KYUNGPOOK Math. J. 55(2015), 327-333
http://dx.doi.org/10.5666/KMJ.2015.55.2.327
pISSN 1225-6951 eISSN 0454-8124


# Certain Subclass of $p-$Valent Meromorphic Functions Associated with Linear Operator

A. O. Mostafa, M. K. Aouf, A. Shamandy and Eman Ahmed Adwan*
*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
*e-mail*: adelaeg254@yahoo.com, mkaouf127@yahoo.com, shamandy16@hotmail.com *and* eman.a2009@yahoo.com

Abstract. In this paper, we introduce a class of $p-$valent meromorphic functions associated with linear operator and derive several interesting results of this class.

## 1. Introduction and Preliminaries

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. Let $P_k(\beta)$ be the class of functions $\phi(z)$ analytic in $U$ satisfying the properties $\phi(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{\phi(z)\} - \beta}{1 - \beta} \right| d\theta \leq k\pi, \tag{1.2}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \beta < 1$. The class $P_k(\beta)$ was introduced by Padmanabhan and Parvatham [11]. For $\beta = 0$, the class $P_k(0) = P_k$ was introduced by Pinchuk [12]. Also we note that $P_2(\beta) = P(\beta)$, where $P(\beta)$ is the class of functions with positive real part greater than $\beta$ and $P_2(0) = P$ , where $P$ is the class of functions with positive real part. From (1.2), we have $h(z) \in P_k(\beta)$ if and

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* Corresponding Author.
Received August 24, 2012; accepted May 9, 2013.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Bazilevic function, p-valent meromorphic functions, linear operator.

only if there exists $h_1, h_2 \in P(\beta)$ such that

$$\phi(z) = \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z) \quad (z \in U). \tag{1.3}$$

It is known that, the class $P_k(\beta)$ is a convex set (see [10]).
For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in \mathbb{N}), \tag{1.4}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$, is defined by

$$(f*g)(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k = (g*f)(z). \tag{1.5}$$

Aouf et al. [3] considered the following linear operator $D^m_{\lambda,p}(f*g)(z) : \sum_p \longrightarrow \sum_p$ as follows:

$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z), \tag{1.6}$$

$$\begin{aligned} D^1_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda}{z^p}(z^{p+1}(f*g)(z))' \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N}), \\ D^2_{\lambda,p}(f*g)(z) &= D_{\lambda,p}\left(D_{\lambda,p}(f*g)\right)(z) \\ &= (1-\lambda)D_{\lambda,p}(f*g)(z) + \frac{\lambda}{z^p}(z^{p+1}D_{\lambda,p}(f*g)(z))' \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N}), \end{aligned} \tag{1.7}$$

and ( in general )

$D^m_{\lambda,p}(f*g)(z) = D_{\lambda,p}(D^{m-1}_{\lambda,p}(f*g)(z))$

$$= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^m a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N};\ m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}). \tag{1.8}$$

From (1.8) it is easy to verify that:

$$\lambda z(D^m_{\lambda,p}(f*g)(z))' = D^{m+1}_{\lambda,p}(f*g)(z) - (\lambda p+1)D^m_{\lambda,p}(f*g)(z) \quad (\lambda > 0). \tag{1.9}$$

It should be remarked that the linear operator $D_{\lambda,p}^{m}(f*g)$ is a generalization of many othear linear operators considered earlier. We have:
(1) If we take $g(z)=\frac{1}{z^p(1-z)}$ $(or\ b_k=1)$, then we have the operator $D_{\lambda,p}^{m}(f)(z)$ which was introduced and studied by Aouf et al. [3];
(2) If we take $g(z)=\frac{1}{z^p(1-z)}$ $(or\ b_k=1)$ and $\lambda=1$, then we have the operator $M_p^m(f)(z)$ which was introduced and studied by Aouf and Hossen [2] and Srivastava and Patel [13];
(3) If we take $m=0$ and $g(z)=z^{-p}+\sum_{k=0}^{\infty}\Psi_k(\alpha_1)z^k$ $(or\ b_k=\Psi_k(\alpha_1))$, where

$$\Psi_k(\alpha_1)=\frac{(\alpha_1)_{k+p}.......(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_k\ (1)_{k+p}}\qquad (q\leq s+1;q,s\in\mathbb{N}_0)\,, \tag{1.10}$$

then the operator $D_{\lambda,p}^{0}\ (f*g)=(f*g)$ reduces to the operator $H_{p,q,s}\,(\alpha_1)$ which was introduced and studied by Liu and Srivastava [8]. The operator $H_{p,q,s}\,(\alpha_1)$ contains the operator $\ell_p(\alpha_1,\beta_1)$ [7] for $q=2,s=1$, and $\alpha_2=1$ and also contains the operator $D^{\nu+p-1}$ (see [1] and [4]) for $q=2,s=1$ and $\alpha_1=\nu+p$ $(\nu>-p;\ p\in\mathbb{N})$, $\alpha_2=1$ and $\beta_1=p$;
(4) If we take $m=0$ and $g(z)=z^{-p}+\sum_{k=0}^{\infty}\left(\frac{l+\gamma(k+p)}{l}\right)^{\mu}z^k$
$(l>0,\gamma\geq 0,p\in\mathbb{N},\mu\in\mathbb{N}_0)$, then the operator $D_{\lambda,p}^{0}\ (f*g)=(f*g)$ reduces to the operator $J_p^{\mu}\,(\gamma,l)$ which was introduced and studied by El-Ashwah [5];
(5) If we take $m=0$ and $g(z)=z^{-p}+\sum_{k=0}^{\infty}\left(\frac{l}{l+\gamma(k+p)}\right)^{\mu}z^k$
$(l>0,\gamma\geq 0,p\in\mathbb{N},\mu\in\mathbb{N}_0)$, then the operator $D_{\lambda,p}^{0}\ (f*g)=(f*g)$ reduces to the operator $\mathcal{L}_p^{\mu}\,(\gamma,l)$ which was introduced and studied by El-Ashwah [6].

Now, by using the linear operator $D_{\lambda,p}^{m}(f*g)(z)$, we introduce class of $p-$valent Bazilevic functions of $\sum_p$ as follows:

**Definition 1.1.** A function $f\,(z)\in\sum_p$ is said to be *in the class* $K_k^m\,(p,\lambda,\alpha,\beta,\gamma)$ if it satisfies the following condition:
(1.11)

$$\left[(1-\gamma)\left(z^pD_{\lambda,p}^{m}(f*g)(z)\right)^{\alpha}+\gamma\left(\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^{m}(f*g)(z)}\right)\left(z^pD_{\lambda,p}^{m}(f*g)(z)\right)^{\alpha}\right]\in\mathcal{P}_k\,(\beta)$$

$$(k\geq 2;\gamma\geq 0;\alpha,\lambda>0;0\leq\beta<1;z\in U)\,.$$

In this paper, we investigate several properties of the class $K_k^m\,(p,\lambda,\alpha,\beta,\gamma)$.

## 2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $k\geq 2,\alpha,\gamma,\lambda>0,0\leq\beta<1$ and all powers are understood as principle values.

To prove our results we need the following lemma.

**Lemma 2.1.**([9]) *Let* $u = u_1 + iu_2$, $v = v_1 + iv_2$ *and* $\Phi(u, v)$ *be a complex-valued function satisfying the conditions:*

(i) $\Phi(u, v)$ *is continuous in a domain* $D \in \mathbb{C}^2$.

(ii) $(0, 1) \in D$ *and* $\Phi(1, 0) > 0$.

(iii) $\Re\{\Phi(iu_2, v_1)\} > 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \le -\frac{1}{2}\left(1 + u_2^2\right)$.

*If* $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$ *is analytic in* $U$ *such that* $\left(h(z), zh'(z)\right) \in D$ *and* $\Re\left\{\Phi\left(h(z), zh'(z)\right)\right\} > 0$ *for* $z \in U$, *then* $\Re\{h(z)\} > 0$ *in* $U$.

**Theorem 2.2.** *If* $f \in K_k^m(p, \lambda, \alpha, \beta, \gamma)$, *then*

$$\left(z^p D_{\lambda,p}^m (f * g)(z)\right)^\alpha \in \mathcal{P}_k(\mu), \tag{2.1}$$

*where* $\mu$ *is given by*

$$\mu = \frac{\gamma\lambda + 2\alpha\beta}{2\alpha + \gamma\lambda}. \tag{2.2}$$

*Proof.* Setting

$$\left(z^p D_{\lambda,p}^m (f * g)(z)\right)^\alpha = H(z) = (1-\mu)\, h(z) + \mu \tag{2.3}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\{(1-\mu)\, h_1(z) + \mu\} - \left(\frac{k}{4} - \frac{1}{2}\right)\{(1-\mu)\, h_2(z) + \mu\},$$

where $h_i(z)$ $(i = 1, 2)$ are analytic in $U$ with $h_i(0) = 1$ $(i = 1, 2)$, and $h(z)$ is given by (1.3). Differentiating both sides of (2.3) with respect to $z$ and using (1.9) in the resulting equation, we obtain

$$\left[(1-\gamma)\left(z^p D_{\lambda,p}^m (f * g)(z)\right)^\alpha + \gamma\left(\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\right)\left(z^p D_{\lambda,p}^m (f * g)(z)\right)^\alpha\right]$$

$$= \left\{(1-\mu)\, h(z) + \mu + \frac{\gamma\lambda(1-\mu)\, zh'(z)}{\alpha}\right\} \in P_k(\beta) \quad (z \in U),$$

which implies that

$$\frac{1}{1-\beta}\left\{\mu - \beta + (1-\mu)\, h_i(z) + \frac{\gamma\lambda(1-\mu)\, zh_i'(z)}{\alpha}\right\} \in P \quad (z \in U; i = 1, 2).$$

We form the function $\Phi(u, v)$ by choosing $u = h_i(z)$, $v = zh_i'(z)$, that is

$$\Phi(u, v) = \mu - \beta + (1-\mu)\, u + \frac{\gamma\lambda(1-\mu)\, v}{\alpha}.$$

Clearly, the first two conditions of Lemma 2.1 are satisfied. Now, we verify the condition (iii) as follows:

$$\begin{aligned}\Re\{\Phi(iu_2, v_1)\} &= \mu-\beta+\Re\left\{\frac{\gamma\lambda(1-\mu)v_1}{\alpha}\right\}\\ &\leq \mu-\beta-\frac{\gamma\lambda(1-\mu)\left(1+u_2^2\right)}{2\alpha}\\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A &= 2\alpha(\mu-\beta)-\gamma\lambda(1-\mu),\\ B &= -\gamma\lambda(1-\mu),\\ C &= 2\alpha.\end{aligned}$$

We note that $\Re\{\Phi(iu_2, v_1)\}<0$ if and only if $A=0, B<0$. From $\mu$ given by (2.2), we have $0\leq\mu<1$, $A=0$ and $B<0$. Therefore applying Lemma 2.1, we have $h_i(z)\in P\,(i=1,2)$ and consequently $H(z)\in P_k(\mu)$ for $z\in U$. This completes the proof of Theorem 2.2. □

**Theorem 2.3**. *If* $f\in K_k^m(p,\lambda,\alpha,\beta,\gamma)$, *then*

$$\left(z^pD^m_{\lambda,p}(f*g)(z)\right)^{\frac{\alpha}{2}}\in\mathcal{P}_k(\eta), \tag{2.4}$$

*where* $\eta$ *is given by*

$$\eta=\frac{\lambda\gamma+\sqrt{n^2\gamma^2\lambda^2+4(\alpha+n\gamma\lambda)\beta\alpha}}{2(\alpha+n\gamma\lambda)}. \tag{2.5}$$

*Proof.* Let $f\in K_k^m(p,\lambda,\alpha,\beta,\gamma)$ and

$$\begin{aligned}\left(z^pD^m_{\lambda,p}(f*g)(z)\right)^{\alpha} &= M(z)=[(1-\eta)h(z)+\eta]^2 \qquad (2.6)\\ &= \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\eta)h_1(z)+\eta]^2-\left(\frac{k}{4}-\frac{1}{2}\right)[(1-\eta)h_2(z)+\eta]^2,\end{aligned}$$

where $h_i(z)$ $(i=1,2)$ are analytic in $U$ with $h_i(0)=1$ $(i=1,2)$ and $h(z)$ is given by (1.3). Differentiating both sides of (2.6) with respect to $z$ and using (1.9) in the resulting equation, we obtain

$$\begin{aligned}&\left[(1-\gamma)\left(z^pD^m_{\lambda,p}(f*g)(z)\right)^{\frac{\alpha}{2}}+\gamma\left(\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\right)\left(z^pD^m_{\lambda,p}(f*g)(z)\right)\right]\\ =\ &\left\{[(1-\eta)h(z)+\eta]^2+[(1-\eta)h(z)+\eta]\,\frac{2\gamma\lambda(1-\eta)zh'(z)}{\alpha}\right\}\in P_k(\beta)\quad(z\in U),\end{aligned}$$

which implies that

$$\frac{1}{1-\beta}\left\{[(1-\eta)h_i(z)+\eta]^2+[(1-\eta)h(z)+\eta]\frac{2\gamma\lambda(1-\eta)zh_i^{'}(z)}{\alpha}-\beta\right\}\in P\quad(i=1,2).$$

We form the function $\Phi(u,v)$ by choosing $u=h_i(z)$, $v=zh_i^{'}(z)$, that is

$$\Phi(u,v)=[(1-\eta)u+\eta]^2+[(1-\eta)u+\eta]\frac{2\gamma\lambda(1-\eta)v}{\alpha}-\beta.$$

Clearly, the conditions (i) and (ii) of Lemma 2.1 are satisfied. Now, we verify the condition (iii) as follows:

$$\begin{aligned}\Re\{\Phi(iu_2,v_1)\} &= \eta^2-(1-\eta)^2u_2^2+\frac{2\gamma\lambda\eta(1-\rho_4)v_1}{\alpha}-\beta\\ &\le \eta^2-\beta-(1-\eta)^2u_2^2-\frac{\gamma\lambda\eta(1-\eta)\left(1+u_2^2\right)}{\alpha}\\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A &= \alpha\left(\eta^2-\beta\right)-\gamma\lambda\eta(1-\eta),\\ B &= -\left[\alpha(1-\eta)^2+n\gamma\lambda\eta(1-\eta)\right],\\ C &= \frac{\alpha}{2}.\end{aligned}$$

We note that $\Re\{\Phi(iu_2,v_1)\}<0$ if and only if $A=0,B<0$. From $\eta$ given by (2.5), we have $0\le\eta<1$, $A=0$ and $B<0$. Therefore applying Lemma 2.1,we have $h_i(z)\in P\,(i=1,2)$ and consequently $M(z)\in P_k(\eta)$ for $z\in U$. This completes the proof of Theorem 2.3 □

**Remark 2.4.** Specializing $m,\lambda$ and $g(z)$ in the above results, we obtain the corresponding results for classes corresponding to the corresponding operators (1-5) defined in the introduction.

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