KYUNGPOOK Math. J. 55(2015), 345-353
http://dx.doi.org/10.5666/KMJ.2015.55.2.345
pISSN 1225-6951 eISSN 0454-8124

Subnormality and Weighted Composition Operators on L^2 Spaces

Mohammad Reza Azimi
Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran
e-mail : mhr.azimi@maragheh.ac.ir

Abstract. Subnormality of bounded weighted composition operators on $L^2(\Sigma)$ of the form $Wf = uf \circ T$, where T is a nonsingular measurable transformation on the underlying space X of a σ-finite measure space (X, Σ, μ) and u is a weight function on X; is studied. The standard moment sequence characterizations of subnormality of weighted composition operators are given. It is shown that weighted composition operators are subnormal if and only if $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x \in X$, where $J_n = h_n E_n(|u|^2) \circ T^{-n}$, $h_n = d\mu \circ T^{-n}/d\mu$ and E_n is the conditional expectation operator with respect to $T^{-n}\Sigma$.

1. Introduction

Weighted composition operators acting on certain function spaces are operators of the form $Wf = uf \circ T$, where T is a measurable transformation on a σ-finite measure space (X, Σ, μ) and u is a weight function on X. These operators include composition operators of the form $Cf = f \circ T$ and multiplication operators with symbol u. So far most general properties of composition operators and major classes of known normal, quasinormal, hyponormal and weak hyponormal composition operators have been widely studied by numerous mathematicians (See [1],[2],[13],[15],[17],[19]). Conditions for weighted composition operators to belong to some certain specific classes are found in [8]. However general properties of weighted composition operators have not been studied as well as composition operators. Originally subnormal composition operators have been studied by A. Lambert in [12] based on the standard moment sequence characterizations. Recently, in [18] subnormality of composition operators have been characterized in the Laplace density case. In this paper we shall use the standard moment sequence

Received July 6, 2013; revised October 19, 2013; accepted October 23, 2013.
2010 Mathematics Subject Classification: Primary 47B20; Secondary 47B38.
Key words and phrases: Subnormal, Weighted composition operators, Conditional expectation, Moment sequence.

characterizations of subnormality of weighted composition operators.

2. Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space and T be a nonsingular measurable transformation from X onto X, i.e. $T^{-1}\Sigma \subseteq \Sigma$ and such that $\mu \circ T^{-1}$ is absolutely continuous with respect to μ which is denoted by $\mu \circ T^{-1} \ll \mu$. Given a real or complex valued measurable weight function u on X, the weighted composition operator on $L^2(\Sigma) := L^2(X, \Sigma, \mu)$ induced by T and u is defined by $Wf = uf \circ T$. It is worth reminding the reader that the nonsingularity of T assures that W is well defined as a mapping of equivalence classes of functions on the support of u. Let $h = h_1 := d\mu \circ T^{-1}/d\mu$ be the Radon-Nikodym derivative and for each $n \geq 1$ let $h_n := d\mu \circ T^{-n}/d\mu$. For $f \in L^2(\Sigma)$ or $f \geq 0$ a.e let $E(f) := E(f|T^{-1}\Sigma)$ be the *conditional expectation* of f with respect to σ-subalgebra $T^{-1}\Sigma$. For each $n \geq 2$, let $E_n(f) := E(f|T^{-n}\Sigma)$. Conditional expectation operator E is defined as an orthogonal projection from $L^2(\Sigma)$ onto $L^2(T^{-1}\Sigma)$ by the assignment $f \to E(f)$. If g is a $T^{-1}\Sigma$-measurable function then $E(fg) = gE(f)$. One may find more noteworthy properties of conditional expectation operator in [6, 16, 2] and the references therein. In [15] it is shown that C defines a bounded operator on $L^2(\Sigma)$ if and only if $h \in L^\infty(\Sigma) := L^\infty(X, \Sigma, \mu)$. Necessary and sufficient condition for W to be bounded on $L^2(\Sigma)$ is that $J := hE(|u|^2) \circ T^{-1} \in L^\infty(\Sigma)$ which is found in [7]. Conditional expectation is an functional projection and we shall use it in this paper frequently. For example observe the recurrent relationship $h_{n+1} = hE(h_n) \circ T^{-1}$ proved in [11].
A measurable transformation $T : X \to X$ is said to be *essentially surjective* if $\mu(X - T(X)) = 0$. It is shown in [9] that a bounded weighted composition operator W is injective if and only if T is essentially bounded. Throughout this paper it is assumed that T is essentially bounded.
Recall that a bounded operator A defined on a Hilbert space is *normal* if and only if $A^*A = AA^*$ (where A^* is a adjoint of A) and A is *subnormal* if A has an extension to a normal operator on a larger Hilbert space. Let $A = U|A|$ be the canonical polar decomposition for A and let $p \in (0, \infty)$. An operator A is *p-hyponormal* if $(A^*A)^p \geq (AA^*)^p$ and A is *p-quasihyponormal* if $A^*(A^*A)^pA \geq A^*(AA^*)^pA$. For all unit vectors $x \in \mathcal{H}$, if $\||A|^pU|A|^px\| \geq \||A|^px\|^2$, then A is called a *p-paranormal* operator(See [5]). For instance consider the known relationships between these classed as follows

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal} \Longrightarrow \text{hyponormal},$$

$$p-\text{hyponormal} \Longrightarrow p-\text{quasihyponormal} \Longrightarrow p-\text{paranormal}.$$

Further information may be found in [1, 8, 2].

3. Subnormal Weighted Composition Operators

The assumption that T is essentially bounded; plays an important role to make use of the following theorem stated in [14] .

Theorem 3.1. *Let A be an operator on a Hilbert space H with kernel$\{A\} = 0$. Then A is subnormal if and only if for every x in H the sequence $\{\|A^n x\|\}_{n=0}^{+\infty}$ is a moment sequence, i.e., there is an interval $I = [0, r]$ such that for each x in H one can find a Borel measure m_x such that for each $n \geq 0$,*

$$\|A^n x\|^2 = \int_I t^n dm_x(t).$$

Remark 3.2. For each $n \in \mathbb{N}$ and $F \in \Sigma$, define the measure λ_u^n by

$$\lambda_u^n(F) = \int_{T^{-n}(F)} |u|^2 d\mu.$$

According to the following chain

$$\lambda_u^n \ll \mu \circ T^{-n} \ll ... \ll \mu \circ T^{-2} \ll \mu \circ T^{-1} \ll \mu,$$

the assumption $\mu \circ T^{-1} \ll \mu$ implies that $\lambda_u^n \ll \mu$. Hence there exists the Radon-Nikodym derivative $J_n := \frac{d\lambda_u^n}{d\mu}$. For $F \in \Sigma$, we have

$$\begin{aligned}\lambda_u^n(F) &= \int_{T^{-n}(F)} |u|^2 d\mu \\ &= \int_{T^{-n}(F)} E_n(|u|^2) d\mu \\ &= \int_F h_n E_n(|u|^2) \circ T^{-n} d\mu.\end{aligned}$$

Then $d\lambda_u^n = h_n[E_n(|u|^2)] \circ T^{-n} d\mu$, since F was chosen arbitrarily. It follows that $J_n = h_n[E_n(|u|^2)] \circ T^{-n}$. Note that $J_1 = J$ and let $J_0 = h_0 = 1$. Moreover, for an arbitrary $F \in \Sigma$,

$$\begin{aligned}\lambda_u^n(F) &= \int_{T^{-n}(T^{-1}F)} |u|^2 d\mu \\ &= \int_{T^{-n}(T^{-1}F)} E_n(|u|^2) d\mu \\ &= \int_{T^{-1}(F)} h_n E_n(|u|^2) \circ T^{-n} d\mu \\ &= \int_{T^{-1}(F)} E(J_n) d\mu \\ &= \int_F hE(J_n) \circ T^{-1} d\mu.\end{aligned}$$

Then $J_{n+1} = hE(J_n) \circ T^{-1}$. If Theorem 3.1 is now applied to a bounded weighted composition operator we have the following.

Theorem 3.3. *W is subnormal if and only if for each* $f \in L^2(\Sigma)$, $\{\int_x J_n|f|^2 d\mu\}_{n=0}^{+\infty}$ *is a moment sequence.*

Proof. Let $f \in L^2(\Sigma)$, by calculating the n-th iteration of W we will have

$$W^n f = \prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n).$$

Thus,

$$\begin{aligned} \|W^n f\|^2 &= \int_X |\prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n)|^2 d\mu \\ &= \int_X \prod_{i=0}^{n-2} |u \circ T^i|^2 |f \circ T^{n-1}|^2 d\lambda_u^1 \\ &= \int_X \prod_{i=0}^{n-3} |u \circ T^i|^2 |f \circ T^{n-2}|^2 d\lambda_u^2 \\ &\vdots \\ &= \int_X |f|^2 d\lambda_u^n \\ &= \int_X J_n |f|^2 d\mu. \end{aligned}$$

□

In what follows we are going to determine the subnormality of the weighted composition operators just by the sequence $\{J_n\}_{n=0}^{+\infty}$ based on the following standard moment sequence characterization.

Theorem 3.4. *Let* $\{\lambda_n\}_{n=0}^{+\infty}$ *be a sequence of positive real numbers. Then* $\{\lambda_n\}_{n=0}^{+\infty}$ *is moment sequence if and only if for some interval I the linear functional* φ *defined on* $P(I)$*- the set of polynomials over* I*- by* $\varphi(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n \lambda_n$ *is positive (i.e.,* $\varphi(p) \geq 0$ *whenever* $P(t) \geq 0$ *on* I*).*

Proof. See [20]. □

The application of above theorem characterizes subnormality of weighted composition operators initially.

Theorem 3.5. *Let* $W \in \mathcal{B}(L^2(\Sigma))$ *and* $I = [0, \|J\|_\infty]$*. Define the linear transformation* $\mathcal{L} : P(I) \to L^\infty(\Sigma)$ *by* $\mathcal{L}(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n J_n$*. Then W is subnormal if and only if* $\mathcal{L}$ *is positive, in a sense that* $\mathcal{L}(p) \geq 0$ *a.e* $[\mu]$ *whenever* $p \geq 0$ *on* I*.*

Proof. Let W be a subnormal and p be an arbitrary nonnegative polynomial on I. Let $K \subseteq \{x \in X : \mathcal{L}(p)(x) < 0\}$ with $\mu(K) < +\infty$. By Theorem 3.3 there is a Borel measure m on I such that for every $n \geq 0$,

$$\int_K J_n d\mu = \int_I t^n dm(t).$$

Then

$$\begin{aligned} \int_K \mathcal{L}(p) d\mu &= \sum_{n=0}^{+\infty} a_n \int_K J_n d\mu \\ &= \sum_{n=0}^{+\infty} a_n \int_I t^n dm(t) \\ &= \int_I p(t) dm(t) \geq 0. \end{aligned}$$

It follows that $\mathcal{L}(p) \geq 0$ a.e, hence $\mathcal{L}$ is positive.
Conversely, suppose that $\mathcal{L}$ is positive. Note that $\mathcal{L}$ is continuous with respect to essential sup norm since $\mathcal{L}$ preserves the unit element of both sides and $J_n(x) \in I = [0, \|J\|_\infty]$ a.e for each $n \geq 0$. Then $\mathcal{L}$ can be normally extended to a positive linear mapping from $C(I)$(the space of all continuous functions on I) into $L^\infty(\Sigma)$. Correspond for each $f \in L^2(\Sigma)$ a linear functional φ_f on $C(I)$ by

$$\varphi_f(g) = \int_X \mathcal{L}(g)|f|^2 d\mu.$$

Since $\mathcal{L}(g) \in L^\infty(\Sigma)$, φ_f is bounded. Actually its norm can not be exceed $\|\mathcal{L}\|\|f\|_2^2$. By the Riesz representation theorem there is a unique Borel measure m_f such that

$$\varphi_f(g) = \int_I g(t) dm_f(t), \qquad g \in C(I).$$

Hence $\int_X \mathcal{L}(g)|f|^2 d\mu = \int_I g(t) dm_f(t)$. For each $n \geq 0$, putting $g(t) = t^n$ in the last equation leads that

$$\int_X J_n |f|^2 d\mu = \int_I t^n dm_f(t).$$

It follows from Theorem 3.3 that W is subnormal. □

Corollary 3.6. *$W \in \mathcal{B}(L^2(\Sigma))$ is subnormal if and only if for every Σ-measurable set A of finite measure the sequence $\{\int_{T^{-n}A} |u|^2 d\mu\}_{n=0}^{+\infty}$ is a moment sequence.*

Proof. First suppose that W is subnormal. Then by Theorem 3.3 we have

$$\begin{aligned}\int_X J_n\chi_A d\mu &= \int_A h_n[E_n(|u|^2)]\circ T^{-n}d\mu\\ &= \int_{T^{-n}A} E_n(|u|^2)d\mu\\ &= \int_{T^{-n}A}|u|^2d\mu.\end{aligned}$$

Conversely, let A be an arbitrary Σ-measurable set of finite measure on which $\{\int_{T^{-n}A}|u|^2d\mu\}_{n=0}^{+\infty}$ is a moment sequence. Let $p(t)\geq 0$ on $I=[0,\|J\|_\infty]$. Then there is a Borel measure m such that for each n, $\int_{T^{-n}A}|u|^2d\mu=\int_I t^n dm(t)$. Then

$$\begin{aligned}\int_A \mathcal{L}(p)d\mu &= \sum_{n=0}^{+\infty} a_n\int_A J_n d\mu\\ &= \sum_{n=0}^{+\infty} a_n\int_{T^{-n}A}|u|^2d\mu\\ &= \sum_{n=0}^{+\infty} a_n\int_I t^n dm(t)\\ &= \int_I p(t)dm(t)\\ &\geq 0.\end{aligned}$$

Hence $\mathcal{L}(p)\geq 0$ a.e, since A was chosen arbitrarily. This fact yields that W is subnormal by Theorem 3.5. □

Proposition 3.7. *$W\in\mathcal{B}(L^2(\Sigma))$ is subnormal if and only if $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x\in X$.*

Proof. The proof follows very closely the proof of Corollary 4 in [12]. Suppose first that W is subnormal. Let $P^+(I)$ be the set of all nonnegative polynomials on I. Let $P_r^+(I)$ be the set of all polynomials with rational coefficients which are nonnegative on I. By Theorem 3.5 for each $p\in P_r^+(I)$, $\mathcal{L}(p)\geq 0$ a.e. Let $X_p:=\{x\in X:\mathcal{L}(p)\geq 0\}$. Then $\mu(X-X_p)=0$. Since $P_r^+(I)$ is a countable dense subset of $P^+(I)$ we have $\mu(X-X^*)=0$ where $X^*=\bigcup\{X_p:p\in P_r^+(I)\}$. It means that for each $p\in P^+(I)$, $\mathcal{L}(p)\geq 0$ a.e. Therefore by Theorem 3.5 it is inferred that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for every $x\in X^*$.
Conversely, suppose that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x\in X$. Let X_0 be a measurable subset of X with $\mu(X-X_0)=0$ in a way that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence on X_0. If p is an arbitrary nonnegative polynomial on I, then by Theorem 3.4, $\mathcal{L}(p)\geq 0$ exactly on X_0. Thus the positivity of $\mathcal{L}$ yields that W is subnormal by Theorem 3.5. □

Example 3.8. Consider the case that $h\circ T=h$ and $E(|u|^2)\circ T=E(|u|^2)$. This case in turn characterizes p-quasihyponormal and p-paranormal weighted composition operators; see [8]. Now the relationship $J_{n+1} = hE(J_n) \circ T^{-1}$ follows that $J_n = h^{n-1}J$ for each $n \geq 0$. Thus $\mathcal{L}(\sum_{n=0}^{+\infty} a_n t^n) = a_0 + J\sum_{n=1}^{+\infty} a_n h^{n-1}$ is positive since $J(x) \in [0, \|J\|_\infty]$ for almost every $x \in X$. Therefore W is subnormal by Theorem 3.5.

Example 3.9. Let X be the set of nonnegative integers, let Σ be the σ-algebra of all subsets of X, and take μ to be the point mass measure determined by the $\mu(k) := m_k,\ \ k \in X$. Define the measurable transformation T by

$$T(k) = \begin{cases} 0 & k = 0, 1, \\ k-1 & k \geq 2 \end{cases}$$

and the weight function u by

$$u(k) = \begin{cases} 1 & k = 0, 1, \\ k & k \geq 2. \end{cases}$$

By routine computations one may verify that

$$\begin{aligned} h(k) &= \frac{1}{m_k} \sum_{j \in T^{-1}(k)} m_j \\ h_n(k) &= \frac{1}{m_k} \sum_{j \in T^{-1}(k)} h_{n-1}(j) m_j \\ J_n(k) &= \frac{1}{m_k} \sum_{j \in T^{-n}(k)} (u(j))^2 m_j. \end{aligned}$$

So by inserting T and u in the above formulas we obtain

$$\begin{aligned} J_1(0) &= \frac{1}{m_0}(m_0 + m_1); \\ J_n(0) &= \frac{1}{m_0}(m_0 + m_1 + 2^2 m_2 + 3^2 m_3 + ... + n^2 m_n); \\ J_n(k) &= \frac{m_{k+n}}{m_k}, \quad k \geq 1. \end{aligned}$$

In particular, fix $a > 0$ and let $\epsilon > 0$ be an arbitrary. Let $m_0 = \epsilon$ and for every $k \geq 1$, let

$$m_k = \frac{1}{k^2} \int_a^{a+\epsilon} (t^k - t^{k-1}) dt.$$

In Proposition 3.7 it is proved that W is subnormal if and only if for every $k \geq 1$ the sequence $\{J_n(k)\}_{n=0}^{+\infty}$ is a moment sequence. Namely the sequences $\{m_0 + \sum_{i=1}^{n} i^2 m_i\}_{n=0}^{+\infty}$ and $\{\frac{m_{k+n}}{m_k}\}_{n=0}^{+\infty}$ are moment sequences. But these assertions are true since for every $k \geq 1$ and $n \geq 0$, we have

$$\begin{aligned} m_0 + \sum_{i=1}^{n} i^2 m_i &= m_0 + \sum_{i=1}^{n} \int_a^{a+\epsilon} (t^i - t^{i-1}) dt \\ &= \int_a^{a+\epsilon} t^n dt \end{aligned}$$

and

$$\begin{aligned} \frac{m_{k+n}}{m_k} &= \frac{1}{(k+n)^2 m_k} \int_a^{a+\epsilon} (t^{k+n} - t^{k+n-1}) dt \\ &= \int_a^{a+\epsilon} t^n \left(\frac{t^k - t^{k-1}}{(k+n)^2 m_k}\right) dt \\ &= \int_a^{a+\epsilon} t^n d\mu(t), \end{aligned}$$

where $d\mu = (\frac{t^k - t^{k-1}}{(k+n)^2 m_k})dt$ is a nonnegative measure on $[a, a+\epsilon]$. Hence W is subnormal.

References

[1] C. Burnap, I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., **337**(2008), 686-694.

[2] C. Burnap, I. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397.

[3] J. B. Conway, *Subnormal Operators*, Pitman Publ. Co., London, 1981.

[4] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr., Amer. Math. Soc., Providence, RI, 1991.

[5] M. Fujii, S. Izumino and R. Nakamoto, *Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder- McCarthy inequality*, Nihonkai Math. J., **5**(1994), 61-67.

[6] J. Herron, *Weighted conditional expectation operators*, Oper. Matrices, **5**(2011), 107-118.

[7] T. Hoover, A. Lambert and J. Quinn, *The Markov process determined by a weighted composition operator*, Studia Math., **72(3)**(1982), 225-235.

[8] M. R. Jabbarzadeh, M. R. Azimi, *Some weak hyponormal classes of weighted composition operators*, Bull. Korean Math. Soc., **47(4)**(2010), 793-803.

[9] R. Kumar, *Ascent and descent weighted composition operators on L^p-spaces*, Math. Vesnik, **60**(2008), 47-51.

[10] A. Lambert, Thomas G. Lucas, *Nagata's principle of idealization in relation to module homomorphisms and conditional expectations*, Kyungpook Math. J., **40**(2000), 327-337.

[11] A. Lambert, *Normal extentions of subnormal compostion operators*, Michigan Math. J., **35**(1988), 443-450.

[12] A. Lambert, *Subnormal composition operators*, Proc. Amer. Math. Soc., **103**(1988), 750-754.

[13] A. Lambert, *Hyponormal composition operators*, Bull. Lond. Math. Soc., **18**(1986), 395-400.

[14] A. Lambert, *Subnormality and weighted shifts*, Bull. Lond. Math. Soc., **14(2)**(1976), 476-480.

[15] E. Nordgren, *Composition operators on Hilbert spaces*, in: Lecture Notes in Math., **693**, Springer-Verlag, Berlin, 1978.

[16] M. M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

[17] R. K. Singh, J. S. Manhas, *Composition Operators on Function Spaces*, Elsevier Science Publishers B. V., North-Holland, Amsterdam, 1993.

[18] J. Stochel, J. B. Stochel, *Seminormal composition operators on L2 spaces induced by matrices: The Laplace density case*, J. Math. Anal. Appl., **375**(2011), 1-7.

[19] R. Whitley, *Normal and quasinormal composition operators*, Proc. Amer. Math. Soc., **70**(1978), 114-118.

[20] D. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946.