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Value Distribution of the Product of a Meromorphic Derivative and a Power of the Function

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Abstract. In the paper we discuss the value distribution of the product of the derivative of a transcendental meromorphic function and a power of the function.

1. Introduction

W. K. Hayman [5] proved the following result.

Theorem A. ([5]) *If $n(\geq 3)$ is an integer and f is a transcendental meromorphic function, then $f^n f'$ assumes all finite values, except possibly zero, infinitely often.*

Hayman [7] also conjectured that Theorem A might be valid for $n = 1$ and $n = 2$. E. Mues [12] settled the conjecture for $n = 2$ and the case $n = 1$ was settled by W. Bergweiler and A. Eremenko [1] and by H. H. Chen and M. L. Fang [3].

In 1999 X. C. Pang and L. Zalcman [13] considered the general order derivative of an entire function. They proved the following result.

Theorem B. ([13]) *Let f be a transcendental entire function, all of whose zeros have multiplicity at least k and let n be a positive integer. Then $f^n f^{(k)}$ assume every nonzero finite value infinitely often.*

Recently J. P. Wang [16] considered the meromorphic case and proved the following theorem.

Theorem C. ([16]) *Let f be a transcendental meromorphic function all of whose zeros have multiplicity at least t. Then for any positive integer $k(\geq 2)$, $ff^{(k)}$ assumes*

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every nonzero finite value infinitely often provided that $t = k + 1$ *for* $2 \leq k \leq 4$, $t = 5$ *for* $k = 5$ *and* $t = 6$ *for* $k \geq 6$.

N. Steinmetz [15] proved that if f is a transcendental meromorphic function, then $f^n f^{(k)}$ assume every nonzero finite value infinitely often, where $n(\geq 2)$ and k are positive integers.

In 1994 Yik Man Chiang asked the question of value distribution of $ff' - a$, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of f i.e., $T(r, a) = S(r, f)$. In response to this question W. Bergweiler [2] proved the following theorem.

Theorem D. ([2]) *Let* f *be a transcendental meromorphic function of finite order and* $a = a(z)(\not\equiv 0)$ *be a polynomial. Then* $ff' - a$ *has infinitely many zeros.*

In 2005 I. Lahiri and S. Dewan [10] observed that if $a = bz^n$, n is a nonnegative integer and b is a nonzero constant, then the order restriction on f can be withdrawn. Their result is as follows.

Theorem E. ([10]) *Let* f *be a transcendental meromorphic function. Then* $f^p f' - bz^n$ *has infinitely many zeros, where* $b(\neq 0)$ *is a constant and* $n(\geq 0)$, $p(\geq 1)$ *are integers.*

If one considers a small function, then following two results are worth mentioning, which follow from two inequalities proved by Q. D. Zhang [18].

Theorem F. *Let* f *be a transcendental meromorphic function with* $\delta(\infty; f) > \frac{7}{9}$. *Then* $ff' - a$ *has infinitely many zeros, where* $a = a(z)(\not\equiv 0, \infty)$ *is a small function of* f.

Theorem G. *Let* f *be a transcendental meromorphic function with* $\delta(\infty; f) + 2\delta(0; f) > 1$. *Then* $ff' - a$ *has infinitely many zeros, where* $a = a(z)(\not\equiv 0, \infty)$ *is a small function of* f.

K. W. Yu [17] treated the small function case without imposing any restriction on f. However he had to consider a small function and its negative as a pair of targets. He proved the following theorem.

Theorem H. ([17]) *If* $a = a(z)(\not\equiv 0, \infty)$ *is a small function of a transcendental meromorphic function* f, *then at least one of* $ff' + a$ *and* $ff' - a$ *has infinitely many zeros.*

In 2003 I. Lahiri and S. Dewan [9] considered the general order derivative and proved the following result.

Theorem I. (cf. Corollary 1 [9]) *Let* f *be a transcendental meromorphic function and* k *be a positive integer. Suppose that* $F_1 = ff^{(k)} - a$ *and* $F_2 = ff^{(k)} + a$, *where* $a = a(z)(\not\equiv 0, \infty)$ *is a small function of* f. *Then* $\Theta(0; F_1) + \Theta(0; F_2) \leq 2 - \frac{2}{(2+k)^2}$.

The problem of value distribution of $f^p f^{(k)} - a$ remains open, where f is a transcendental meromorphic function, $a = a(z)(\not\equiv 0, \infty)$ is a small function of f and p, k are positive integers. In the paper we deal with this problem.

We respectively denote by $N_{k)}(r, 0; f)$ and $\overline{N}_{k)}(r, 0; f)$ the counting function and the reduced counting function of those zeros of f which have multiplicities less than or equal to k, where k is a positive integer.

For standard definitions and notations of the value distribution theory we refer the reader to [6].

We now state the main result of the paper.

Theorem 1. *Let f be a transcendental meromorphic function and $\alpha = \alpha(z)(\not\equiv 0, \infty)$ be a small function of f such that the zero-pole sets of f and α are disjoint. Suppose that the zeros of f have multiplicity at least t, where $t = [\frac{k+1}{p}] + 1$ if $p \geq 2$, $t = k + 1$ if $p = 1$ and $1 \leq k \leq 4$ and $t = \min\{k, 6\}$ if $p = 1$ and $k \geq 5$. If $F = f^p f^{(k)} - \alpha$, then one of the following holds:*

(i) $\Theta(0; F) \leq 1 - \dfrac{p}{k+p+1}(1 - \dfrac{k+1}{pt})$ *if* $p \geq 2$;

(ii) $\Theta(0; F) \leq 1 - \dfrac{k}{4t(k+2)}$ *if* $p = 1$ *and* $1 \leq k \leq 4$;

(iii) $\Theta(0; F) \leq 1 - \dfrac{(t-4)(k+1) - t}{4t(k+1)(k+2)}$ *if* $p = 1$ *and* $k \geq 5$.

2. Lemmas

In this section we state two necessary lemmas. Let f be a transcendental meromorphic function and n, p be positive integers. A differential polynomial P of f is defined by $P(z) = \sum_{k=1}^{n} \phi_k(z)$, where $\phi_k(z) = \alpha_k(z) \prod_{j=0}^{p} (f^{(j)}(z))^{S_{kj}}$, $\alpha_k(z) \not\equiv 0$, S_{kj} are nonnegative integers and $T(r, \alpha_k) = S(r, f)$.

If we suppose only $m(r, \alpha_k) = S(r, f)$, then $P(z)$ is called a quasi-differential polynomial.

The quantities $\overline{d}(P) = \max_{1 \leq k \leq n} \{\sum_{j=0}^{p} S_{kj}\}$ and $\underline{d}(P) = \min_{1 \leq k \leq n} \{\sum_{j=0}^{p} S_{kj}\}$ are respectively called the degree and lower degree of $P(z)$. If, in particular, $\overline{d}(P) = \underline{d}(P)$, then $P(z)$ is called homogeneous.

Lemma 1. ([8]) *Let f be a transcendental meromorphic function and $P = P(z)$ be a nonconstant differential polynomial in f with $\underline{d}(P) > 1$. Suppose that $Q = \max_{1 \leq k \leq n} \{\sum_{j=1}^{p} jS_{kj}\}$. Then*

$$T(r, f) \leq \frac{Q+1}{\underline{d}(P) - 1} \overline{N}(r, 0; f) + \frac{1}{\underline{d}(P) - 1} \overline{N}(r, 1; P) + S(r, f).$$

Lemma 2. (p.39 [11]) *Let f be a nonconstant meromorphic function and Q_1, Q_2 be quasi-differential polynomials in f with $Q_2 \not\equiv 0$. Let n be a positive integer and $f^n Q_1 = Q_2$. If $\overline{d}(Q_2) \leq n$, then $m(r, Q_1) = S(r, f)$.*

3. Proof of Theorem 1

First we suppose that $p \geq 2$. We put $P = \frac{1}{\alpha} f^p f^{(k)}$. Then $\underline{d}(P) = \overline{d}(P) = p + 1$ and $Q = k$. So by Lemma 1 we get

$$\begin{aligned} T(r, f) &\leq \frac{k+1}{p} \overline{N}(r, 0; f) + \frac{1}{p} \overline{N}(r, 1; P) + S(r, f) \\ &\leq \frac{k+1}{pt} T(r, f) + \frac{1}{p} \overline{N}(r, 1; P) + S(r, f) \end{aligned}$$

and so

$$p(1 - \frac{k+1}{pt}) T(r, f) \leq \overline{N}(r, 1; P) + S(r, f). \tag{3.1}$$

We note that {cf. [4], [14]}

$$T(r, f) + S(r, f) \leq CT(r, F) + S(r, F) \tag{3.2}$$

and

$$T(r, F) \leq (k + p + 1) T(r, f) + S(r, f), \tag{3.3}$$

where C is a nonzero constant.

From (3.2) and (3.3) we see that $S(r, f)$ and $S(r, F)$ are mutually interchangeable. So from (3.1) and (3.3) we get

$$\frac{p}{k+p+1} (1 - \frac{k+1}{pt}) T(r, F) \leq \overline{N}(r, 0; F) + S(r, F).$$

This implies $\Theta(0; F) \leq 1 - \frac{p}{k+p+1} (1 - \frac{k+1}{pt})$.

Now we suppose that $p = 1$. Let us put

$$F = f f^{(k)} - \alpha \tag{3.4}$$

and

$$a = \frac{f' f^{(k)}}{f} + f^{(k+1)} - f^{(k)} \frac{F'}{F}. \tag{3.5}$$

Then

$$f a = \alpha (\frac{\alpha'}{\alpha} - \frac{F'}{F}). \tag{3.6}$$

Let $\frac{\alpha'}{\alpha}-\frac{F'}{F}\equiv 0$. Then on integration we get $F=c\alpha$, where $c(\neq 0)$ is a constant. Hence we get from (3.4)

$$ff^{(k)}=(1+c)\alpha. \tag{3.7}$$

Since $ff^{(k)}\not\equiv 0$, we have $1+c\neq 0$. From (3.7) we get

$$N(r,0;f)\leq N(r,0;\alpha)=S(r,f). \tag{3.8}$$

Also from (3.7) we obtain $\dfrac{1}{f^2}=\dfrac{1}{(1+c)\alpha}\dfrac{f^{(k)}}{f}$ and so $m(r,\frac{1}{f^2})=S(r,f)$. This implies

$$m(r,0;f)=S(r,f). \tag{3.9}$$

From (3.8), (3.9) and the first fundamental theorem we get $T(r,f)=S(r,f)$, a contradiction. Therefore $\frac{\alpha'}{\alpha}-\frac{F'}{F}\not\equiv 0$. So from (3.6) we get by Lemma 2

$$m(r,a)=S(r,f). \tag{3.10}$$

Let z_1 be a pole of f with multiplicity $q(\geq 2)$. Then z_1 is a simple pole of $\alpha(\frac{\alpha'}{\alpha}-\frac{F'}{F})$ as $\alpha(z_1)\neq 0,\infty$ and so z_1 is a zero of a with multiplicity $q-1$. Hence

$$N_{(2}(r,\infty;f)\leq 2N(r,0;a), \tag{3.11}$$

where $N_{(2}(r,\infty;f)$ denotes the counting function of multiple poles of f.

Let z_2 be a zero of f with multiplicity $q(\geq k+1)$. Then z_2 is a zero of $F'+\alpha'=f'f^{(k)}+ff^{(k+1)}$ with multiplicity at least $2q-(k+1)$.

Since $F+\alpha=ff^{(k)}$, we see that z_2 is a zero of $F+\alpha$ with multiplicity $2q-k$.

From (3.6) we get $fa=(F'+\alpha')-\frac{F'(F+\alpha)}{F}$. So z_2 is a zero of fa with multiplicity at least $2q-(k+1)$. Therefore z_2 is not a pole of a.

Also from (3.6) we see that a simple pole of f is not a pole of a. Hence the poles of a are contributed by the zeros of F and by zeros of f with multiplicities less than or equal to k and the poles of α. Therefore

$$N(r,\infty;a)=\overline{N}(r,\infty;a)\leq \overline{N}_{k)}(r,0;f)+\overline{N}(r,0;F)+S(r,f). \tag{3.12}$$

By (3.10) and (3.12) we get

$$T(r,a)\leq \overline{N}_{k)}(r,0;f)+\overline{N}(r,0;F)+S(r,f). \tag{3.13}$$

From (3.6) we obtain

$$\begin{aligned} m(r,f) &\leq m(r,0;a)+S(r,f)\\ &= T(r,a)-N(r,0;a)+S(r,f). \end{aligned} \tag{3.14}$$

Let z_0 be a simple pole of f. Then from (3.6) we see that $a(z_0) \neq 0, \infty$. Now in some neighbourhood of z_0 we get

$$f(z) = \frac{c_1}{z - z_0} + c_0 + O(z - z_0), \tag{3.15}$$

$$a(z) = a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2 \tag{3.16}$$

and

$$\alpha(z) = \alpha(z_0) + \alpha'(z_0)(z - z_0) + O(z - z_0)^2, \tag{3.17}$$

where $c_1 \neq 0$ and $\alpha(z_0) \neq 0, \infty$.

Differentiating (3.15) we get

$$f^{(j)}(z) = \frac{(-1)^j j! c_1}{(z - z_0)^{j+1}} + O(1) \ \ for \ \ j = 1, 2, 3, \ldots. \tag{3.18}$$

From (3.5) and (3.6) we have

$$\alpha f a = \alpha f' f^{(k)} + \alpha f f^{(k+1)} + f^2 f^{(k)} a - \alpha' f f^{(k)}. \tag{3.19}$$

Now by (3.15) – (3.19) we get

$$\begin{aligned}
& \{\alpha(z_0) + \alpha'(z_0)(z - z_0) + O(z - z_0)^2\}\{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\} \\
& \{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\} \\
= \ & \{\alpha(z_0) + \alpha'(z_0)(z - z_0) + O(z - z_0)\}\{\frac{-c_1}{(z - z_0)^2} + O(1)\} \\
& \{\frac{(-1)^k c_1 k!}{(z - z_0)^{k+1}} + O(1)\} + \{\alpha(z_0) + \alpha'(z_0)(z - z_0) + O(z - z_0)^2\} \\
& \{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\}\{\frac{(-1)^{k+1} c_1 (k+1)!}{(z - z_0)^{k+2}} + O(1)\} \\
& + \{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)^2\}^2\{\frac{(-1)^k c_1 k!}{(z - z_0)^{k+1}} + O(1)\} \\
& \{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\} - \{\alpha'(z_0) + O(z - z_0)\} \\
& \{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\}\{\frac{(-1)^k c_1 k!}{(z - z_0)^{k+1}} + O(1)\}.
\end{aligned} \tag{3.20}$$

Equating the coefficients of $\frac{1}{(z - z_0)^{k+3}}$ and $\frac{1}{(z - z_0)^{k+2}}$ of both sides of (3.20) we respectively get

$$c_1 a(z_0) = \alpha(z_0)(k + 2) \tag{3.21}$$

and

$$\frac{c_0}{c_1} = \frac{\alpha'(z_0)}{\alpha(z_0)} - \frac{(k+2)a'(z_0)}{(k+3)a(z_0)}. \tag{3.22}$$

From (3.15) we have

$$\frac{f'}{f} = \frac{-1}{z-z_0} + \frac{c_0}{c_1} + O(z-z_0). \tag{3.23}$$

Also from (3.6) we obtain

$$\begin{aligned}\alpha(\frac{\alpha'}{\alpha} - \frac{F'}{F}) &= fa \\ &= \{\frac{c_1}{z-z_0} + c_0 + O(z-z_0)\}\{a(z_0) + a'(z_0)(z-z_0) + O(z-z_0)^2\} \\ &= c_1 a(z_0)\{\frac{1}{z-z_0} + \frac{c_0}{c_1} + \frac{a'(z_0)}{a(z_0)}\} + O(z-z_0). \qquad (3.24)\end{aligned}$$

From (3.21) – (3.24) we get in some neighbourhood of z_0

$$\alpha(\frac{\alpha'}{\alpha} - \frac{F'}{F}) = \alpha(z_0)(k+2)\{\frac{2\alpha'(z_0)}{\alpha(z_0)} - \frac{f'}{f} - \frac{(k+1)a'(z_0)}{(k+3)a(z_0)}\} + O(z-z_0). \tag{3.25}$$

Let

$$h = (k+2)\frac{f'}{f} - \frac{F'}{F} - \frac{(2k+3)\alpha'}{\alpha} + \frac{(k+1)(k+2)a'}{(k+3)a}. \tag{3.26}$$

First we suppose that $h(z) \equiv 0$. Then on integration we get

$$f^{(k+2)(k+3)}a^{(k+1)(k+2)} = A\alpha^{(2k+3)(k+3)}F^{k+3}, \tag{3.27}$$

where $A(\neq 0)$ is a constant.

Let z_3 be a zero of f with multiplicity $p(\geq 1)$. Then from (3.27) we see that z_3 is a pole of a with multiplicity q such that $(k+2)(k+3)p = (k+1)(k+2)q$ and so $q = \frac{k+3}{k+1}p > 1$. This is impossible because from (3.5) we see that a zero of f is at most a simple pole of a. Hence f has no zero. Since a zero of F is a possible pole of a and f has no zero, we have from (3.27) $\overline{N}(r,0;F) = S(r,f)$. Therefore from (3.13) we get

$$T(r,a) = S(r,f). \tag{3.28}$$

Now by the first fundamental theorem we get

$$\begin{aligned} N(r,0;f^{(k)}) &= N(r,0;\frac{f^{(k)}}{f}) \\ &\le T(r,\frac{f^{(k)}}{f})+O(1) \\ &= N(r,\frac{f^{(k)}}{f})+S(r,f) \\ &= k\overline{N}(r,\infty;f)+S(r,f) \\ &= N(r,\infty;f^{(k)})-N(r,\infty;f)+S(r,f). \end{aligned} \tag{3.29}$$

From (3.5) we obtain $\frac{1}{f^{(k)}}=\frac{1}{a}\left(\frac{f'}{f}+\frac{f^{(k+1)}}{f^{(k)}}-\frac{F'}{F}\right)$ and so in view of (3.28) we get $m(r,0;f^{(k)})=S(r,f)$. Hence by the first fundamental theorem we get

$$T(r,f^{(k)})=N(r,0;f^{(k)})+S(r,f). \tag{3.30}$$

From (3.29) and (3.30) we see that

$$N(r,\infty;f)=S(r,f). \tag{3.31}$$

So from (3.14), (3.28) and (3.31) we get

$$T(r,f)\le T(r,a)-N(r,0;a)+S(r,f)=S(r,f),$$

a contradiction.

Let $h(z)\not\equiv 0$. Then from (3.25) we see that $h(z_0)=0$. Hence

$$\begin{aligned} N_{1)}(r,\infty;f) &\le N(r,0;h) \\ &\le T(r,h)+O(1) \\ &= N(r,h)+m(r,h)+O(1) \\ &= N(r,h)+S(r,f). \end{aligned} \tag{3.32}$$

The possible poles of h are (i) zeros and poles of α, (ii) zeros and poles of a, (iii) zeros and poles of f and (iv) zeros and poles of F. We further note that

(I) A pole of F is either a pole of f or a pole of α or of both;

(II) A zero of f with multiplicity $\ge k+2$ is a zero of a. Also a zero of f with multiplicity $k+1$ is a nonzero regular point of a;

(III) A simple pole of f is a zero of h and a multiple pole of f is a zero of a;

(IV) A zero of F, which is not a zero or a pole of α, is a pole of a.

Let z_4 be a zero of f with multiplicity k. Then $F(z_4) \neq 0, \infty$ and $f^{(k)}(z_4) \neq 0, \infty$. Then from (3.5) we see that z_4 is a simple pole of a. Therefore we have

$$N(r,h) \leq \overline{N}(r,0;a) + \overline{N}(r,\infty;a) + \overline{N}_{k-1)}(r,0;f) + \overline{N}_{k+1}(r,0;f) + S(r,f), \tag{3.33}$$

where $\overline{N}_{k+1}(r,0;f)$ denotes the reduced counting function of those zeros of f which have multiplicity exactly equal to $k+1$.

Now considering (3.11), (3.13), (3.14), (3.32) and (3.33) we get

$$\begin{aligned} T(r,f) &= m(r,f) + N_{1)}(r,\infty;f) + N_{(2}(r,\infty;f) \\ &\leq T(r,a) - N(r,0;a) + 2N(r,0;a) + \overline{N}(r,0;a) \\ &\quad + \overline{N}(r,\infty;a) + \overline{N}_{k-1)}(r,0;f) + \overline{N}_{k+1}(r,0;f) + S(r,f) \\ &\leq 4T(r,a) + \overline{N}_{k-1)}(r,0;f) + \overline{N}_{k+1}(r,0;f) + S(r,f) \\ &\leq 4\overline{N}_{k)}(r,0;f) + \overline{N}_{k-1)}(r,0;f) + \overline{N}_{k+1}(r,0;f) + 4\overline{N}(r,0;F) + S(r,f). \quad (3.34) \end{aligned}$$

Let $1 \leq k \leq 4$. then from (3.34) and (3.3), for $p=1$, we get by the hypothesis $T(r,f) \leq \frac{1}{k+1}T(r,f) + 4\overline{N}(r,0;F) + S(r,F)$ and so $\frac{1}{k+2}(1 - \frac{1}{k+1})T(r,F) \leq 4\overline{N}(r,0;F) + S(r,F)$. This implies $\Theta(0;F) \leq 1 - \frac{k}{4(k+1)(k+2)} = 1 - \frac{k}{4t(k+2)}$.

Next let $k \geq 5$. Then from (3.34) and (3.3), for $p=1$, we get by the hypothesis $T(r,f) \leq \frac{4}{t}T(r,f) + \frac{1}{k+1}T(r,f) + 4\overline{N}(r,0;F) + S(r,f)$ and so $\frac{1}{k+2}(1 - \frac{4}{t} - \frac{1}{k+1})T(r,F) \leq 4\overline{N}(r,0;F) + S(r,F)$. Therefore $\Theta(0;F) \leq 1 - \frac{(t-4)(k+1)-t}{4t(k+1)(k+2)}$. This proves the theorem. □

References

[1] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana, **11**(1995), 355-373.

[2] W. Bergweiler, *On the product of a meromorphic function and its derivative*, Bull. Hong Kong Math. Soc., **1**(1997), 97-101.

[3] H. H. Chen and M. L. Fang, *The value distribution of $f^n f'$*, Sci. China Ser. A, **38**(1995), 789-798.

[4] W. Doeringer, *Exceptional values of differential polynomials*, Pacific J. Math., **98**(1982), 55-62.

[5] W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. of Math., **70**(1959), 9-12.

[6] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).

[7] W. K. Hayman, Research Problems in Function Theory, The Athlone Press, University of London, London (1967).

[8] J. D. Hinchliffe, *On a result of Chuang related to Hayman's alternative*, Computational Methods Function Theory (CMFT), **2(1)**(2002), 293-297.

[9] I. Lahiri and S. Dewan, *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J., **26(1)**(2003), 95-100.

[10] I. Lahiri and S. Dewan, *Value distribution of meromorphic derivatives*, Kyungpook Math. J., **45(2005)**, 339-348.

[11] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berline, New York (1993).

[12] E. Mues, *Über ein Problem von Hayman*, Math. Z., **164**(1979), 239-259.

[13] X. C. Pang and L. Zalcman, *On theorems of Hayman and Clunie*, New Zealand J. Math., **28**(1999), 71-75.

[14] A. P. Singh, *On order of homogeneous differential polynomials*, Indian J. Pure Appl. Math., **16**(1985), 791-795.

[15] N. Steinmetz, *Über die Nullstellen von Differentialpolynomen*, Math. Z., **176**(1981), 255-264.

[16] J. P. Wang, *On the value distribution of* $ff^{(k)}$, Kyungpook Math. J., **46**(2006), 169-180.

[17] K. W. Yu, *A note on the product of meromorphic functions and its derivatives*, Kodai Math. J., **24**(2001), 339-343.

[18] Q. D. Zhang, *The value distribution of* $\phi(z)f(z)f'(z)$, Acta Math. Sinica, **37**(1994), 91-98.