k)-(\delta_1+k\beta)\psi_n]}{(1-\delta_1-k(1-\beta)} a_{n,1}a_{n,2} \le 1. \tag{3.4}$$

From Theorem 2.2, and using the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)} \sqrt{a_{n,1}a_{n,2}} \le 1. \tag{3.5}$$

Thus it is sufficient to show that

$$\frac{[\phi_n(1+k)-(\delta_1+k\beta)\psi_n]}{(1-\delta_1-k(1-\beta)} a_{n,1}a_{n,2} \le \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)} \sqrt{a_{n,1}a_{n,2}},\ n \ge 2 \tag{3.6}$$

that is

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n](1-\delta_1-k(1-\beta)}{[\phi_n(1+k)-(\delta_1+k\beta)\psi_n](1-\alpha-k(1-\beta))},\ n \ge 2. \tag{3.7}$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1-\alpha-k(1-\beta))}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}, \; n \geq 2. \tag{3.8}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha-k(1-\beta))}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]} \leq \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n](1-\delta_1-k(1-\beta)}{[\phi_n(1+k)-(\delta_1+k\beta)\psi_n](1-\alpha-k(1-\beta))}, \tag{3.9}$$

or equivalently

$$\delta_1 \leq \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-[\phi_n(1+k)-k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-\psi_n(1-\alpha-k(1-\beta))^2} = \Delta(n). \tag{3.10}$$

Since $\Delta(n)$ is an increasing function of $n(n \geq 2)$, letting $n=2$ in (3.10) we obtain

$$\delta_1 \leq \Delta(2) = \frac{[\phi_2(1+k)-(\alpha+k\beta)\psi_2]^2-[\phi_2(1+k)-k\beta\psi_2](1-\alpha-k(1-\beta))^2}{[\phi_2(1+k)-(\alpha+k\beta)\psi_2]^2-\psi_2(1-\alpha-k(1-\beta))^2} \tag{3.11}$$

which proves the main assertion of Theorem 3.1. The result is sharp for the functions defined by (2.9). □

Theorem 3.2. *Let the function $f_j(z)(j=1,2)$ defined by (3.1) be in the class $\mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$. If the sequence $\{\phi_n(1+k)-(k\beta+\alpha)\psi_n\}$ is non-decreasing. Then the function*

$$h(z) = z - \sum_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{3.12}$$

belongs to the class $\mathcal{VS}_\eta(\lambda,\delta_2,k)$ where

$$\delta_2 = \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-2[\phi_n(1+k)-k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-2\psi_n(1-\alpha-k(1-\beta))^2}.$$

Proof. By virtue of Theorem 2.2, for $f_j(z)(j=1,2) \in \mathcal{VS}_\eta(\lambda,\alpha,\beta,k)$, one could get

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2(a_{n,1}^2+a_{n,2}^2) \leq 1. \tag{3.13}$$

Therefore we need to find the largest δ_2, such that

$$\frac{[\phi_n(1+k)-(\delta_2+k\beta)\psi_n]}{(1-\delta_2-k(1-\beta)} \leq \frac{1}{2}\left[\frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2, \quad n \geq 2$$

that is

$$\delta_2 \leq \frac{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-2[\phi_n(1+k)-k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\phi_n(1+k)-(k\beta+\alpha)\psi_n]^2-2\psi_n(1-\alpha-k(1-\beta))^2} = \Psi(n).$$

Since $\Psi(n)$ is an increasing function of $n, (n \geq 2)$, we readily have

$$\delta_2 \leq \Psi(2) = \frac{[\phi_2(1+k) - (\alpha + k\beta)\psi_2]^2 - 2[\phi_2(1+k) - k\psi_2](1-\alpha-k(1-\beta))^2}{[\phi_2(1+k) - (\alpha+k)\psi_2]^2 - 2\psi_2(1-\alpha-k(1-\beta))^2}$$

which completes the proof. □

Remark 3.1. By employing the techniques as used in Theorems 2.5, 2.6, 2.7, 2.8, 3.1 and 3.2, one can restate the above theorems for the class $k - \mathcal{V}UCV_\eta(\lambda, \alpha, \beta, k)$. So we omitted the details to the readers interest.

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