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Approximation of Common Fixed Points of Two Strictly Pseudononspreading Multivalued Mappings in ℝ-Trees

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Abstract. In this paper, we introduce and study a new multivalued mapping in ℝ-trees, called *k-strictly pseudononspreading*. We also introduce a new two-step iterative process for two k-strictly pseudononspreading multivalued mappings in ℝ-trees. Strong convergence theorems of the proposed iteration to a common fixed point of two k-strictly pseudononspreading multivalued mappings in ℝ-trees are established. Our results improve and extend the corresponding results existing in the literature.

1. Introduction

ℝ-trees were introduced by Tits [19] in 1977. Fixed point theory for single-valued mappings in ℝ-trees was first studied by Kirk [10]. He proved that every continuous single-valued mappings defined on a geodesically bounded complete ℝ-tree always has a fixed point. Since then fixed point theorems for various types of single-valued and multivalued mappings in ℝ-trees has been rapidly developed and many of papers have appeared (see e.g. [1],[2],[3],[8],[12]). It is worth mentioning that fixed point theorems in ℝ-trees can be applied to graph theory, biology and computer science (see e.g., [4],[6],[11],[17]).

In 2009, Shahzad and Zegeye [18] proved strong convergence theorems of the Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying the end-point condition in Banach spaces. Later in 2010, Puttasontiphot [15] obtained similar results in complete CAT(0) spaces. In 2012, Samanmit and Panyanak [16] introduced a condition on mappings in ℝ-trees which is more general than the end-point condition, call it the *gate condition*, and proved strong convergence theorems of a modified Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying such condition.

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In 2011, Osilike and Isiogugu [14] introduced a new single-valued mapping in a Hilbert space, namely k-strictly pseudononspreading. Recall that a single-valued mapping T is called *k-strictly pseudononspreading* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.1}$$

for all $x, y \in D(T)$. In a Hilbert space, it is easy to show that (1.1) is equivalent to

$$(2-k)\|Tx - Ty\|^2 \leq k\|x - y\|^2 + (1-k)\|y - Tx\|^2 + (1-k)\|x - Ty\|^2 + k\|x - Tx\|^2 + k\|y - Ty\|^2$$

for all $x, y \in D(T)$. Osilike and Isiogugu also proved weak and strong convergence theorems for approximating fixed points of k-strictly pseudononspreading mappings in Hilbert spaces.

However, up to now, no researchers have studied fixed point theorems for multivalued k-strictly pseudononspreading mappings even in Hilbert spaces, Banach spaces, CAT(0) spaces and $\mathbb{R}$-trees setting. The propose of this paper is to introduce the concept of k-strictly pseudononspreading multivalued mappings and to obtain the strong convergence theorems for approximating a common fixed point of those multivalued mappings in the framework of $\mathbb{R}$-trees under the gate condition.

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. For each $x, y \in X$ and $\alpha \in (0, 1)$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1-\alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$.

A nonempty subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points. A nonempty subset D of X is said to be *gated* if for any point $x \notin D$ there is a unique point y_x such that for any $z \in D$,

$$d(x, z) = d(x, y_x) + d(y_x, z).$$

Clearly gated sets in a complete geodesic space are always closed and convex. The point y_x is called the *gate* of x in D. It is easy to see that y_x is also the unique nearest point of x in D.

Definition 2.1. An $\mathbb{R}$-tree is a geodesic metric space X satisfying

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$;

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

It follows by (i) and (ii) that

(iii) if $u, v, w \in X$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in X$.

An $\mathbb{R}$-tree is a special case of a CAT(0) space. Note that a metric space X is a complete $\mathbb{R}$-tree if and only if X is hyperconvex with unique metric segments; see [9].

Let $T : D \to 2^D$ be a multivalued mapping. An element $z \in D$ is called *fixed point* of T if $z \in Tz$. An element $p \in D$ is said to be an *endpoint* of T if p is a fixed point of T and $Tp = \{p\}$ (see [20]). We shall denote by $F(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. It is easy to see that for each T, $End(T) \subseteq F(T)$ and the converse is not true in general. If $End(T) = F(T)$ then we say that T satisfies the *endpoint condition*.

We shall denote the family of nonempty closed bounded subsets of D by $CB(D)$, the family of nonempty closed convex subsets of D by $CC(D)$, and the family of nonempty compact convex subsets of D by $KC(D)$. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\right\} \quad \text{for } A, B \in CB(D).$$

The multivalued mapping $T : D \to CB(D)$ is called

(i) *nonexpansive* if $H(Tx, Ty) \le d(x, y)$ for all $x, y \in D$;

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tz) \le d(x, z)$ for all $x \in D$ and $z \in F(T)$;

(iii) *L-Lipschitzian* if there exists a constant $L > 0$ such that $H(Tx, Ty) \le Ld(x, y)$ for all $x, y \in D$;

(iv) *hemicompact* if for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$. We note that if D is compact, then every multivalued mapping $T : D \to CB(D)$ is hemicompact.

It is clear that every nonexpansive multivalued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive but there exist quasi-nonexpansive mappings that are not nonexpansive.

Let $T : D \to CC(D)$ be a multivalued mapping with $F(T) \neq \emptyset$. We say that a point $u \in D$ is a *key* of T if, for each $x \in F(T)$, x is the gate of u in Tx. We say that T satisfies the *gate condition* if T has a key in D. It is clear that the endpoint condition implies the gate condition but the converse is not true.

We now collect some basic properties of $\mathbb{R}$-trees.

Lemma 2.2. *Let X be a complete $\mathbb{R}$-tree. Then the following statements hold:*

(i) [7] *if $x, y, z \in X$ and $\alpha \in [0, 1]$, then*

$$d(z, \alpha x \oplus (1-\alpha)y)^2 \le \alpha d(z, x)^2 + (1-\alpha)d(z, y)^2 - \alpha(1-\alpha)d(x, y)^2;$$

(ii) [7] *if $x, y, z \in X$, then $d(x,z) + d(z,y) = d(x,y)$ if and only if $z \in [x,y]$;*

(iii) [12] *if A and B are bounded closed convex subsets of X, then*

$$d(P_A(u), P_B(u)) \leq H(A,B)$$

for any $u \in X$, where $P_A(u), P_B(u)$ are respectively the unique nearest points of u in A and B;

(iv) [8] *the gate subsets of X are precisely its closed and convex subsets.*

We state the following conditions in $\mathbb{R}$-trees:

A multivalued mapping $T : D \to CB(D)$ is said to satisfy *condition (I)* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\operatorname{dist}(x, Tx) \geq f(\operatorname{dist}(x, F(T)))$ for all $x \in D$.

Two multivalued mappings $T_1, T_2 : D \to CB(D)$ are said to satisfy *condition (II)* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that either $\operatorname{dist}(x, T_1x) \geq f(\operatorname{dist}(x, F(T_1) \cap F(T_2)))$ or $\operatorname{dist}(x, T_2x) \geq f(\operatorname{dist}(x, F(T_1) \cap F(T_2)))$ for all $x \in D$.

The following results are needed for proving our results.

Lemma 2.3.([13]) *Let X be a complete metric space, A be a bounded closed subset of X, and B be a compact subset of X. If $x \in A$ then there exists $y \in B$ such that $d(x,y) \leq H(A,B)$.*

Proposition 2.4.([5]) *Let (X,d) be a complete metric space and F be a nonempty closed subset of X. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F$ and $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to some point in F if and only if $\lim_{n\to\infty}$ dist(x_n, F).*

3. Main Results

Definition 3.1. Let D be a nonempty subset of a complete $\mathbb{R}$-tree X. A multivalued mapping $T : D \to CB(D)$ is called

(i) *nonspreading* if

$$2H(Tx,Ty)^2 \leq \operatorname{dist}(y,Tx)^2 + \operatorname{dist}(x,Ty)^2$$

for all $x, y \in D$.

(ii) *k-strictly pseudononspreading* if there exists $k \in [0,1)$ such that

$$(2-k)H(Tx,Ty)^2 \leq kd(x,y)^2 + (1-k)\operatorname{dist}(y,Tx)^2 + (1-k)\operatorname{dist}(x,Ty)^2 + k\operatorname{dist}(x,Tx)^2 + k\operatorname{dist}(y,Ty)^2$$

for all $x, y \in D$.

Clearly every nonspreading multivalued mapping is 0-strictly pseudononspreading. It is clear that if T is k-strictly pseudononspreading and has a fixed point, then for all $x \in D$ and $p \in F(T)$ we have

$$H(Tx, Tp)^2 \leq d(x,p)^2 + k \operatorname{dist}(x, Tx)^2.$$

So, T may not be quasi-nonexpansive. It is easy to show that if T is a k-strictly pseudononspreading multivalued mapping with $F(T) \neq \emptyset$, then $F(T)$ is closed. Indeed, we let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. By the definition of T, we have

$$\begin{aligned} H(Tx, Tx_n)^2 &\leq d(x, x_n)^2 + k \operatorname{dist}(x, Tx)^2 \\ &\leq (d(x, x_n) + \sqrt{k} \operatorname{dist}(x, Tx))^2. \end{aligned}$$

Then

$$H(Tx, Tx_n) \leq d(x, x_n) + \sqrt{k} \operatorname{dist}(x, Tx).$$

It follows that

$$\begin{aligned} \operatorname{dist}(x, Tx) &\leq \operatorname{dist}(x, Tx_n) + H(Tx_n, Tx) \\ &\leq 2d(x, x_n) + \sqrt{k} \operatorname{dist}(x, Tx). \end{aligned}$$

By letting $n \to \infty$ in above inequality, we have $(1 - \sqrt{k}) \operatorname{dist}(x, Tx) \leq 0$. Since $k \in [0, 1)$, we get $\operatorname{dist}(x, Tx) = 0$. Hence, $x \in Tx$ so that $F(T)$ is closed.

In order to prove our main results, the following lemma is needed.

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Assume that $T : D \to KC(D)$ is a k-strictly pseudononspreading multivalued mapping. If $\{x_n\}$ is a sequence in D such that $x_n \to x$ and dist$(x_n, Tx_n) \to 0$ as $n \to \infty$, then $x \in Tx$.*

Proof. Let $\{x_n\}$ be a sequence in D such that $x_n \to x$ and $\operatorname{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. For each n, let $y_n \in Tx_n$ such that $d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n)$. Since Tx is compact, by Lemma 2.3, there exist $z_n \in Tx$ such that $d(y_n, z_n) \leq H(Tx_n, Tx)$. Then, we have

$$\begin{aligned} (2-k)d(y_n, z_n)^2 &\leq (2-k)H(Tx_n, Tx)^2 \\ &\leq kd(x_n, x)^2 + (1-k)\operatorname{dist}(x, Tx_n)^2 + (1-k)\operatorname{dist}(x_n, Tx)^2 \\ &\quad + k \operatorname{dist}(x_n, Tx_n)^2 + k \operatorname{dist}(x, Tx)^2 \\ &\leq kd(x_n, x)^2 + (1-k)(d(x, y_n)^2 + d(x_n, z_n)^2) + k \operatorname{dist}(x_n, Tx_n)^2 \\ &\quad + k \operatorname{dist}(x, Tx)^2 \\ &\leq kd(x_n, x)^2 + (1-k)(d(x, y_n) + d(x_n, z_n))^2 + k \operatorname{dist}(x_n, Tx_n)^2 \\ &\quad + k \operatorname{dist}(x, Tx)^2 \\ &\leq kd(x_n, x)^2 + (1-k)(d(x, x_n) + 2d(x_n, y_n) + d(y_n, z_n))^2 \\ &\quad + k \operatorname{dist}(x_n, Tx_n)^2 + k \operatorname{dist}(x, Tx)^2. \end{aligned} \tag{3.1}$$

Compactness of Tx implies that there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to z \in Tx$ as $i \to \infty$. Then, it follows by (3.1) that

$$\lim_{i\to\infty} d(y_{n_i}, z_{n_i}) \leq \sqrt{k}\,\mathrm{dist}(x, Tx).$$

This implies

$$\begin{aligned} d(x,z) &= \lim_{i\to\infty} d(x_{n_i}, z) \\ &\leq \lim_{i\to\infty} d(x_{n_i}, y_{n_i}) + d(y_{n_i}, z_{n_i}) + d(z_{n_i}, z) \\ &= \lim_{i\to\infty} d(y_{n_i}, z_{n_i}) \\ &\leq \sqrt{k}\,\mathrm{dist}(x, Tx) \\ &\leq \sqrt{k}\,d(x,z). \end{aligned}$$

Thus, $(1-\sqrt{k})d(x,z) = 0$. Since $k \in [0,1)$, it implies that $x = z \in Tx$. □

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T_1 : D \to KC(D)$ be a k-strictly pseudononspreading multivalued mapping and $T_2 : D \to KC(D)$ be a k-strictly pseudononspreading and L-Lipschitzian multivalued mapping with $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that T_1, T_2 satisfy the gate condition. Let u_1, u_2 be keys of T_1, T_2, respectively. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n^{(1)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(1)}$ is the gate of u_1 in $T_1 x_n$, and

$$x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n^{(2)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(2)}$ is the gate of u_2 in $T_2 y_n$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1-k$. If one of the following is satisfied:

(i) *T_1, T_2 satisfy condition (II),*

(ii) *T_1 or T_2 is hemicompact,*

then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof. For each $p \in \mathcal{F}$, we obtain by the gate condition and Lemma 2.2(i) that

$$\begin{aligned}
d(x_{n+1},p)^2 &\le (1-\beta_n)d(y_n,p)^2 + \beta_n d(z_n^{(2)},p)^2 - \beta_n(1-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\beta_n)d(y_n,p)^2 + \beta_n d(P_{T_2y_n}(u_2),P_{T_2p}(u_2))^2 - \beta_n(1-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\beta_n)d(y_n,p)^2 + \alpha_n H(T_2y_n,T_2p)^2 - \beta_n(1-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\beta_n)d(y_n,p)^2 + \beta_n(d(y_n,p)^2 + k\,\mathrm{dist}(y_n,T_2y_n)^2) \\
&\quad\; - \beta_n(1-\beta_n)d(y_n,z_n^{(2)})^2 \\
&= d(y_n,p)^2 - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(z_n^{(1)},p)^2 - \alpha_n(1-\alpha_n)d(x_n,z_n^{(1)})^2 \\
&\quad\; - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(P_{T_1x_n}(u_1),P_{T_1p}(u_1))^2 - \alpha_n(1-\alpha_n)d(x_n,z_n^{(1)})^2 \\
&\quad\; - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n H(T_1x_n,T_1p)^2 - \alpha_n(1-\alpha_n)d(x_n,z_n^{(1)})^2 \\
&\quad\; - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \\
&\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n(d(x_n,p)^2 + k\,\mathrm{dist}(x_n,T_1x_n)^2) \\
&\quad\; - \alpha_n(1-\alpha_n)d(x_n,z_n^{(1)})^2 - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \\
&= d(x_n,p)^2 - \alpha_n(1-k-\alpha_n)d(x_n,z_n^{(1)})^2 - \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2. \qquad (3.2)
\end{aligned}$$

This implies by $\alpha_n, \beta_n < 1-k$ that $d(x_{n+1},p) \le d(x_n,p)$ for all $n \in \mathbb{N}$. Hence, $\{d(x_n,p)\}$ is nonincreasing and bounded below. It follows that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in \mathcal{F}$. Using (3.2), we get that

$$\alpha_n(1-k-\alpha_n)d(x_n,z_n^{(1)})^2 + \beta_n(1-k-\beta_n)d(y_n,z_n^{(2)})^2 \le d(x_n,p)^2 - d(x_{n+1},p)^2.$$

This implies from $0 < a \le \alpha_n, \beta_n \le b < 1-k$ that

$$\lim_{n\to\infty} d(x_n,z_n^{(1)}) = 0 \text{ and } \lim_{n\to\infty} d(y_n,z_n^{(2)}) = 0. \qquad (3.3)$$

Also

$$\mathrm{dist}(x_n,T_1x_n) \le d(x_n,z_n^{(1)}) \to 0 \text{ as } n \to \infty. \qquad (3.4)$$

By the definition of T_2, we get that

$$\begin{aligned}\operatorname{dist}(x_n, T_2x_n) &\le \operatorname{dist}(x_n, T_2y_n) + H(T_2y_n, T_2x_n)\\ &\le d(x_n, z_n^{(2)}) + Ld(y_n, x_n)\\ &\le (1+L)d(x_n, y_n) + d(y_n, z_n^{(2)})\\ &\le (1+L)(d(x_n, z_n^{(1)}) + d(z_n^{(1)}, y_n)) + d(y_n, z_n^{(2)})\\ &= (1+L)(d(x_n, z_n^{(1)}) + (1-\alpha_n)d(z_n^{(1)}, x_n)) + d(y_n, z_n^{(2)})\\ &= (1+L)(2-\alpha_n)d(x_n, z_n^{(1)}) + d(y_n, z_n^{(2)})\\ &\le (1+L)(2-a)d(x_n, z_n^{(1)}) + d(y_n, z_n^{(2)}).\end{aligned}$$

This implies by (3.3) that

$$\operatorname{dist}(x_n, T_2x_n) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Case (i): T_1, T_2 satisfy the condition (II). Then, we have by (3.4) and (3.5) that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By the closedness of $\mathcal{F}$ and Proposition 2.4, we have $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.

Case (ii): T_1 or T_2 is hemicompact. Without loss of generality, we assume that T_1 is hemicompact. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $z \in D$. By (3.4) and (3.5), it follows by Lemma 3.2 that $z \in \mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$, it implies that $\{x_n\}$ converges strongly to $z \in \mathcal{F}$. □

Since any nonspreading multivalued mapping is 0-strictly pseudononspreading multivalued mapping, the next corollary is obtained immediately from Theorem 3.3.

Corollary 3.4. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T_1 : D \to KC(D)$ be a nonspreading multivalued mapping and $T_2 : D \to KC(D)$ be a nonspreading and L-Lipschitzian multivalued mapping with $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that T_1, T_2 satisfy the gate condition. Let u_1, u_2 be keys of T_1, T_2, respectively. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n^{(1)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(1)}$ is the gate of u_1 in T_1x_n, and

$$x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n^{(2)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(2)}$ is the gate of u_2 in T_2y_n. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. If one of the following is satisfied:

(i) *T_1, T_2 satisfy condition (II),*

(ii) *T_1 or T_2 is hemicompact,*

then $\{x_n\}$ *converges strongly to an element of* $\mathcal{F}$.

Using the same arguments as in the proof of Theorem 3.3, we obtain the following result for approximating a fixed point of a k-strictly pseudononspreading multivalued mapping in $\mathbb{R}$-trees.

Theorem 3.5. *Let* D *be a nonempty closed convex subset of a complete* $\mathbb{R}$*-tree* X*. Let* $T : D \to KC(D)$ *be a* k*-strictly pseudononspreading multivalued mapping with* $F(T) \neq \emptyset$*. Suppose that* T *satisfies the gate condition. Let* u *be a key of* T*. For* $x_1 \in D$*, the sequence* $\{x_n\}$ *generated by*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n z_n \text{ for all } n \in \mathbb{N},$$

where z_n *is the gate of* u *in* Tx_n*. Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$ *such that* $0 < a \leq \alpha_n \leq b < 1 - k$*. If either* T *satisfy condition* (I) *or* T *is hemicompact, then* $\{x_n\}$ *converges strongly to an element of* $F(T)$*.*

Proof. In Theorem 3.3, put $T_1 = T$ and $T_2 = I$ (identity mapping). Hence, we obtain the desired result from Theorem 3.3. □

Remark 3.6. Theorem 3.3 extends [16] to the case of two k-strictly pseudononspreading multivalued mappings and our iteration process is different from the iteration process defined by Samanmit and Panyanak [16].

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References

[1] A. G. Aksoy and M. A. Khamsi, *A selection theorem in metric trees*, Proc. Amer. Math. Soc., **134**(2006), 2957-2966.

[2] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Some fixed point results on a metric space with a graph*, Topology Appl., **159**(2012), 659-663.

[3] A. Amini-Harandi and A. P. Farajzadeh, *Best approximation, coincidence and fixed point theorems for set-valued maps in* $\mathbb{R}$*-trees*, Nonlinear Anal., **71**(2009), 1649-1653.

[4] I. Bartolini, P. Ciaccia and M. Patella, *String matching with metric trees using an approximate distance*, SPIR Lecture notes in Comput. Sci. 2476, Springer, Berlin, 1999.

[5] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, NY, USA, 2011.

[6] M. Bestvina, $\mathbb{R}$*-trees in topology, geometry, and group theory*, in: Handbook of Geometric Topology, North-Holland, Amsterdam, 2002, pp. 55-91.

[7] S. Dhompongsa and B. Panyanak, *On* Δ*-convergence theorems in CAT(0) spaces*, Comp. Math. Appl., **56(10)**(2008), 2572-2579.

[8] R. Espínola, W. A. Kirk, *Fixed point theorems in* $\mathbb{R}$*-trees with applications to graph theory*, Topology and its Applications, **153**(2006), 1046-1055.

[9] W. A. Kirk, Hyperconvexity of $\mathbb{R}$-trees, Fundamenta Mathematicae, **156(1)**(1998), 67-72.

[10] W. A. Kirk, *Fixed point theorems in CAT(0) spaces and* $\mathbb{R}$*-trees*, Fixed Point Theory Appl., **2004**(2004), 309-316.

[11] W. A. Kirk, *Some recent results in metric fixed point theory*, J. Fixed Point Theory Appl., **2**(2007), 195-207.

[12] J. T. Markin, *Fixed points, selections and best approximation for multivalued mappings in* $\mathbb{R}$*-trees*, Nonlinear Anal., **67**(2007), 2712-2716.

[13] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.

[14] M. O. Osilike and F. O. Isiogugu, *Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces*, Nonlinear Anal., **74**(2011), 1814-1822.

[15] T. Puttasontiphot, *Mann and Ishikawa iteration schemes for multivalued mappings in CAT(0) spaces*, Applied Mathematical Sciences, **4(61)**(2010), 3005-3018.

[16] K. Samanmit and B. Panyanak, *On multivalued nonexpansive mappings in* $\mathbb{R}$*-trees*, Journal of Applied Mathematics, Volume 2012, Article ID 629149, 13 pages.

[17] C. Semple and M. Steel, *Phylogenetics*, Oxford Lecture Ser. Math. Appl., vol. 24, Oxford Univ. Press, Oxford, 2003.

[18] N. Shahzad and H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal., **71**(2009), 838-844.

[19] J. Tits, *A Theorem of LieKolchin for Trees. Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin*, Academic Press, New York, 1977.

[20] K. Wlodarczyk, D. Klim and R. Plebaniak, *Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces*, J. Math. Anal. Appl., **328**(2007), 46-57.